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Deformation of a renormalization-group equation applied to infinite-order phase transitions

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ABSTRACT

By adding a linear term to a renormalization-group equation in a system exhibiting infinite-order phase transitions, asymptotic behavior of running coupling constants are derived in an algebraic manner. A benefit of this method is presented explicitly using several examples.

I. INTRODUCTION

Renormalization-group (RG) technique is one of the most powerful methods for investigating critical phenomena in statistical physics[1]. In general, RG transformation (RGT) consists of a coarse graining and a rescaling. It reduces many-body effects in a statistical model to an ordinary differential equation of coupling constants. The differential equation is called the RG equation (RGE), and has generally the following form:

$$\frac{d\boldsymbol{g}}{dt} = \boldsymbol{V}(\boldsymbol{g}), \tag{1}$$

where $\boldsymbol{g} = (g_1, \cdots, g_n)$, a collection of coupling constants depending on t, and $t = \log L$ with L giving the length scale of the coarse graining in the RG. One obtains a beta function $\boldsymbol{V}(\boldsymbol{g}) = (V_1(\boldsymbol{g}), \cdots, V_n(\boldsymbol{g}))$ by applying the RGT explicitly to a statistical model. We can derive universal exponents that characterize critical phenomena from asymptotic behavior of solutions of Eq. (1) for large t.

Since the asymptotic behavior is determined by vicinity of a fixed point $\boldsymbol{g}^*$, linearization of $\boldsymbol{V}(\boldsymbol{g})$ about $\boldsymbol{g}^*$ is effective enough to obtain the exponents. For example, in a second-order phase transition, the correlation length ξ typically behaves as

$$\xi = \text{const.}/|T - T_c|^{\nu}, \tag{2}$$

where ν is the correlation-length exponent and T is a parameter specifying a state in a statistical model (e.g., the temperature). In the language of RG, T parametrizes initial

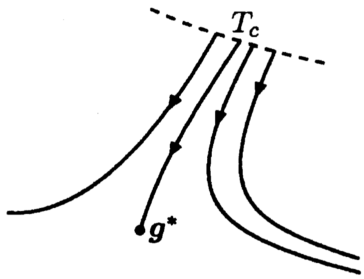

FIG. 1: Typical RG trajectories near a phase transition. As T changes, an initial value moves on the dashed line. The trajectory with $T = T_c$ is absorbed into the fixed point $\boldsymbol{g}^*$.

values of RGE. The trajectory starting from the initial value at $T = T_c$ is absorbed into the fixed point. Other trajectories approach $\boldsymbol{g}^*$ once but leave the fixed point subsequently, as shown in Fig. 1. This implies that it takes longer for $\boldsymbol{g}$ to leave the fixed point as T approaches T_c.

Let $\bar{t}$ be time satisfying $|\boldsymbol{g}(\bar{t}) - \boldsymbol{g}^*| \sim O(1)$. Then the correlation length ξ is related to $\bar{t}$ by the following formula[1]:

$$\xi = \text{const.} \times e^{\bar{t}}. \tag{3}$$

Therefore, if the scaling matrix $M(\boldsymbol{g}^*)$, where

$$M_{ij}(\boldsymbol{g}^*) \equiv \frac{\partial V_i}{\partial g_j}(\boldsymbol{g}^*) \tag{4}$$

has a unique positive eigenvalue α, $\bar{t}$ behaves as

$$|\boldsymbol{g}(\bar{t}) - \boldsymbol{g}^*| \sim |\boldsymbol{a}(T)| e^{\alpha \bar{t}}, \tag{5}$$

as $\bar{t} \to \infty$. Here $\boldsymbol{a}(T)$ is an initial value parametrized by T. Expanding $\boldsymbol{a}(T)$ about $T = T_c$, we have

$$\bar{t} \sim |T - T_c|^{-1/\alpha} \tag{6}$$

as $\bar{t} \to \infty$. From Eqs.(2), (3) and (6), we get

$$\nu = \frac{1}{\alpha}. \tag{7}$$

In this way, we do not need to find an explicit solution of nonlinear RGE (1).

On the other hand, in the case of infinite-order phase transitions, ξ has the following essential singularity:

$$\xi = \text{const.} \times \exp\left(A/|T - T_c|^{\sigma}\right), \tag{8}$$

where σ is a universal exponent and $A > 0$ is a non-universal constant. Such behavior is observed when all the coupling constants are marginal, i.e. when the canonical dimensions of the coupling constants are zero at $\boldsymbol{g}^*$. Since the linear term in $\boldsymbol{V}(\boldsymbol{g})$ is proportional to the canonical dimensions of $\boldsymbol{g}$, $M_{ij}(\boldsymbol{g}^*) = 0$ for all i and j in the case of infinite-order phase transitions.

The essential singularity in Eq.(8) is understood from the following simple example:

$$\frac{dg(t)}{dt} = (g - g^*)^2. \tag{9}$$

The solution starting from $a(T)$ at $t = 0$ is

$$g(t) - g^* = \frac{1}{\frac{1}{a(T)-g^*} - t}. \tag{10}$$

We assume that $a(T_c) = g^*$ and that

$$a(T) = g^* + b(T - T_c) + \mathrm{O}\left((T - T_c)^2\right). \tag{11}$$

If $\bar{t}$ satisfies

$$|g(\bar{t}) - g^*| = 1, \tag{12}$$

then

$$\xi = \text{const.} \exp \bar{t} \sim \text{const.} \exp\left(\frac{1}{|b(T - T_c)|}\right), \tag{13}$$

as $T \to T_c$. Thus $\sigma = 1$ and $A = 1/|b|$ in this example. One finds that the essential singularity originates from the rational form of the solution Eq.(10). Note that A depends on a functional form of $a(T)$ while σ does not, as long as we do not consider the exceptional case: $b = 0$.

Since the scaling matrix $M(\boldsymbol{g}^*)$ vanishes in an infinite-order phase transition, we cannot extract σ from the usual linearization, in contrast to a second-order one. As we have shown in the above example, explicit solutions were traditionally required in the case of an infinite-order phase transition such as the BKT phase transition[2].

This difficulty has been recently overcome in Ref.[3], where an RG for RGE (1) is used for deriving asymptotic behavior of solutions. A general idea of RG, applied as a tool for asymptotic analysis of non-linear differential equations, is developed in Refs.[4, 5].

In this report, we present another method. Namely, we derive σ from the following deformed RGE:

$$\frac{d\boldsymbol{g}}{dt} = \epsilon\,(\boldsymbol{g} - \boldsymbol{g}^*) + \boldsymbol{V}(\boldsymbol{g}) \equiv \bar{\boldsymbol{V}}(\boldsymbol{g}), \tag{14}$$

where ϵ is a real number but not necessarily small. As we will see in the next section, the RG equation (22) for the RGE (1) has a complicated form compared with the deformed RGE. Hence, using the deformed RGE makes derivation of the critical exponent simple. Another benefit of this approach is as follows: suppose that an infinite-order phase transition occurs when the spatial dimensions of the original statistical model are d_c. Then, the deformed RGE can be derived when they are $d_c-\epsilon$, under the condition that all the coupling constants have a common canonical dimension. This condition is satisfied by various field-theoretical models, e.g., an effective theory of antiferromagnets[6], a model containing several gauge fields[7], a model describing true self-avoiding random walks[8], and a model of nematic elastomers[9]. In Ref.[9], infrared asymptotic behavior in d_c dimensions and $d_c - \epsilon$ dimensions are analyzed separately because of the problem of the vanishing scaling matrix explained above. Our method enables us to obtain universal quantities in both cases simultaneously. We will show this advantage in the last example of Sec.IV.

II. RGE FOR RGE

Here we summarize the results of Ref.[3] that will be used later. We consider an RGE (1) for infinite-order phase transitions that are controlled by a fixed point $\boldsymbol{g}^*$. In what follows, we put $\boldsymbol{g}^* = 0$ for convenience. Suppose that we have obtained $\boldsymbol{V}(\boldsymbol{g})$ by the lowest-order perturbation. Since linear terms vanish in infinite-order phase transitions, components of $\boldsymbol{V}(\boldsymbol{g})$ are quadratic in $\boldsymbol{g}$. Hence the scaling property

$$\boldsymbol{V}(k\boldsymbol{g}) = k^2 \boldsymbol{V}(\boldsymbol{g}) \tag{15}$$

holds in this case.

In order to evaluate the asymptotic behavior of a solution of Eq.(1), we define another renormalization group on an $n-1$ dimensional sphere. We denote the solution $\boldsymbol{g}$ of Eq.(1) with the initial condition $\boldsymbol{a}_0 = (a_{01}, \cdots, a_{0n})$ as

$$\boldsymbol{g}(t, \boldsymbol{a}_0), \tag{16}$$

namely, $\boldsymbol{g}(0, \boldsymbol{a}_0) = \boldsymbol{a}_0$. The function $e^{\tau}\boldsymbol{g}(e^{\tau}t, \boldsymbol{a}_0)$ is a solution of the RGE (1) as well, because of its scale invariance. Let S be the $n-1$ dimensional sphere whose center is at the origin with the radius $|\boldsymbol{a}_0| \equiv a_0$. We define a new renormalization-group transformation

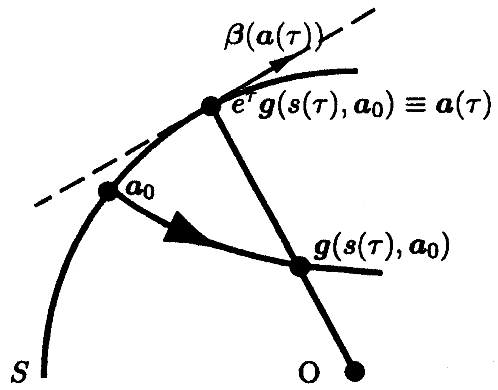

FIG. 2: Illustration for $\mathcal{R}_\tau$ and the beta function defined in Eq.(22). For simplicity, we take $n = 2$. Dashed line represents the tangent space at $\boldsymbol{a}(\tau) \in S$.

$\mathcal{R}_\tau : S \rightarrow S$ as follows:

$$\mathcal{R}_\tau \boldsymbol{a}_0 \equiv e^{\tau}\boldsymbol{g}(s(\tau),\boldsymbol{a}_0) \equiv \boldsymbol{a}(\tau). \tag{17}$$

Using Eq.(15), one finds that $\mathcal{R}_\tau$ has a semi-group property:

$$\mathcal{R}_{\tau_1+\tau_2} = \mathcal{R}_{\tau_2} \circ \mathcal{R}_{\tau_1}. \tag{18}$$

The meaning of $\mathcal{R}_\tau$ is as follows: first, choose τ. Then move $\boldsymbol{a}_0$ along the solution $\boldsymbol{g}(t,\boldsymbol{a}_0)$ during the time $s(\tau)$. Here $s(\tau)$ is determined by the condition $\boldsymbol{g}(s(\tau),\boldsymbol{a}_0)e^{\tau} \in S$. See Fig. 2.

Next let us derive the beta function of $\mathcal{R}_\tau$. Noting that $\boldsymbol{V}(\boldsymbol{g})$ is quadratic, we have

$$\begin{aligned}\frac{d\boldsymbol{a}}{d\tau} &= \boldsymbol{a} + e^{\tau}\boldsymbol{V}(\boldsymbol{g}(s,\boldsymbol{a}_0))\frac{ds}{d\tau} \\ &= \boldsymbol{a} + e^{-\tau}\boldsymbol{V}(\boldsymbol{a})\frac{ds}{d\tau}.\end{aligned} \tag{19}$$

The length-preserving condition

$$\boldsymbol{a} \cdot \frac{d\boldsymbol{a}}{d\tau} = 0 \tag{20}$$

leads to the following differential equation for $s(\tau)$:

$$\frac{ds}{d\tau} = -\frac{e^{\tau}a_0^2}{\boldsymbol{a}\cdot\boldsymbol{V}(\boldsymbol{a})} \tag{21}$$

with the initial condition $s(0) = 0$. Inserting Eq.(21) into Eq.(19), we obtain the beta function for $\mathcal{R}_\tau$.

$$\beta_i(\boldsymbol{a}) \equiv \frac{da_i}{d\tau} = \frac{a_i\boldsymbol{a}\cdot\boldsymbol{V}(\boldsymbol{a}) - V_i(\boldsymbol{a})a_0^2}{\boldsymbol{a}\cdot\boldsymbol{V}(\boldsymbol{a})}. \tag{22}$$

Note that $\boldsymbol{\beta}$ can be written as

$$\boldsymbol{\beta}(\boldsymbol{a}) = -\frac{a_0^2}{\boldsymbol{a}\cdot\boldsymbol{V}(\boldsymbol{a})}P(\boldsymbol{a})\boldsymbol{V}(\boldsymbol{a}), \tag{23}$$

where P is the $n\times n$ matrix that projects $\boldsymbol{V}(\boldsymbol{a}(\tau))$ onto the tangent space at $\boldsymbol{a}(\tau)\in S$:

$$P_{ij}(\boldsymbol{a}) \equiv \delta_{ij} - \frac{a_i a_j}{a_0^2}. \tag{24}$$

Since $\boldsymbol{\beta}(\boldsymbol{a})$ is perpendicular to $\boldsymbol{a}$, a solution of the new RGE is restricted on S. Introducing the polar coordinates $\{\theta_\alpha\}_{1\le\alpha\le n-1}$ on S and the corresponding orthonormal basis,

$$\tilde{\boldsymbol{e}}_\alpha \equiv f_\alpha(\boldsymbol{a})^{-1}\frac{\partial\boldsymbol{a}}{\partial\theta_\alpha},\ f_\alpha(\boldsymbol{a}) \equiv \left|\frac{\partial\boldsymbol{a}}{\partial\theta_\alpha}\right|, \tag{25}$$

we can expand $\boldsymbol{\beta}(\boldsymbol{a})$ as

$$\boldsymbol{\beta}(\boldsymbol{a}) = \sum_{\alpha=1}^{n-1}\tilde{\beta}_\alpha(\boldsymbol{a})\,\tilde{\boldsymbol{e}}_\alpha. \tag{26}$$

The new RGE is represented as

$$\frac{d\theta_\alpha}{d\tau}(\boldsymbol{a}) = f_\alpha^{-1}(\boldsymbol{a})\,\tilde{\beta}_\alpha(\boldsymbol{a}) \tag{27}$$

in the polar-coordinate representation.

It is easily found that $\boldsymbol{a}^*\in S$ is a fixed point of the new RGE (22) if $\boldsymbol{g}(t,\boldsymbol{a}^*)$ is a straight flow line. In particular, a fixed point on an incoming straight flow line satisfying $\boldsymbol{a}^*\cdot\boldsymbol{V}(\boldsymbol{a}^*)<0$ plays an important role, because trajectories near this fixed point correspond to trajectories of Eq. (1) approaching $\boldsymbol{g}^*$. Unlike the original RGE, the new RGE can be linearized about $\boldsymbol{a}^*$. In Ref.[3], it is shown that the scaling matrix of the new RGE

$$\mu_{\alpha\beta}(\boldsymbol{a}^*) \equiv f_\alpha^{-1}(\boldsymbol{a}^*)\frac{\partial\tilde{\beta}_\alpha}{\partial\theta_\beta}(\boldsymbol{a}^*) \tag{28}$$

plays a similar role to $M(\boldsymbol{g}^*)$ in the original RGE describing a second-order phase transition. Namely, if the matrix $\mu(\boldsymbol{a}^*)$ has a unique positive eigenvalue λ, in which typical trajectories of the original RGE are as in Fig. 3 (a), we can observe divergence of the correlation length by one-parameter tuning and

$$\sigma = \frac{1}{\lambda} \tag{29}$$

in Eq. (8). On the other hand, if all the eigenvalues of $\mu(\boldsymbol{a}^*)$ are negative, where typical trajectories are in Fig. 3 (b), $\boldsymbol{g}(t,\boldsymbol{a}_0)$ behaves as

$$\boldsymbol{g}(t,\boldsymbol{a}_0) \sim \frac{1}{C(\boldsymbol{a}^*)\,t}\boldsymbol{e}^*. \tag{30}$$

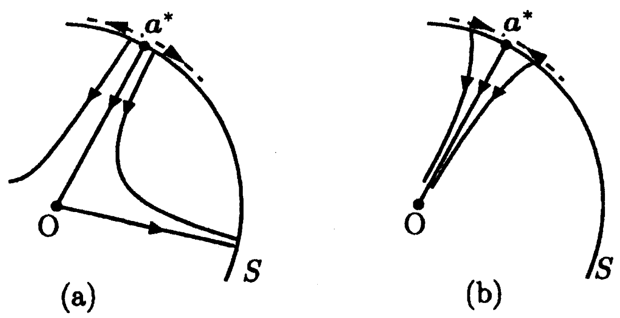

FIG. 3: Schematic trajectories of RGE. The solid lines are for the original RGE (1), while the dashed lines are for the new RGE (22) defined on S. Here (a) is the case where a unique positive eigenvalue exists in $\mu(\boldsymbol{a}^*)$. (b) is the case where all the eigenvalues of $\mu(\boldsymbol{a}^*)$ are negative.

In this formula, $\boldsymbol{e}^* \equiv \boldsymbol{a}^*/a_0$ and $C(\boldsymbol{g})$ is defined by the relation

$$C(\boldsymbol{g})\,|\boldsymbol{g}|^3 = -\boldsymbol{g}\cdot\boldsymbol{V}(\boldsymbol{g})\,. \tag{31}$$

The asymptotic behavior in Eq. (30) is important for investigating finite-size scaling in a statistical system for example.

III. DEFORMED RGE

Next, we consider the deformed RGE (14), putting $\boldsymbol{g}^* = \boldsymbol{0}$. We can take $\epsilon > 0$ without loss of generality. A fixed point $\boldsymbol{c}^*$ of the deformed RGE solves

$$\bar{\boldsymbol{V}}(\boldsymbol{c}^*) = \epsilon\boldsymbol{c}^* + \boldsymbol{V}(\boldsymbol{c}^*) = \boldsymbol{0}. \tag{32}$$

A key feature of the deformed RGE is that $\boldsymbol{c}^*$ in Eq. (32) and a fixed point $\boldsymbol{a}^*$ of the new RGE (22) on an incoming straight flow line has one-to-one correspondence via

$$\boldsymbol{a}^* = \frac{a_0}{c^*}\boldsymbol{c}^* \tag{33}$$

as depicted in Fig. 4. Writing $\boldsymbol{V}(\boldsymbol{g})$ as

$$\boldsymbol{V}(\boldsymbol{g}) = \sum_{\alpha=1}^{n-1}\tilde{V}_\alpha(\boldsymbol{g})\,\tilde{e}_\alpha + \tilde{V}_n(\boldsymbol{g})\,\tilde{e}_n, \tag{34}$$

where $\tilde{e}_n \equiv \boldsymbol{g}/g$, we have the deformed RGE in the polar coordinates:

$$\begin{aligned}\frac{d\theta_\alpha}{dt}(\boldsymbol{g}) &= f_\alpha^{-1}(\boldsymbol{g})\,\tilde{V}_\alpha(\boldsymbol{g})\\ \frac{dg}{dt}(\boldsymbol{g}) &= \epsilon g + \tilde{V}_n(\boldsymbol{g})\,.\end{aligned} \tag{35}$$

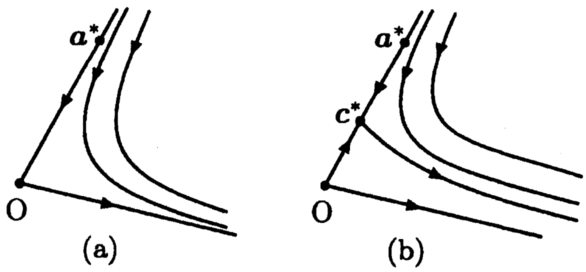

FIG. 4: (a) Schematic trajectories for the original RGE. (b) Those for the deformed RGE.

Expanding the above formula about the fixed point c^*, we have the following scaling matrix $\bar{M}(c^*)$:

$$\begin{aligned}
\bar{M}_{\alpha\beta}(c^*) &= f_\alpha^{-1}(c^*)\frac{\partial\tilde{V}_\alpha}{\partial\theta_\beta}(c^*)\\
\bar{M}_{\alpha n}(c^*) &= f_\alpha^{-1}(c^*)\frac{\partial\tilde{V}_\alpha}{\partial g}(c^*)\\
\bar{M}_{n\alpha}(c^*) &= \frac{\partial\tilde{V}_n}{\partial\theta_\beta}(c^*)\\
\bar{M}_{nn}(c^*) &= \left(\epsilon+\frac{\partial\tilde{V}_n}{\partial g}(c^*)\right),
\end{aligned} \tag{36}$$

where α and β run from 1 to $n-1$. Since $\tilde{V}_\alpha(g)$ is a component perpendicular to g, one finds that $\tilde{V}_\alpha(kc^*)=0$ for all k with the help of Eqs.(15) and (32). This means that $\partial_g\tilde{V}_\alpha(c^*)=0$. On the other hand, $\partial_g\tilde{V}_n(c^*)=2\tilde{V}_n(c^*)/g^*=-2\epsilon$ because $\tilde{V}_n(g)$ is quadratic in g. Therefore,

$$\bar{M}_{\alpha n}=0,\quad \bar{M}_{nn}=-\epsilon \tag{37}$$

in Eq. (36). Furthermore, we can rewrite $\bar{M}_{\alpha\beta}(c^*)$ in terms of $\mu_{\alpha\beta}(a^*)$. In fact, $\mu(a^*)$ in Eq. (28) is written as

$$\mu_{\alpha\beta}(a^*) = f_\alpha^{-1}(a^*)\frac{1}{C(a^*)\,a_0}\frac{\partial\tilde{V}_\alpha}{\partial\theta_\beta}(a^*). \tag{38}$$

Employing the following scaling properties:

$$\begin{aligned}
C(kg) &= C(g)\\
f_\alpha(kg) &= kf_\alpha(g)\\
\frac{\partial\tilde{V}_\alpha}{\partial\theta_\beta}(kg) &= k^2\frac{\partial\tilde{V}_\alpha}{\partial\theta_\beta}(g),
\end{aligned} \tag{39}$$

we get

$$\bar{M}_{\alpha\beta}\left(\boldsymbol{c}^{*}\right)=\epsilon\mu_{\alpha\beta}\left(\boldsymbol{a}^{*}\right). \tag{40}$$

Eqs.(37) and (40) shows that $\bar{M}\left(\boldsymbol{c}^{*}\right)$ has a form of

$$\bar{M}\left(\boldsymbol{c}^{*}\right)=\left(\begin{array}{ccc|c} & & & 0 \\ & \epsilon\mu\left(\boldsymbol{a}^{*}\right) & & \vdots \\ & & & 0 \\ \hline * & \cdots & * & -\epsilon \end{array}\right). \tag{41}$$

in the polar coordinates. It readily follows from this formula that $\bar{M}(\boldsymbol{c}^{*})\tilde{\mathbf{e}}_n=-\epsilon\tilde{\mathbf{e}}_n$. Thus we can derive all the eigenvalues of $\mu(\boldsymbol{a}^{*})$ from $\bar{M}(\boldsymbol{c}^{*})$ by removing $-\epsilon$, which is the eigenvalue corresponding to the eigenvector $\tilde{\mathbf{e}}_n$, from the set of the eigenvalues of $\bar{M}(\boldsymbol{c}^{*})$, and, by multiplying by $1/\epsilon$, the remaining eigenvalues. Further, if all the eigenvalues of $\bar{M}\left(\boldsymbol{c}^{*}\right)$ are negative, $\boldsymbol{g}(t,\boldsymbol{a}_0)$ behaves as

$$\boldsymbol{g}(t,\boldsymbol{a}_0)\sim\frac{1}{C\left(\boldsymbol{a}^{*}\right)t}\boldsymbol{e}^{*}=\frac{1}{\epsilon t}\boldsymbol{c}^{*}, \tag{42}$$

according to Eq. (30) and the scaling property of $C\left(\boldsymbol{a}^{*}\right)$ in Eq. (39).

IV. EXAMPLE

Here are several examples. The first example is taken from the two-dimensional classical XY model[2]. Here, the beta function $\boldsymbol{V}\left(\boldsymbol{g}\right)$ is given as

$$\boldsymbol{V}\left(\boldsymbol{g}\right)=\begin{pmatrix}-g_2^2\\ -g_1g_2\end{pmatrix}, \tag{43}$$

for $g_1,g_2>0$. The deformed RGE has the fixed point $\boldsymbol{c}^{*}=(\epsilon,\epsilon)$. The scaling matrix $\bar{M}\left(\boldsymbol{c}^{*}\right)$ of the deformed RGE is easily computed in terms of the cartesian coordinates as

$$\bar{M}\left(\boldsymbol{c}^{*}\right)=\begin{pmatrix}\epsilon & -2\epsilon\\ -\epsilon & 0\end{pmatrix}. \tag{44}$$

It has the eigenvalues $-\epsilon$ and 2ϵ. Employing Eq. (29), we get

$$\sigma=\frac{\epsilon}{2\epsilon}=\frac{1}{2}, \tag{45}$$

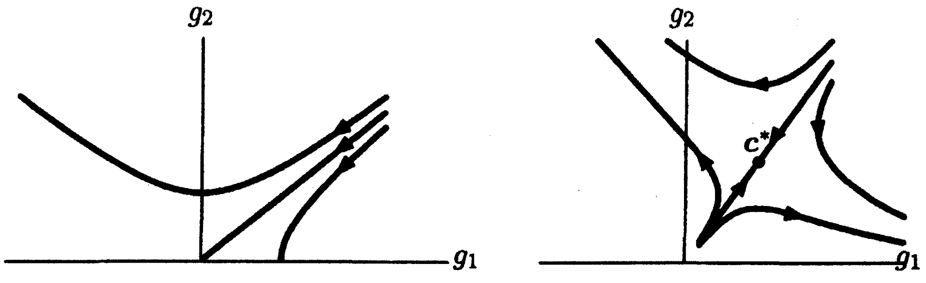

FIG. 5: (a) Schematic trajectories for the original RGE of the XY model. (b) Those for the deformed RGE of the XY model.

which is a well-known result. As we have explained in the previous section, the other eigenvalue, $-\epsilon$, always appears in a deformed RGE (14), which corresponds to the eigenvector $\mathbf{c}^*/c^*$.

The next example is the RGE in a one-dimensional quantum spin chain, studied by Itoi and Kato[10]; it is defined by

$$V(g) = \begin{pmatrix} g_1 (Ng_1 + 2g_2) \\ -g_2 (2g_1 + Ng_2) \end{pmatrix}. \tag{46}$$

The deformed RGE has the following three nontrivial fixed points:

$$\mathbf{c}_1^* = \left(-\frac{\epsilon}{N}, 0\right), \ \mathbf{c}_2^* = \left(0, \frac{\epsilon}{N}\right), \ \mathbf{c}_3^* = \frac{\epsilon}{N-2}(-1, 1). \tag{47}$$

The corresponding scaling matrices are

$$\bar{M}_1 = \begin{pmatrix} -\epsilon & -\frac{2\epsilon}{N} \\ 0 & \frac{N+2}{N}\epsilon \end{pmatrix}, \ \bar{M}_2 = \begin{pmatrix} \frac{N+2}{N}\epsilon & 0 \\ -\frac{2\epsilon}{N} & -\epsilon \end{pmatrix},$$
$$\bar{M}_3 = \begin{pmatrix} \frac{N\epsilon}{2-N} & \frac{2\epsilon}{2-N} \\ \frac{2\epsilon}{2-N} & \frac{N\epsilon}{2-N} \end{pmatrix}. \tag{48}$$

The eigenvalues of those matrices are, respectively,

$$\frac{N+2}{N}\epsilon, \ \frac{N+2}{N}\epsilon, \ \text{and} \ \frac{2+N}{2-N}\epsilon, \tag{49}$$

up to the common eigenvalue $-\epsilon$. The other eigenvalues divided by ϵ are equal to those of the scaling matrices derived from the new RGE (22), which is computed in Ref.[3]. It should be noted that the deformed RGEs in the above two examples do not correspond to those in $2-\epsilon$ and $1-\epsilon$ dimensions respectively. However, the derivation presented here is much simpler than the method using Eq.(22).

The last example is the RGE in a field-theoretical model for nematic elastomers, proposed in Ref.[9]. In contrast to the previous examples, the deformed RGE is obtained exactly in $3-\epsilon$ dimensions with

$$\boldsymbol{V}(\boldsymbol{g}) = \frac{-1}{8(4g_1+g_2)} \begin{pmatrix} g_1(40g_1^2+68g_1g_2+13g_2^2) \\ 2g_2(4g_1^2+32g_1g_2+7g_2^2) \end{pmatrix}. \tag{50}$$

Although $\boldsymbol{V}(\boldsymbol{g})$ is not quadratic polynomial, our result is applicable because all we need

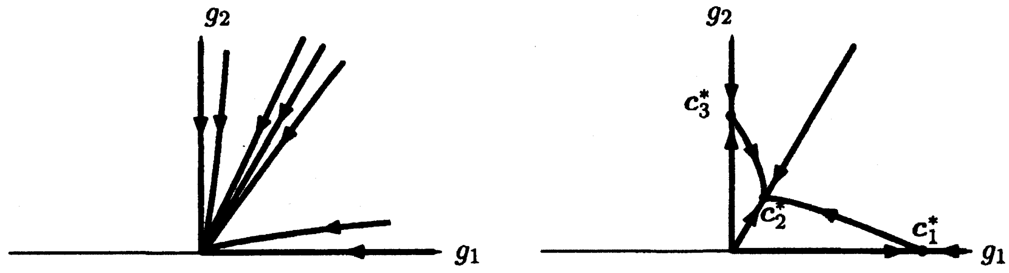

FIG. 6: (a) Schematic trajectories for the original RGE of the model of nematic elastomers. (b) Those for the deformed RGE of the model of nematic elastomers.

to apply the present method is the scaling property of $\boldsymbol{V}(\boldsymbol{g})$, Eq. (15). The deformed RGE has the three fixed points

$$\boldsymbol{c}_1^* = \left(\frac{4\epsilon}{5}, 0\right), \quad \boldsymbol{c}_2^* = \left(\frac{4\epsilon}{59}, \frac{32\epsilon}{59}\right), \quad \boldsymbol{c}_3^* = \left(0, \frac{4\epsilon}{7}\right). \tag{51}$$

One can check that the scaling matrices have the following respective eigenvalues

$$4\epsilon/5, \quad -4\epsilon/59, \quad \text{and } \epsilon/14 \tag{52}$$

in addition to the common eigenvalue $-\epsilon$. Now we turn to the case of just three dimensions. If $g_1, g_2 > 0$, infrared behavior of a system is governed by the fixed point $\boldsymbol{c}_2^*$[9]. Since the eigenvalue at $\boldsymbol{c}_2^*$ is negative, $\boldsymbol{g}(t, \boldsymbol{a}_0)$ behaves as

$$\boldsymbol{g}(t, \boldsymbol{a}_0) \sim \frac{1}{\epsilon t}\boldsymbol{c}_2^* = \frac{1}{t}\left(\frac{4}{59}, \frac{32}{59}\right) \tag{53}$$

for sufficiently large t, according to Eq. (42). The result is consistent with that in Ref.[9].

V. SUMMARY

We have shown how to derive asymptotic behavior of a solution of RGE for infinite-order phase transition, by adding a linear term to this RGE. This method can allow us to apply a result of the ϵ expansion to the case where $\epsilon = 0$.

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