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Kyoto University
Supersymmetric analysis of
discrete magnetic Schrödinger operators

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Abstract
In the last symposium (Jul. 2001), T. Shirai talked about the spectrum of the infinitely extended Sierpinski lattice [7, 3]. Their results are based on some relations between the spectra of an infinite regular graph and its line-graph. In this report, we extend their results to the cases of discrete magnetic Schrödinger operators on infinite regular graphs.

1 Definitions
A graph $G = (V(G), A(G))$ is a pair of the vertex set $V(G)$ and the oriented edge set $A(G)$. We say that two vertices $x, y$ are adjacent if there exists an edge

![Graph Diagram]

Figure 1: a graph $G$

which connects them. We denote $x \sim y$ if $x$ and $y$ are adjacent. Let $\alpha \in A(G)$,
which has direction from the origin $x \in V(G)$ to the terminus $y \in V(G)$. Then, we denote $\alpha = xy$, $o(\alpha) = x$ and $t(\alpha) = y$. We denote $\overline{\alpha} = yx$ the reverse edge of $\alpha$. We assume that $\overline{\alpha} \in A(G)$ provided $\alpha \in A(G)$.

Let

$$A_x(G) = \{ \alpha \in A(G); o(\alpha) = x \},$$

$$\deg(x) = \#A_x(G).$$

We call $\deg(x)$ the degree of $x$. If there exists a constant $d$ such that $\deg(x) = d$ for all $x \in V(G)$, then the graph $G$ is called $d$-regular. Regularity has important role in this report.

Throughout this report, we assume that (i) $G$ is locally finite, that is, $\deg(x) < \infty$ for all $x \in V(G)$; (ii) $G$ has no loop and multiple edge.

We define the discrete Laplacian $\triangle_G$ on a graph $G$. We work on the Hilbert space

$$l^2(G) = \left\{ f : V(G) \to \mathbb{C}; \sum_{x \in V(G)} |f(x)|^2 < \infty \right\}.$$

The discrete Laplacian $\triangle_G$ acts $f \in l^2(G)$ as follow:

$$(\triangle_G f)(x) = \frac{1}{\deg(x)} \sum_{\alpha \in A_x(G)} [f(t(\alpha)) - f(x)]$$

$$= \frac{1}{\deg(x)} \left[ \sum_{\alpha \in A_x(G)} f(t(\alpha)) \right] - f(x)$$

$$= \frac{1}{\deg(x)} \left[ \sum_{y \sim x} f(y) \right] - f(x).$$

We denote $\text{Spec}(-\triangle_G)$ the spectrum of $-\triangle_G$.

Remark 1. It is a well-known fact that $\text{Spec}(-\triangle_G) \subset [0,2]$. We remark that all of the operators in this report is bounded.

We use three graphs associated with a given graph $G$. First one is the subdivision graph $S(G)$. See Figure 2. We make $S(G)$ by adding one vertex $|\alpha|$ at the midpoint of each edges $\alpha \in A(G)$. We note $|\overline{\alpha}| = |\alpha|$. Formally, we give

$$V(S(G)) = V(G) \cup E(G),$$

$$A(S(G)) = \{ x\alpha, \alpha x; \; x \in V(G), \alpha \in A_x(G) \}.$$ 

Here, we put

$$E(G) = \{ |\alpha|; \; \alpha \in A(G) \}.$$ 

We call $E(G)$ the (unoriented) edge set of $G$. 
Second one is the line graph $L(G)$. See Figure 3. A vertex of $L(G)$ is an edge of $G$;

$$V(L(G)) = E(G).$$

The vertices $|\alpha|, |\beta| \in V(L(G))$ are adjacent on $L(G)$ if and only if $\alpha, \beta \in A(G)$ are adjacent on $G$;

$$A(L(G)) = \{\alpha\beta; \alpha, \beta \in A(G), \alpha \sim \beta\}.$$

Last one is the para-line graph $P(G)$ introduced by Yu. Higuchi [2]. See Figure 4. To construct $P(G)$, we add two vertices $x'$ and $y'$ on each edges $xy \in A(G)$ in this order and then connect $x'$ and $y'$. Moreover, if $o(\alpha) = o(\beta)$, then we connect $o(\alpha)'$ and $o(\beta)'$.

As a result, we have

$$P(G) = L(S(G)), \quad (1)$$

that is, the para-line graph is the line graph of the subdivision graph. It seems that this $P(G)$ has more information of $G$ than $S(G)$ or $L(G)$. 

Figure 2: a graph $G$ and its subdivision graph $S(G)$

Figure 3: a graph $G$ and its line graph $L(G)$

Figure 4: a graph $G$ and its para-line graph $P(G)$
We have a natural question on the relation among the spectra of the four laplacians, \( \Delta_G, \Delta_S(G), \Delta_L(G), \Delta_P(G) \). Yu. Higuchi and T. Shirai gave the answer. In the next section, we review their results (See, [3, 7]).

2 Higuchi and Shirai’s results

\[
\text{Spec}(-\Delta_{S_n}) = \bigcup_{k=0}^{\infty} \left[ g^{-k}\left(\frac{n+1}{2n}\right) \cup g^{-k}\left(\frac{n+3}{2n}\right) \right] \cup \left\{ \frac{n+1}{n} \right\}
\]

Here, \( g(x) = -2nx^2 + (n+3)x \).
They proved this theorem using approximation by finite lattices. Higuchi and Shirai gave new proof based on some relations between the spectra of an infinite regular graph and its line-graph, without any approximation. They proved Theorem 2 as a conclusion of the following four lemmas.

**Lemma 3 (Shirai [6]).** Let $G$ be a $d$-regular graph with $d \geq 3$. We have
\[
\text{Spec}(-\triangle_{S(G)}) = \psi^{-1}(\text{Spec}(-\triangle_{G})) \cup \{1\}
\]
Here, $\psi(x) = 2(2x - x^2)$.

**Lemma 4 (Shirai [6]).** Let $G$ be a $d$-regular graph with $d \geq 3$. We have
\[
\text{Spec}(-\triangle_{L(G)}) = \frac{2}{2d - 2} \text{Spec}(-\triangle_{G}) \cup \left\{\frac{d+2}{d}\right\}.
\]

Shirai proved Lemma 3 and Lemma 4 using the weak Weyl criterion on essential spectrum; He constructed a weak sequence for $-\triangle_{S(G)}$ from an eigenvector of $-\triangle_{G}$ and vice versa.

**Lemma 5 (Shirai [6]).** Let $G$ be $d$-regular with $d \geq 3$. We have
\[
\text{Spec}(-\triangle_{P(G)}) = \phi^{-1}(\text{Spec}(-\triangle_{G})) \cup \{1\} \cup \left\{\frac{n+3}{n+1}\right\}
\]
Here, $\phi(x) = -dx^2 + (d+2)x$.

This Lemma 5 can be obtained from Lemmas 3 and 4 and the fact Eq. (1).

**Lemma 6 (Higuchi and Shirai [3]).** Let $S_n$ be $n$-dim. Sierpinski lattice. Then, there exists a $(n+1)$-regular graph $G_n$ such that
\[
P(G_n) = G_n \quad \text{and} \quad S_n = L(G_n).
\]

**Outline of HS's proof of Theorem 2.** By Lemmas 5 and 6, we have the equation of the set Spec($-\triangle_{G_n}$),
\[
\text{Spec}(-\triangle_{G_n}) = \phi^{-1}(\text{Spec}(-\triangle_{G_n})) \cup \{1\} \cup \left\{\frac{n+3}{n+1}\right\}.
\]
Since the map from Spec($-\triangle_{G_n}$) to RHS is a contraction map, there exists a unique solution of this equation and we can derive Spec($-\triangle_{G_n}$) exactly. Since $S_n = L(G_n)$, Lemmas 4 implies the desired result. For more detail, see Refs. [3, 7, 6].

Our goals in this report are (i) we give another simplest proof of Lemma 3 and Lemma 4 using supersymmetry; (ii) we extend these to magnetic Schrödinger case. We devote the next section to (i) and do the last section to (ii).

**Remark 7.** Shirai proved that 1 is always the infinitely degenerate eigenvalue of $-\triangle_{S(G)}$ and $(d+2)/d$ is always the infinitely degenerate eigenvalue of $-\triangle_{L(G)}$. Though we are interested in these eigenvalues, we omit the discussions on these (Remark 18).
3 Supersymmetry

In this section, we give our new proof on the relations among the spectra of $G$, $S(G)$ and $L(G)$.

We summarize the facts on supersymmetry. Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces and $A$ be a densely defined closed linear operator from $\mathcal{H}_1$ to $\mathcal{H}_2$.

Theorem 8 (Deift [1]). We have

$$\text{Spec} (AA^*) \setminus \{0\} = \text{Spec} (A^*A) \setminus \{0\}$$

with taking account of multiplicity.

Corollary 9 (I. Shigekawa [5]). Let

$$D = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} \text{ on } \mathcal{H}_1 \oplus \mathcal{H}_2$$

and

$$H = D^2 = \begin{pmatrix} A^*A & 0 \\ 0 & A^*A \end{pmatrix}.$$

Then, we have that

$$\text{Spec} (H) \setminus \{0\} = \text{Spec} (A^*A) \setminus \{0\} = \text{Spec} (AA^*) \setminus \{0\}$$

and

$$\text{Spec} (D) \setminus \{0\} = (\sqrt{\text{Spec} (H)} \cup -\sqrt{\text{Spec} (H)}) \setminus \{0\}.$$

In physics literatures, $D$ is called a supercharge and $H$ is called a SUSY-Hamiltonian. We remark that we can ignore the condition, which $A$ must be densely defined and closed, since all of our operators is bounded (Remark 1).
3.1 the spectra of bipartite graph

We start to prove a well-known fact on spectrum of graph using supersymmetry. Let $G$ be bipartite, that is,

$V(G) = V_1 \cup V_2,
V_1 \cap V_2 = \emptyset,
x \not\sim y \text{ for all } x, y \in V_i \quad (i = 1, 2).$

![Bipartite graph](image)

Figure 7: bipartite graph and non-bipartite graph

**Lemma 10.** If $G$ is bipartite, then $\text{Spec}(-\triangle_G)$ is symmetric w.r.t. 1.

**Proof.** We have $l^2(G) = l^2(V_1) \oplus l^2(V_2)$. Let $\phi_{12}$ be an operator from $l^2(V_1)$ to $l^2(V_2)$ defined by

$$(\phi_{12}f)(y) = \frac{1}{\deg(y)} \sum_{x \sim y} f(x).$$

Then, we have

$$\triangle_G + 1 = \begin{pmatrix} 0 & \phi_{12}^* \\ \phi_{12} & 0 \end{pmatrix}.$$

Thus, $\text{Spec}(-\triangle_G - 1)$ is symmetric w.r.t. 0. \hfill \Box

Similarly, we define the operator $\phi_{21}$ from $l^2(V_2)$ to $l^2(V_1)$ by

$$(\phi_{21}g)(x) = \frac{1}{\deg(x)} \sum_{y \sim x} g(y).$$

Then we have $\phi_{21} = \phi_{12}^*$.

These operators $\phi_{12}$ and $\phi_{21}$ are used in our new proofs in the following.
3.2 the spectra of subdivision graph

We consider the relation between $G$ and $S(G)$.

Lemma 11 (SUSY version of Lemma 3). For arbitrary graph $G$, we have

$$\text{Spec}(-\triangle_{S(G)}) = \psi^{-1}(\text{Spec}(-\triangle_{G})) \cup \{1\}$$

Here, $\psi(x) = 2(2x - x^2)$.

![Figure 8: the subdivision graph as a bipartite graph](image)

Proof. $S(G)$ is bipartite (See, Fig. 8). In fact, we can take $V_1 = V(G)$ and $V_2 = E(G)$;

$$V(S(G)) = V(G) \cup E(G) = V_1 \cup V_2.$$ 

Thus, we have

$$\Delta_{S(G)} + 1 = \begin{pmatrix} 0 & \phi_{21} \\ \phi_{12} & 0 \end{pmatrix}.$$ 

Moreover, we can see

$$(\Delta_{S(G)} + 1)^2 = \begin{pmatrix} \frac{1}{2}(\Delta_{G} + 2) & 0 \\ 0 & \phi_{12}\phi_{21} \end{pmatrix}.$$ 

Indeed, we can write $\phi_{12}$ and $\phi_{21}$ as follows:

$$(\phi_{12}f)(|\alpha|) = \frac{1}{2}[f(t(\alpha)) + f(o(\alpha))],$$ 

$$(\phi_{21}g)(x) = \frac{1}{\deg(x)} \sum_{\alpha \in A_{x}(G)} g(|\alpha|).$$
Therefore,
\[
(\phi_{21}\phi_{12}f)(x) = \frac{1}{\deg(x)} \sum_{\alpha \in A_x(G)} [\phi_{12}f](|\alpha|) = \frac{1}{\deg(x)} \sum_{\alpha \in A_x(G)} \frac{1}{2} [f(t(\alpha)) + f(o(\alpha))] = \frac{1}{2} \left( \frac{1}{\deg(x)} \sum_{\alpha \in A_x(G)} f(t(\alpha)) + f(x) \right) = \frac{1}{2} (\triangle_G + 2)f(x)
\]

Thus, we obtain
\[
\text{Spec} \left( \Delta_{S(G)} + 1 \right) \setminus \{0\} = \pm \sqrt{\text{Spec} \left( \frac{1}{2} (\triangle_G + 2) \right) \setminus \{0\}}.
\]

Thus, the spectral mapping theorem implies Lemma 1.

Via supersymmetry, we can not see that 1 is an infinitely degenerate eigenvalue of $-\Delta_{S(G)}$. We need another discussion, but omit it here (Remark 18).

We remark that we do not need the regularity condition as in Lemma 3.

### 3.3 the spectra of line graph

We consider the relation between $G$ and $L(G)$.

**Lemma 12 (SUSY version of Lemma 4).** Let $G$ be $d$-regular with $d \geq 3$. Then, we have

\[
\text{Spec} \left( -\Delta_{L(G)} \right) = \frac{2}{2d-2} \text{Spec} \left( -\Delta_G \right) \cup \left\{ \frac{d+2}{d} \right\}.
\]

**Proof.** We use same $\phi_{21}$ and $\phi_{12}$ as in Lemma 1 and we have

\[
\Delta_{S(G)} + 1 = \begin{pmatrix} 0 & \phi_{21} \\ \phi_{12} & 0 \end{pmatrix} \text{ on } l^2(V_1) \oplus l^2(V_2).
\]

Here, $V_1 = V(G)$, $V_2 = E(G)$. We can identify $E(G)$ and $V(L(G))$. (See, Fig. 9.) If $G$ is $d$-regular, then $L(G)$ is $2d - 2$-regular. Therefore, $l^2(L(G))$ and $l^2(V_2)$ is unitary equivalent through the unitary operator $U$ defined by

\[
U : l^2(L(G)) \to l^2(V_2), \quad Uf = \sqrt{d-1} f.
\]
Using this $U$, we obtain
\[(\Delta_{S(G)} + 1)^2 = \begin{pmatrix} \frac{1}{2}(\Delta_{G} + 2) & 0 \\ 0 & U \left[ \frac{d-1}{d} (\Delta_{L(G)} + \frac{d}{d-1}) \right] U^* \end{pmatrix} \]
by direct computations. Thus,
\[
\text{Spec} \left( \frac{1}{2}(\Delta_{G} + 2) \right) \setminus \{0\} = \text{Spec} \left( \frac{d-1}{d} (\Delta_{L(G)} + \frac{d}{d-1}) \right) \setminus \{0\}.
\]

Via supersymmetry, we can not see that $(d+2)/d$ is an infinitely degenerate eigenvalue of $-\Delta_{L(G)}$. We need another discussion, but omit it here (Remark 18).

# 4 discrete magnetic Schrödinger operator

For simplicity, we assume that the transition probability on $G$ is isotropic. We can remove this restriction.

We introduce the space of 1-forms (vector potentials) on graph $G$.

\[C^1(G) = \{ \theta : A(G) \to \mathbb{R}; \ \theta(\overline{\alpha}) = -\theta(\alpha) \}\]

We define the discrete magnetic Schrödinger operator $H_{\theta,G}$ with a 1-form $\theta$ by
\[
H_{\theta,G}f(x) = \frac{1}{\deg(x)} \sum_{\alpha \in A_{2}(G)} e^{\theta(\alpha)} f(t(\alpha)) - f(x) = \frac{1}{\deg(x)} \left[ \sum_{\alpha \in A_{2}(G)} e^{\theta(\alpha)} f(t(\alpha)) \right] - f(x).
\]
Our problem is whether we can extend Lemma 11 and Lemma 12 for $H_{\theta,G}$.

Remark 13. In ordinary, $H_{\theta,G}$ is defined with the opposite sign. Then $H_{\theta,G}$ is non-negative. But, in this report, we want to compare it to the discrete Laplacian, so we choose this sign.

For later use, we introduce a quantity related to 1-form. Let $C$ be an oriented cycle on $G$, i.e.,

$$C = \{\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}\} \subset A(G)$$

such that $t(\alpha_i) = o(\alpha_{i+1})$ ($\alpha_n = \alpha_0$). For this cycle $C$ and $\theta$, we set

$$\Psi(\theta, C) = \sum_{\alpha \in C} \theta(\alpha).$$

We call this $\Psi(\theta, C)$ the magnetic flux through the cycle $C$.

### 4.1 the spectra of subdivision graph

Lemma 14 (magnetic case of Lemma 11). Let $G$ be an arbitrary graph. Assume that $\theta \in C^1(G)$ and $\theta_8 \in C^1(S(G))$ satisfy that

$$\theta(\alpha) = \theta_S(o(\alpha)|\alpha|) + \theta_S(|\alpha|t(\alpha)) \quad \text{for all } \alpha \in A(G).$$

Then,

$$\text{Spec}(-H_{\theta_S,S(G)}) = \psi^{-1}(\text{Spec}(-H_{\theta,G})) \cup \{1\}$$

Here, $\psi(x) = 2(2x - x^2)$.

**Proof.** Let

$$(\phi_{12}f)(|\alpha|) = \frac{1}{2} \sum_{\beta \in \{\alpha, \beta\}} e^{i\theta_S(|\alpha|t(\beta))} f(t(\beta)),$$

$$(\phi_{21}g)(x) = \frac{1}{\deg(x)} \sum_{\alpha \in A_x(G)} e^{i\theta_S(x|\alpha|)} g(|\alpha|).$$

Then, by direct computations, we obtain that

$$H_{\theta_S,S(G)} + 1 = \begin{pmatrix} 0 & \phi_{21} \\ \phi_{12} & 0 \end{pmatrix},$$

$$(H_{\theta_S,S(G)} + 1)^2 = \begin{pmatrix} -\frac{1}{2}(H_{\theta,G} + 2) & 0 \\ 0 & \phi_{12}\phi_{21} \end{pmatrix}.$$
Remark 15. The assumption of this Lemma 14 is natural. These $\theta$ and $\theta_S$ has same magnetic flux for same cycle. Let

$$C = \{\alpha_0, \alpha_1, \ldots, \alpha_n\},$$
$$C_S = \{\alpha_{01}, \alpha_{02}, \alpha_{10}, \alpha_{11}, \ldots, \alpha_{n,0}, \alpha_{n,1}\}.$$

See Figure 10. Then, we have

$$\Psi(\theta_S, C_S) = \sum_{\alpha \in C_S} \theta_S(\alpha) = \sum_{i=0}^{n} (\theta_S(\alpha_{i,0}) + \theta_S(\alpha_{i,1}))$$
$$= \sum_{i=0}^{n} \theta(\alpha_i) = \sum_{\alpha \in C} \theta(\alpha) = \Psi(\theta, C).$$

Of course, via supersymmetry, we can not see that 1 is an infinitely degenerate eigenvalue of $-H_{\theta,S(G)}$. We need another discussion, but omit it here. (cf. Remark 18).

4.2 the spectra of line graph

Lemma 16 (magnetic case of Lemma 12 (Lemma 4)). Let $G$ be $d$-regular with $d \geq 3$. Assume that $\theta \in C^1(G)$, $\theta_S \in C^1(S(G))$, $\theta_L \in C^1(L(G))$ satisfy that

$$\theta(\alpha) = \theta_S(o(\alpha)|\alpha|) + \theta_S(|\alpha|t(\alpha)) \text{ for all } \alpha \in A(G),$$
$$\theta_L(\alpha) = \theta_S(|\alpha|x) + \theta_S(x|\beta|) \text{ for all } \alpha \beta \in A((L(G))).$$

Then

$$\text{Spec } (-H_{\theta_L,L(G)}) = \frac{2}{2d-2} \text{Spec } (-H_{\theta,G}) \cup \{\frac{d+2}{d}\}.$$

Proof. We use same identification between $E(G)$ and $V(L(G))$, $l^2(V_2)$ and $l^2(L(G))$ using $U$. Then, using same $\phi_{12}$ and $\phi_{21}$ in the proof of Lemma 14, we have

$$(-H_{\theta_S,S(G)} + 1)^2 = \begin{pmatrix}
\frac{1}{2}(-H_{\theta,G} + 2) & 0 \\
0 & U \left[ \frac{d-1}{d} \left( -H_{\theta_L,L(G)} + \frac{d}{d-1} \right) \right] U^* \end{pmatrix}.$$
Thus, we can obtain the desired result.

Figure 11: Same cycle on $G$ and $L(G)$

Remark 17. The assumption of this Lemma 16 is natural. These $\theta$ and $\theta_S$ has same magnetic flux for same cycle. Let

$$C = \{\alpha_0, \alpha_1, \ldots, \alpha_n\},$$
$$C_L = \{\alpha_1\alpha_2, \alpha_2\alpha_3, \ldots, \alpha_n\alpha_1\}.$$ 

Then, it holds that $\Psi(\theta_L, C_L) = \Psi(\theta, C)$. The pair of $C = \{\alpha_0, \alpha_1, \alpha_2\}$ and $C_L = \{\alpha_0\alpha_1, \alpha_1\alpha_2, \alpha_2\alpha_0\}$ in Figure 11 is an example. But, $L(G)$ maybe has some cycles, which has no corresponding cycles on $G$. The cycle $\{\alpha_2\alpha_0, \alpha_0\beta, \beta\alpha_2\}$ in Figure 11 is an example. These cycles have zero magnetic flux.

Remark 18. As in Remark 7, though we omit the discussions on the eigenvalue 1 of $-H_G$ and the eigenvalue $(d + 2)/d$ of $-H_{\theta,L(G)}$, these are corresponding to $\ker \phi_{12}$ and $\ker \phi_{21}$. In other words, these eigenvalues are zero-modes in SUSY context. So, we must investigate these states in detail [4].

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References


