Anderson Localization of the Green's Function with Complex Random Potentials and the 2D $O(N)$ Spin Models

Keiichi R. Ito

Department of Mathematics and Physics,

*Setsunan University, Neyagawa Osaka 572-8508, Japan**

(Dated: October 30, 2003)

We investigate Green's function of the lattice Laplacian $-\Delta + m^2 + i\alpha\psi$, where ψ are real valued random variables and $m > 0$ is an arbitrary small constant. This arises from the Fourier transform of $O(N)$ invariant classical spin model on Z^2. We show that the averaged Green's function behaves like $(-\Delta + m^2 + \alpha^2|\log\alpha|)^{-1}$ for sufficiently small α. This enables us to improve the upper bound for the critical inverse temperature β_c at which a phase transition takes place in the 2D $O(N)$ spin model.

I.   INTRODUCTION AND SUMMARY

In this paper, we argue properties of Green's function of Laplacian which depends on real random potentials $\{\psi(x); x \in Z^2\}$ with pure imaginary coefficients:

$$G^\psi(x, y) = \frac{1}{-\Delta + m^2 + i\alpha\psi}(x) \tag{1.1}$$

where Δ is the Laplacian defined on the lattice space Z^2 $((\Delta)_{xy} = -4\delta_{x,y} + \delta_{|x-y|,1})$ and $\{\psi(x); x \in Z^2\}$ are random variables which obey the Gaussian probability distributions $d\nu(\psi)$. We then apply our analysis to $O(N)$ symmetric spin models in two dimensions. The Gaussian probability distributions $d\nu(\psi)$ we consider here are:

case 1:   locally and identically independently distributed:

$$d\nu(\psi) = \prod_x \frac{e^{-\frac{1}{2}\psi^2(x)}}{\sqrt{2\pi}} d\psi(x) \tag{1.2}$$

*Electronic address: ito@mpg.setsunan.ac.jp, ito@kurims.kyoto-u.ac.jp; Also at: Division of Mathematics, College of Human and Environmental Studies, Kyoto University, Kyoto 606, Japan

case 2: correlating via Yukawa potential:

$$d\nu(\psi) = \exp[-\frac{1}{2}\sum_{x,y} G_0(x,y)\psi(x)\psi(y)] \prod_x d\psi(x)$$

$$= \exp[-\frac{1}{2} <\psi, G_0\psi>] \prod_x d\psi(x)$$

where $G_0(x,y) = (-\Delta + m_0^2)^{-1}(x,y)$ is the Yukawa potential where $m_0^2 > 0$ is an arbitrarily small constant which may be set at zero after all calculations.

Theorem A *Let*

$$G^{(ave)}(x,y) \equiv \int G^{(\psi)}(x,y)d\nu(\psi) \tag{1.3}$$

(averaged Green's functions). Assume $|\log m| \exp[-\alpha^{-1}] << 1$ *in case 2. (No assumption is needed in case 1.) Then*

$$G^{(ave)}(x,y) \sim \frac{1}{-\Delta + m_{eff}^2}(x,y) \tag{1.4}$$

where $m_{eff}^2 = O(\alpha^2|\log\alpha|)$ *for case 1 and* $m_{eff}^2 = O(\alpha^2)$ *for case 2.*

This problem arises from the study of $O(N)$ spin models in two dimensions [5, 6] and this theorem is closely related to non-existence of phase transitions in two-dimensional $O(N)$ spin models with $N \geq 3$, the problem which remains unsolved since the last century. The parameter α is equal to $(N\beta)^{-1}$ in case 1 and equal to $(N\beta)^{-1/2}$ in case 2, where $N\beta$ is the inverse temperature of the system. Main conclusion which is derived from these bounds is (I admit that this is still provisional since some parts remain proved rigorously)

Quasi Theorem *Let* β_c *be the inverse critical temperature of the 2D* $O(N)$ *spin model (the real inverse temperature is* $N\beta$*). Then*

$$\beta_c \geq N^\delta, \quad \delta > 0 \tag{1.5}$$

This is an extension of our previous work [7, 9]. See also [3] which established an existence of first order phase transitions in 2D $O(N)$ spin models which have exotic interactions like $-\sum_{(x,y)}(\phi(x) \cdot \phi(y))^p$, $p >> 1$.

II. AUXILIARY FIELDS AND SPIN MODEL

In models such as $O(N)$ spin modes and $SU(N)$ lattice gauge models [11, 14], the field variables form compact manifolds and the block spin transformations break the structures.

In some cases, this can be avoided by introducing an auxiliary field ψ [1] which may be regarded as a complex random field. The ν dimensional $O(N)$ spin (Heisenberg) model at the inverse temperature $N\beta$ is defined by the Gibbs expectation values

$$< f > \equiv \frac{1}{Z_\Lambda(\beta)} \int f(\phi) \exp[-H_\Lambda(\phi)] \prod_i \delta(\phi_i^2 - N\beta) d\phi_i \tag{2.1}$$

Here Λ is an arbitrarily large square with center at the origin. Moreover $\phi(x) = (\phi(x)^{(1)}, \cdots, \phi(x)^{(N)})$ is the vector valued spin at $x \in \Lambda$, Z_Λ is the partition function defined so that $< 1 >= 1$. The Hamiltonian H_Λ is given by

$$H_\Lambda \equiv -\frac{1}{2} \sum_{|x-y|_1=1} \phi(x)\phi(y), \tag{2.2}$$

where $|x|_1 = \sum_{i=1}^\nu |x_i|$.

We substitute the identity $\delta(\phi^2 - N\beta) = \int \exp[-ia(\phi^2 - N\beta)]da/2\pi$ into eq.(2.1) with the condition that $\text{Im} a_i < -\nu$ [1], and set

$$\text{Im}\, a_i = -(\nu + \frac{m^2}{2}), \quad \text{Re}\, a_i = \frac{1}{\sqrt{N}}\psi_i \tag{2.3}$$

where $m > 0$ will be determined soon. Thus we have

$$Z_\Lambda = c^{|\Lambda|} \int \cdots \int \exp[-\frac{1}{2} < \phi, (m^2 - \Delta + \frac{2i}{\sqrt{N}}\psi)\phi > + \sum_j i\sqrt{N}\beta\psi_j] \prod \frac{d\phi_j d\psi_j}{2\pi}$$

$$= c^{|\Lambda|} \det(m^2 - \Delta)^{-N/2} \int \cdots \int F(\psi) \prod \frac{d\psi_j}{2\pi} \tag{2.4}$$

where c's are constants being different on lines, $\Delta_{ij} = -2\nu\delta_{ij} + \delta_{|i-j|,1}$ is the lattice Laplacian and

$$F(\psi) = \det(1 + \frac{2iG}{\sqrt{N}}\psi)^{-N/2} \exp[i\sqrt{N}\beta \sum_j \psi_j]. \tag{2.5}$$

Moreover $G = (m^2 - \Delta)^{-1}$ is the covariant matrix discussed later. In the same way, the two-point function is given by

$$< \phi_0\phi_x > = \frac{1}{\tilde{Z}} \int \cdots \int (m^2 - \Delta + \frac{2i}{\sqrt{N}}\psi)_{0x}^{-1} F(\psi) \prod \frac{d\psi_j}{2\pi} \tag{2.6}$$

namely by an average of Green's function which includes complex fields $\psi(x)$, $x \in Z^2$, where the constant $\tilde{Z}$ is chosen so that $< \phi_0^2 >= N\beta$. We choose the mass parameter $m > 0$ so that $G(0) = \beta$, where

$$G(x) = \int \frac{e^{ipx}}{m^2 + 2\sum(1 - \cos p_i)} \prod_{i=1}^\nu \frac{dp_i}{2\pi} \tag{2.7}$$

This is possible for any β if $\nu \le 2$, and we easily find that

$$m^2 \sim 32e^{-4\pi\beta} \text{ for } \nu = 2 \tag{2.8}$$

as $\beta \to \infty$. Thus for $\nu = 2$, we can rewrite

$$F(\psi) = \det{}_3^{-N/2}(1 + \frac{2iG}{\sqrt{N}}\psi)\exp[- <\psi, G^{\circ 2}\psi >], \tag{2.9}$$

$$\det{}_3(1 + A) \equiv \det[(1 + A)e^{-A + A^2/2}] \tag{2.10}$$

where $G^{\circ 2}(x, y) = G(x, y)^2$ so that $\mathrm{Tr}(G\psi)^2 = <\psi, G^{\circ 2}\psi >$.

A. Feshbach-Krein Decomposition of The Determinant

Assume $\Lambda = \Delta_1 \cup \Delta_2$, where Δ_i are squares of large size such that $\Delta_1 \cap \Delta_2 = \emptyset$. Introduce notation $G_\Delta = \chi_\Delta G \chi_\Delta$, $G_{\Delta_i, \Delta_j} = \chi_{\Delta_i} G \chi_{\Delta_j}$ and $\psi_\Delta = \chi_\Delta \psi \chi_\Delta$. Then we have

$$\det{}^{-N/2}(1 + i\kappa G_\Lambda \psi_\Lambda) = \det{}^{-N/2}\left(1 + i\kappa \sum_{i,j} G_{\Delta_i, \Delta_j}\psi_{\Delta_j}\right)$$

$$= \det{}^{-N/2}(1 + W)\prod \det{}^{-N/2}(1 + i\kappa G_{\Delta_i}\psi_{\Delta_i})$$

where $\kappa = 2/\sqrt{N}$ and W has the following expression which depends on the variables ψ

$$W = W(\Delta_1, \Delta_2)$$

$$\equiv -(i\kappa)^2 G_{\Delta_1, \Delta_2}\psi_{\Delta_2}\frac{1}{1 + i\kappa G_{\Delta_2}\psi_{\Delta_2}}G_{\Delta_2, \Delta_1}\psi_{\Delta_1}\frac{1}{1 + i\kappa G_{\Delta_1}\psi_{\Delta_1}}$$

This is an immediate consequence of the Feshbach-Krein formula discussed in the Remark added below. This can be easily generalized. Put

$$\Lambda = \cup_{i=1}^n \Delta_i, \quad \Lambda_k = \cup_{i=k+1}^n \Delta_i$$

Then we have

$$\det{}^{-N/2}(1 + i\kappa G_\Lambda \psi_\Lambda)$$

$$= \left[\prod_{i=1}^{n-1} \det{}^{-N/2}(1 + W(\Delta_i, \Lambda_i))\right]\prod_{i=1}^n \det{}^{-N/2}(1 + i\kappa G_{\Delta_i}\psi_{\Delta_i}) \tag{2.11}$$

where

$$W(\Delta_i, \Lambda_i) = -(i\kappa)^2 \frac{1}{1 + i\kappa G_{\Delta_i}\psi_{\Delta_i}}G_{\Delta_i, \Lambda_i}\psi_{\Lambda_i}\frac{1}{1 + i\kappa G_{\Lambda_i}\psi_{\Lambda_i}}G_{\Lambda_i, \Delta_i}\psi_{\Delta_i} \tag{2.12}$$

$$= -(i\kappa)^2 \frac{1}{[G_{\Delta_i}]^{-1} + i\kappa\psi_{\Delta_i}}[G_{\Delta_i}]^{-1}G_{\Delta_i, \Lambda_i}\psi_{\Lambda_i}\frac{1}{[G_{\Lambda_i}]^{-1} + i\kappa\psi_{\Lambda_i}}[G_{\Lambda_i}]^{-1}G_{\Lambda_i, \Delta_i}\psi_{\Delta_i} \tag{2.13}$$

Since $[G_\Delta]^{-1}$ is a Laplacian restricted to the square Δ with suitable boundary conditions, we regard $([G_\Delta]^{-1} + i\kappa\psi_\Delta)^{-1}$ as massive Green's functions which decrease fast, and moreover we regard ψ be the Gaussian random variable of zero mean and covariance $[G^{\circ 2}]^{-1}$.

Remark 1 *The Feshbach-Krein formula of matrices is*

$$X = \begin{pmatrix} A & D \\ C & B \end{pmatrix} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B - CA^{-1}D \end{pmatrix} \begin{pmatrix} I & A^{-1}D \\ 0 & I \end{pmatrix} \tag{2.14}$$

which holds for matrices A of size $\ell \times \ell$, B of size $m \times m$, C of size $m \times \ell$ and D of size $\ell \times m$ respectively.

B. The local measure

Let us consider the measure localized on each block:

$$d\mu_\Delta = \det{}_3^{-N/2}(1 + \frac{2i}{\sqrt{N}}G_\Delta\psi_\Delta)\exp[-(\psi_\Delta, G_\Delta^{\circ 2}\psi_\Delta)]\prod_{x\in\Delta} d\psi(x) \tag{2.15}$$

Since the norm of G_Δ is of order $O(|\Delta|\beta) \gg 1$, it is still impossible to expand the determinant. However this comes with the factor $\exp[-(\psi_\Delta, G_\Delta^{\circ 2}\psi_\Delta))]$, so there is a chance to make the norm of $\frac{2i}{\sqrt{N}}G_\Delta\psi_\Delta$ small.

Note that $G(x,y) = \beta - \delta G(x,y)$, $\delta G(x,y) \sim \log(|x-y|+1)$. Then $G_\Delta^{\circ 2} \sim \beta^2 - 2\beta\delta G(x,y) \sim 2\beta G(x,y) - \beta^2$. Then $[G_\Delta^{\circ 2}]^{-1} \sim G^{-1}/2\beta = (m^2 - \Delta)/2\beta$ since $(\beta \cdot \delta G) \sim 0$ where we regard $\beta(x,y)$ as a matrix whose components are β independent of x, y.

The most difficult problem in this approach is that the norm of $G = (-\Delta + m^2)^{-1}$ is $m^{-2} \sim e^{4\pi\beta} \gg 1$ and the determinant cannot be expanded. Put

$$d\mu_\Delta = \det{}_2^{-N/2}(1 + i\kappa G_\Delta\psi_\Delta)\prod_{x\in\Delta} d\psi(x)$$

$$= \det{}_3^{-N/2}(1 + i\kappa G_\Delta\psi_\Delta)\exp[- < \psi_\Delta, G_\Delta^{\circ 2}\psi_\Delta >]\prod_{x\in\Delta} d\psi(x) \tag{2.16}$$

and introduce new variables $\tilde{\psi}_\Delta(x)$ by

$$\psi_\Delta(x) = \frac{1}{\sqrt{2}}\sum_{y\in\Delta}[G_\Delta^{\circ 2}]^{-1/2}(x,y)\tilde{\psi}(y) \tag{2.17}$$

so that $d\mu_\Delta$ is rewritten

$$d\mu_\Delta = \det_3^{-N/2}(1 + i\kappa K_\Delta) \prod_{x \in \Delta} \exp[-\frac{1}{2}\tilde{\psi}(x)^2]d\tilde{\psi}(x), \tag{2.18}$$

$$K_\Delta = \frac{1}{\sqrt{2}}G_\Delta^{1/2}([G_\Delta^{\circ 2}]^{-1/2}\tilde{\psi})G_\Delta^{1/2} \tag{2.19}$$

Put

$$d\nu_\Delta = \prod e^{-\psi^2(x)/2}\frac{d\psi(x)}{\sqrt{2\pi}} \tag{2.20}$$

and define

$$||K||_p = \left(\int \mathrm{Tr}(K^*K)^{p/2}d\nu_\Delta\right)^{1/p} \tag{2.21}$$

The following lemma means that K_Δ is approximately diagonal but not so much:

Lemma 1 *It holds that*

$$\int \mathrm{Tr}K_\Delta^2 d\nu_\Delta = \frac{1}{2}|\Delta|, \tag{2.22}$$

$$||K_\Delta||_p \le (p-1)||K_\Delta||_2, \quad \text{for all } p \ge 2 \tag{2.23}$$

Proof. The first equation is immediate. See [13] for the second inequality. Q.E.D.

Thus we see that κK_{Δ_i} are a.e. bounded with respect to $d\nu_\Delta$, and converges to 0 as $N \to \infty$. To see to what extent K_Δ is diagonal, we estimate

$$\int \mathrm{Tr}K_\Delta^4 d\nu_\Delta = \sum_{x_i \in \Delta} \frac{1}{4}\prod_{i=1}^4 G_\Delta(x_i, x_{i+1})$$
$$\times [2[G^{\circ 2}]^{-1}(x_1, x_2)[G^{\circ 2}]^{-1}(x_3, x_4) + [G^{\circ 2}]^{-1}(x_1, x_3)[G^{\circ 2}]^{-1}(x_2, x_4)]$$

where $x_5 = x_1$. As we will show in the next section,

$$[G_\Delta^{\circ 2}]^{-1}(x, y) = \frac{1}{2\beta}G_\Delta^{-1} - \hat{B}_\Delta,$$
$$\hat{B}_\Delta(x, y) = O(\beta^{-2})$$

The main contribution comes from the term containing $2[G^{\circ 2}]^{-1}(x_1, x_2)\cdots$. To bound this, set $G_\Delta(x_i, x_{i+1}) = \beta D - \delta G(x_i, x_{i+1})$, $(i = 1, 3)$ where D is the matrix of size $|\Delta| \times |\Delta|$ such that $D(x, y) = 1$ for all x, y. Moreover $\delta G(x, x) = 0$, $\delta G(x, x + e_\mu) = 0.25 - O(\beta m^2)$, $(-\Delta)_{xy} = 0$ unless $|x - y| \le 1$. Thus we have

$$\int \mathrm{Tr}K_\Delta^4 d\nu_\Delta \ge \text{const.} \sum_{x_1 \in \Delta} \frac{1}{4\beta^2}\left\{\beta^2 \sum_{x_4} \delta_{x_1, x_4} + \sum_{x_4} G^2(x_1, x_4)\right\}$$
$$\ge \text{const.}(|\Delta| + |\Delta|^2)$$

which means that K_Δ is approximately diagonal but off-diagonal parts are still considerably large.

C. Inverses of Green's Functions

We define $(-\Delta + m^2)_\Delta^{(D)}$, the Laplacian operator satisfying the Dirichlet boundary condition at the exterior boundary of Δ, i.e. at $\partial^+\Delta \equiv \{x \in \Delta^c \,|\, \text{dist}(x, \Delta) = 1\}$, by

$$(-\Delta)_\Delta^{(D)}(x, y) = \chi_\Delta(x)(-\Delta + m^2)\chi_\Delta \qquad (2.24)$$

and the lattice Laplacian satisfying the free boundary conditions at the inner boundary of Δ, i.e. at $\partial\Delta \equiv \{x \in \Delta; \text{dist}(x, \Delta^c) = 1\}$ by

$$(f, (-\Delta)_\Delta^{(F)}g) = \sum_{|x-y|=1} (f(x) - f(y))(g(x) - g(y)), \quad x, y \in \Delta. \qquad (2.25)$$

The Green's function $G_\Delta^{(D)} = [(-\Delta + m^2)_\Delta^{(D)}]^{-1}$ is obtained as the inverse of $(-\Delta + m^2)$ with $m = \infty$ for $x \notin \Delta$. Thus $G_\Delta^{(D)}(x, y) = 0$ if $x \in \partial^+\Delta$ or $y \in \partial^+\Delta$ where

$$\partial^+\Delta = \partial\bar{\Delta}, \quad \bar{\Delta} = \{x; \text{dist}(x, \Delta) \leq 1\}.$$

We first show that G_Δ^{-1} is almost equal to $(-\Delta + m^2)$ on $\ell^2(\Delta)$ with free boundary conditions at $\partial\Delta$. This can be again shown by the Feshbach-Krein formula eq.(2.14). Then we see that

$$[G_\Delta]^{-1} = \chi_\Delta G^{-1}\chi_\Delta - \chi_\Delta G^{-1}\chi_{\Delta^c} \frac{1}{\chi_{\Delta^c}G^{-1}\chi_{\Delta^c}} \chi_{\Delta^c}G^{-1}\chi_\Delta \qquad (2.26)$$

$$= \chi_\Delta G^{-1}\chi_\Delta - E \frac{1}{\chi_{\Delta^c}G^{-1}\chi_{\Delta^c}} E^* \qquad (2.27)$$

$$\chi_{\Delta^c}G\chi_\Delta\chi_\Delta G^{-1}\chi_\Delta = \chi_{\Delta^c}G\chi_{\Delta^c}\chi_{\Delta^c}G^{-1}\chi_\Delta \qquad (2.28)$$

$$= \chi_{\Delta^c}G\chi_{\Delta^c}E^* \qquad (2.29)$$

where $G^{-1} = -\Delta + m^2$ and

$$E = \chi_\Delta G^{-1}\chi_{\Delta^c} = \chi_\Delta(-\Delta)\chi_{\Delta^c} \qquad (2.30)$$

The following two theorems are proved in [6, 8].

Theorem 2 *Let $G_\Lambda = \chi_\Lambda G \chi_\Lambda$. Then*

$$G_\Lambda^{-1} = (-\Delta + m^2)_\Lambda^{(D)} - B_{\partial\Lambda} \tag{2.31}$$

$$= (-\Delta + m^2)_\Lambda^{(F)} + B_{\partial\Lambda}^{(F)} + \delta_{\partial\Lambda} \tag{2.32}$$

where $B_{\partial\Lambda}(x,y) \neq 0$ if and only if $x \in \partial\Lambda$, $y \in \partial\Lambda$, and $B_{\partial\Lambda}^{(F)}$ is a Laplacian defined by

$$B_{\partial\Lambda}^{(F)}(x,y) \equiv \delta_{x,y} \left[\sum_{\zeta \in \partial\Lambda} B(x,\zeta) \right] - (1 - \delta_{x,y}) B(x,y) \tag{2.33}$$

namely by

$$< f, B_{\partial\Lambda}^{(F)} g > = \frac{1}{2} \sum_{x,y \in \partial\Lambda} B_{\partial\Lambda}(x,y)(f(x) - f(y))(g(x) - g(y)) \tag{2.34}$$

Moreover

$$B_{\partial\Lambda}(x,y) = O(\frac{1}{1 + |x - y|^2}) \geqq 0 \tag{2.35}$$

and $\delta_{\partial\Lambda}$ is a strictly positive diagonal matrix defines by

$$\sum_{y \in \partial\Lambda} B_{\partial\Lambda}(x,y) = \sum_{y \in \Lambda^c} -E(x,y) - \delta_{\partial\Lambda}(x), \quad \delta_{\partial\Lambda}(x) = O\left(\frac{1}{\beta|\Lambda|^{1/2}}\right) \tag{2.36}$$

Theorem 3 *Let $G_\Delta^{\circ 2} = (G_\Delta)^{\circ 2} = \chi_\Delta (G^{\circ 2}) \chi_\Delta$. Then*

$$[G_\Delta^{\circ 2}]^{-1} = \frac{1}{2\beta} G_\Delta^{-1} - \hat{B}_{\partial\Delta} \tag{2.37}$$

where

$$\tilde{B}_{\partial\Delta}(x,y) = O(\beta^{-2}) O\left(\frac{1}{1 + |x - y|^2}\right) \tag{2.38}$$

III. GREEN'S FUNCTION WITH COMPLEX FIELDS $i\psi$

A. Decay of $[-\Delta + m^2 + i\alpha\psi]^{-1}$ with i.i.d. ψ's

We set

$$G^{(\psi)}(x,y) = \left(\frac{1}{-\Delta + m^2 + i\alpha\psi} \right)(x,y) \tag{3.1}$$

and define the averaged Green's function $G^{(ave)}$ by

$$G^{(ave)}(x,y) = \int G^{(\psi)}(x,y) d\mu_0 \tag{3.2}$$

$$d\mu_0 = \prod \exp[-\frac{1}{2}\psi^2(x)] \frac{d\psi(x)}{\sqrt{2\pi}} \tag{3.3}$$

We will denote $G^{(ave)}$ simply by $G^{(\alpha)}$ when the dependence on α is specified, and there is no danger of confusion. Put

$$(-\Delta + m^2) + i\alpha\psi = 4 + m^2 + i\alpha\psi - J \tag{3.4}$$

where $J_{x,y} = \delta_{1,|x-y|}$. Then

$$\frac{1}{4 + m^2 + i\alpha\psi - J}(x,y) = \sum_{\omega:x\to y} \prod_{k\in\omega} \frac{1}{(4 + m^2 + i\alpha\psi(k))^{n(k)}} \tag{3.5}$$

where ω are random walks on Z^2 which start at x and end at y and visit $k \in Z^2$ $n(k)$times.

Lemma 4

$$\int \frac{1}{(4 + m^2 + i\alpha\psi)^n} d\mu_0 = \frac{1}{(n-1)!} \int_0^\infty t^{n-1} \exp[-(4 + m^2 + i\alpha\psi)t] dt d\mu_0$$

$$= \frac{1}{(n-1)!} \int_0^\infty t^{n-1} \exp[-(4 + m^2)t - \alpha^2 t^2] dt$$

$$\leq \left(\frac{1}{4 + m^2 + c\alpha^2 n}\right)^n$$

where $c \sim 1/8$

This is easily proved by the steepest descent method. Since $(1+i\alpha\psi)^n \sim e^{in\alpha\psi}$, the role of $i\alpha\psi$ is to yield oscillating integrals which cancel divergences and thus improve the convergence. Thus we have obtained

Lemma 5 *The averaged Green's function $G^{(ave)}(x,y)$ obeys the bound*

$$G^{(ave)}(x,y) \leq \left(\frac{1}{-\Delta + m^2_{eff}}\right)(x,y) \tag{3.6}$$

where

$$m^2_{eff} = m^2 + c(\alpha)\alpha^2 \tag{3.7}$$

and $c(\alpha) > 0$ is a strictly positive function which tends to ∞ as $\alpha \to 0$.

Remark 2 *Our argument is close to the Anderson localization [4]. Our results in this section are very similar to those in [2] and [12]. It is shown in [2] that the spectrum of the random lattice Hamiltonian $-\Delta + \lambda\omega$ (ω are i.i.d.) has a localization in $\leq \lambda^2|\log\lambda|$ for $D=3$, and in [12], it is shown that the spectrum of $-\Delta + \lambda\omega$ localizes on shells of thickness $\lambda^{2-\delta}$ in two dimensions.*

Theorem 6 *With the above definitions, it holds that*

$$\int \frac{1}{-\Delta + m^2 + i\alpha\psi}(x,y)\frac{1}{-\Delta + m^2 + i\alpha\psi}(w,z)d\mu_0 \leq G^{(\alpha)}(x,y)G^{(\alpha)}(w,z), \qquad (3.8)$$

$$\int \frac{1}{-\Delta + m^2 + i\alpha\psi}(x,y)\frac{1}{-\Delta + m^2 + i\alpha\psi}(w,z)d\mu_0 \geq G^{(\sqrt{2}\alpha)}(x,y)G^{(\sqrt{2}\alpha)}(w,z) \qquad (3.9)$$

where $G^{(\sqrt{2}\alpha)}$ is the averaged Green's function with α replaced by $\sqrt{2}\alpha$.

Proof. We expand the left hand side by random walk, and we show that

$$\int \frac{1}{(4 + m^2 + i\alpha\psi)^{\ell+k}}d\mu_0 \leq \int \frac{1}{(4 + m^2 + i\alpha\psi)^{\ell}}d\mu_0 \int \frac{1}{(4 + m^2 + i\alpha\psi)^k}d\mu_0 \qquad (3.10)$$

where the right hand side is

$$\frac{1}{(k-1)!(\ell-1)!}\int_0^\infty \int_0^\infty s^{k-1}t^{\ell-1}dsdt \exp[-(4+m^2)(s+t) - \frac{1}{2}\alpha^2(s^2+t^2)] \qquad (3.11)$$

We then set $X = s + t$ and use $s^2 + t^2 = X^2 - 2st < X^2$ and

$$\int_0^1 (1-s)^{k-1}s^{\ell-1}ds = \frac{(k-1)!(\ell-1)!}{(k+\ell-1)!}$$

to complete the first inequality. The second is immediate from the above.   Q.E.D.

We note that

$$\sum_\zeta \int (-\Delta + m^2 + i\alpha\psi)_{x,\zeta}\frac{1}{-\Delta + m^2 + i\alpha\psi}(\zeta,y)d\mu_0$$

$$= [(-\Delta + m^2)G^{(\alpha)}](x,y) + i\alpha\int \psi(x)\frac{1}{-\Delta + m^2 + i\alpha\psi}(x,y)d\mu_0$$

$$= [(-\Delta + m^2)G^{(\alpha)}](x,y)$$
$$+ \alpha^2\int \frac{1}{-\Delta + m^2 + i\alpha\psi}(x,x)\frac{1}{-\Delta + m^2 + i\alpha\psi}(x,y)d\mu_0$$

$$= \delta_{xy} \qquad (3.12)$$

where we have used integration by parts. Taking the sum over x on the both sides of (3.12), we have

$$\alpha^2\sum_x \int \frac{1}{-\Delta + m^2 + i\alpha\psi}(x,x)\frac{1}{-\Delta + m^2 + i\alpha\psi}(x,y)d\mu_0$$

$$= 1 - m^2\sum_x G^{(\alpha)}(x,y) = 1 - m^2 O(\frac{1}{m^2 + \alpha^2}) \qquad (3.13)$$

Then from this and Theorem 6 (3), we obtain

$$\alpha^2\sum_x G^{(ave)}(x,x)G^{(ave)}(x,y) \leq 2(1 - m^2 O(\frac{1}{m^2 + \alpha^2})) \qquad (3.14)$$

We now rewrite (3.12) as

$$[(-\Delta + m^2 + \alpha^2 G^{(\alpha)}(0))G^{(\alpha)}](x,y) = \delta_{xy} + \alpha^2 \delta G^{(\alpha)}(x,y) \tag{3.15}$$

where

$$
\begin{aligned}
&\delta G^{(\alpha)}(x,y) \\
&= \int \left(G^{(\alpha)}(0)G^{(\alpha)}(x,y) - \frac{1}{-\Delta + m^2 + i\alpha\psi}(x,x)\frac{1}{-\Delta + m^2 + i\alpha\psi}(x,y) \right) d\mu_0 \tag{3.16} \\
&= \sum_{\omega:x\to x} \sum_{\eta:x\to y} \prod_{\zeta\in\omega\cup\eta} \frac{1}{(\ell(\zeta)-1)!(m(\zeta)-1)!} \int \prod_{\zeta\in\omega\cup\eta} s_\zeta^{\ell(\zeta)-1} t_\zeta^{m(\zeta)-1} ds_\zeta dt_\zeta \\
&\quad \times \exp[-(4+m^2)(s_\zeta + t_\zeta) - \tfrac{1}{2}\alpha^2(s_\zeta^2 + t_\zeta^2)](1 - \exp[-\alpha^2 \sum s_\zeta t_\zeta]) \geqq 0 \tag{3.17}
\end{aligned}
$$

Put $y = 0$ and multiply e^{ipx} and take the sum over x in (3.15). Then we have

$$\tilde{G}^{(ave)}(p) = \frac{1 + \alpha^2 \delta\tilde{G}(p)}{4 - 2\sum\cos p_i + m^2 + \alpha^2 G^{(ave)}(0)} \tag{3.18}$$

where $\alpha^2\delta\tilde{G}(p)$ is bounded uniformly in α and tends to 0 as $\alpha \to 0$ $(G^{(\alpha)} = G^{(ave)})$. Thus we have

$$
\begin{aligned}
G^{(ave)}(0) &= \int \frac{1 + \alpha^2\delta\tilde{G}(p)}{[4 - 2\sum\cos p_i + m^2 + \alpha^2 G^{(ave)}(0)]} \frac{d^2p}{(2\pi)^2} \\
&= \frac{1}{4\pi}\left[\log\left(\frac{1}{m^2 + \alpha^2 G^{(ave)}(0)}\right) + 3\log 2 + o(1)\right]. \tag{3.19}
\end{aligned}
$$

Theorem 7

$$
\begin{aligned}
G^{(ave)}(x,x) &= \frac{1}{2\pi}\log\frac{1}{|\alpha|} + O(\log\log\frac{1}{|\alpha|}), \\
m_{eff}^2 &= m^2 + \alpha^2 G^{(ave)}(0) \\
&= m^2 + \frac{\alpha^2}{2\pi}\left(\log\frac{1}{|\alpha|} + O(\log\log\frac{1}{|\alpha|})\right)
\end{aligned}
$$

B.   Case of $\Delta = Z^2$ or of $d\mu = \exp[- <\psi, G\psi>]\prod d\psi$

For very large Δ, we may regard Δ as Z^2 and replace $\prod d\mu_\Delta$ by the following Gaussian measure whose covariance is just the Laplacian $(-\Delta + m_0^2)$:

$$d\mu = \frac{1}{Z}\exp[-\frac{1}{2} <\psi, G\psi>]\prod d\psi(x), \tag{3.20}$$

$$G(x,y) = \frac{1}{-\Delta + m_0^2}, \quad (m_0 \leq m) \tag{3.21}$$

where we have scaled ψ by $2\sqrt{\beta}$ and $m_0 = \sqrt{2}m$ in the actual system, but here m_0 may be put 0 after the calculation. In this case, we have

$$G^{(\alpha)}(x,y) = \frac{1}{-\Delta + m^2 + i\tilde{\alpha}\psi}(x,y), \quad \tilde{\alpha} = \sqrt{2}\alpha = \frac{1}{\sqrt{N\beta}} \tag{3.22}$$

Thus we have

Theorem 8 *Assume* $|\log m|e^{-O(\sqrt{N\beta})} < \infty$. *Then under the same assumption,* $G^{(ave)}$ *obeys the following bound:*

$$G^{(ave)}(x,y) \leq \left(-\Delta + m_{eff}^2\right)^{-1}(x,y),$$

$$m_{eff}^2 = m^2 + \varepsilon\alpha^2$$

where $\alpha^2 = \tilde{\alpha}^2/2 = (2N\beta)^{-1}$ *and* $\varepsilon = O(1)$ *is a strictly positive constant.*

Proof. We estimate

$$G^{(ave)}(x,y) = \sum_{\omega:x\to y} \prod \frac{1}{(n(\zeta)-1)!} \int_0^\infty \prod s_\zeta^{n(\zeta)-1} \exp[-(4+m^2+\varepsilon\alpha^2)s(\zeta)]$$

$$\times \exp[\varepsilon\alpha^2 \sum s(\zeta) - \alpha^2 \sum_{|\zeta-\xi|=1}(s(\zeta)-s(\xi))^2] \prod ds(\zeta)$$

where $\varepsilon > 0$ is a constant determined later. Replace $(4+m^2+\varepsilon\alpha^2)s(\zeta)$ by new variables $s(\zeta)$ so that

$$G^{(ave)}(x,y) = \sum_{\omega:x\to y} \left(\frac{1}{T}\right)^{|\omega|} \left(\prod_{\zeta\in\text{supp}\omega} \frac{1}{(n(\zeta)-1)!} \int_0^\infty s(\zeta)^{n(\zeta)-1}e^{-s(\zeta)}ds(\zeta)\right)$$

$$\times \exp\left[\frac{\varepsilon\alpha^2}{T}\sum_{\zeta\in\text{supp}\omega}s(\zeta) - \frac{\alpha^2}{T^2}\sum_{|\zeta-\xi|=1}(s(\zeta)-s(\xi))^2\right] \tag{3.23}$$

where

$$T = T(\alpha) = 4 + m^2 + \varepsilon\alpha^2 \tag{3.24}$$

We set $s(\zeta) = n(\zeta) + \sqrt{n(\zeta)}\tilde{s}(\zeta)$ so that

$$G^{(ave)}(x,y) = \sum_{\omega:x\to y} \left(\frac{1}{T}\right)^{|\omega|} \exp\left[\frac{\varepsilon\alpha^2}{T}\sum n(\zeta) - \frac{\alpha^2}{T^2}\sum(n(\zeta)-n(\xi))^2\right]$$

$$\times \int \left(\prod_{|\zeta-\xi|=1} g(\tilde{s}(\zeta),\tilde{s}(\xi))\right) \prod_{\zeta\in\text{supp}\omega} f(\tilde{s}(\zeta))d\nu_\zeta(\tilde{s}(\zeta)) \tag{3.25}$$

where

$$d\nu(\tilde{s}) = \frac{1}{(n(\zeta) - 1)!} s^{n(\zeta)-1} \exp[-s(\zeta)] ds(\zeta), \tag{3.26}$$

$$f(\tilde{s}(\zeta)) = \exp\left[\frac{\varepsilon\alpha^2}{T}\sqrt{n(\zeta)}\tilde{s}(\zeta) - \frac{4\alpha^2 n(\zeta)}{T^2}\tilde{s}(\zeta)^2\right], \tag{3.27}$$

$$g(\tilde{s}(\zeta), \tilde{s}(\xi)) = \exp\left[\frac{\alpha^2}{T}\left(2(n(\zeta) - n(\xi))(\sqrt{n(\zeta)}\tilde{s}(\zeta) - \sqrt{n(\xi)}\tilde{s}(\xi))\right.\right.$$
$$\left.\left. + \sqrt{n(\zeta)}\sqrt{n(\xi)}\tilde{s}(\zeta)\tilde{s}(\xi)\right)\right]. \tag{3.28}$$

We note that

$$\int d\nu(\tilde{s}) = 1, \quad \int \tilde{s} d\nu(\tilde{s}) = 0$$

$(-\sqrt{n} \le \tilde{s} \le \infty)$ and so on, and

$$d\nu(\tilde{s}) = \frac{(n + \sqrt{n}\tilde{s})^{n-1}}{(n-1)!}e^{-(n+\sqrt{n}\tilde{s})}\sqrt{n}d\tilde{s} \sim \exp[-\frac{1}{2}\tilde{s}^2]\frac{1}{\sqrt{2\pi}}d\tilde{s} \tag{3.29}$$

It is enough to consider the contributions of walk whose visiting numbers $n(\zeta)$ at $\zeta \in$ suppω satisfy $|n(\zeta) - n(\xi)| \le \max\{\sqrt{n(\zeta)}, \sqrt{n(\xi)}\}$. Otherwise we can extract the factor $e^{-\alpha^2 n(\zeta)}$ or $e^{-\alpha^2 n(\xi)}$ from $e^{-\alpha^2(n(\zeta)-n(\xi))^2}$. Thus we can apply the standard techniques of polymer expansion. It is again proved that δG is small, see [8]. Q.E.D.

C. Case of finite Δ or of $\prod d\mu_\Delta$

Let Δ_i be squares of size $L \times L$ $(L > 1)$, such that $\cup_i \Delta_i = Z^2$ and $\Delta_i \cap \Delta_j = \emptyset$. We again define

$$G^{(ave)}(x, y) \equiv \int G^{(\psi)}(x, y) d\mu_0 \tag{3.30}$$

where

$$G^{(\psi)}(x, y) \equiv \left(\frac{1}{G^{-1} + i\kappa\psi}\right)(x, y) \tag{3.31}$$

$$d\mu_0 = \prod_{\Delta \subset Z^2} \frac{1}{Z_\Delta} \exp[-(\psi_\Delta, G_\Delta^{o2}\psi_\Delta)] \prod_{x \in \Delta} d\psi(x). \tag{3.32}$$

We estimate

$$\int \prod_{x \in \Delta} \frac{1}{(4 + m^2 + i\kappa\psi(x))^{n(x)}} d\mu_\Delta(\psi)$$

$$= \prod \frac{1}{(n_x - 1)!}\int_0^\infty \prod t_x^{n_x-1} \exp[-(4 + m^2)\sum t_x - \kappa^2 < t_\Delta, [G_\Delta^{o2}]^{-1}t_\Delta >]\prod dt_x$$

where $[G_\Delta^{o2}]^{-1} \sim (-\Delta)_\Delta/2\beta$. Then we have [6, 8]

Lemma 9 *Assume $\beta > |\Delta|$. Let m_{eff} be given by*

$$m_{eff}^2 = m^2 + c\frac{1}{N\beta} \tag{3.33}$$

where $c > 0$ is a constant. Then

$$G^{(ave)}(x,y) \leq \frac{1}{-\Delta + m_{eff}^2}(x,y) \tag{3.34}$$

More precise bounds are obtained by applying $d\mu_\Delta$ to

$$\sum_\zeta (-\Delta + m^2 + i\kappa\psi)_{x,\zeta} \left(\frac{1}{-\Delta + m^2 + i\kappa\psi}\right)(\zeta,y) = \delta_{x,y} \tag{3.35}$$

Then we have

$$[(-\Delta + m^2 + \Gamma^{(ave)})G^{(ave)}](x,y) = \delta_{x,y} + \delta G(x,y) \tag{3.36}$$

where

$$\Gamma^{(ave)}(\zeta,\xi) = \frac{\kappa^2}{2}[G_\Delta^{o2}]^{-1}(\zeta,\xi)G^{(ave)}(\zeta,\xi) \tag{3.37}$$

is a strictly positive block-wise diagonal matrix and $\delta G(x,y)$ is the remainder given by

$$\frac{\kappa^2}{2}\sum_\zeta [G_\Delta^{o2}]^{-1}(x,\zeta)\int [G^{(ave)}(x,\zeta)G^{(ave)}(\zeta,y) - G^{(\psi)}(x,\zeta)G^{(\psi)}(\zeta,y)]\,d\mu_\Delta \tag{3.38}$$

which tends to 0 as $\alpha \to 0$.

We decompose $\Gamma^{(ave)}$ into a differential operator part and a diagonal part. Note that

$$\Gamma^{(ave)}(\zeta,\xi) = \frac{\kappa^2}{2}\left\{\frac{1}{2\beta}(-\Delta)_\Delta^{(F)}(\zeta,\xi) + \chi_{\partial\Delta}(\zeta)\delta_{\zeta,\xi}\delta_{\partial\Delta}(\zeta)\right\}G^{(ave)}(\zeta,\xi)$$

$$+\frac{\kappa^2}{4\beta}\chi_{\partial\Delta}(\zeta)\chi_{\partial\Delta}(\xi)B_{\partial\Delta}^{(F)}(\zeta,\xi)G^{(ave)}(\zeta,\xi) + O(\beta^{-2})(\zeta,\xi)G^{(ave)}(\zeta,\xi) \tag{3.39}$$

Since

$$(-\Delta)_\Delta^{(F)}(\zeta,\xi) = \begin{cases} 4,3,2,1 & \text{if } \zeta = \xi \in \Delta \\ -1 & \text{if } |\zeta - \xi| = 1 \\ 0 & \text{otherwise} \end{cases} \tag{3.40}$$

we see that

$$(-\Delta)_\Delta^{(F)}(\zeta,\xi)G^{(ave)}(\zeta,\xi) = \gamma_1(-\Delta)_\Delta^{(F)}(\zeta,\xi) + \delta_{\zeta,\xi}(-\Delta)_\Delta^{(F)}(\zeta,\zeta)(G^{(ave)}(\zeta,\zeta) - \gamma_1) \tag{3.41}$$

is strictly positive, where

$$\gamma_1 = \min_{|\zeta-\xi|\leq 1}\{G^{(ave)}(\zeta,\xi)\} = \log\beta - O(1) \tag{3.42}$$

Similarly, since $B_{\partial\Delta}^{(F)}(\zeta,\xi) = \delta_{\zeta,\xi}[\sum_{\zeta'\in\partial\Delta}B_{\partial\Delta}(\zeta,\zeta')] - (1-\delta_{\zeta,\xi})B_{\partial\Delta}(\zeta,\xi)$, we have

$$\left(\delta_{\zeta,\xi}\left[\sum_{\xi'\in\partial\Delta}B_{\partial\Delta}(\zeta,\xi')\right] - (1-\delta_{\zeta,\xi})B_{\partial\Delta}(\zeta,\xi)\right)G^{(ave)}(\zeta,\xi) = \tag{3.43}$$

$$= \delta_{\zeta,\xi}\gamma_2\left[\sum_{\xi'\in\partial\Delta}B_{\partial\Delta}(\zeta,\xi')\right] + \delta_{\zeta,\xi}\sum_{\xi'\in\partial\Delta}B_{\partial\Delta}(\zeta,\xi')(G^{(ave)}(\zeta,\xi') - \gamma_2)$$

$$-(1-\delta_{\zeta,\xi})B_{\partial\Delta}(\zeta,\xi)(G^{(ave)}(\zeta,\xi) - \gamma_2) \tag{3.44}$$

which is strictly positive, where

$$\gamma_2 = \min_{\zeta,\xi\in\partial\Delta}\{G^{(ave)}(\zeta,\xi)\} = \log\beta - O(1) \tag{3.45}$$

Theorem 10 $\Gamma^{(ave)}$ *has the natural decomposition*

$$\Gamma^{(ave)} = \Gamma_d^{(ave)} + \Gamma_0^{(ave)} \tag{3.46}$$

$$\Gamma_d^{(ave)} \geq \frac{c}{N\beta}, \quad \Gamma_0^{(ave)} \geq 0 \tag{3.47}$$

where $\Gamma_d^{(ave)}$ is a diagonal matrix, and $\Gamma_0^{(ave)}$ is a positive differential operator:

$$\sum_y \Gamma_0^{(ave)}(x,y) = 0 \tag{3.48}$$

Theorem 11 *Let*

$$p_{xy} = \frac{1}{4+m^2+\Gamma^{(ave)}(x,x)}(\delta_{|x-y|,1} - \Gamma^{(ave)}(x,y)) \tag{3.49}$$

Then

$$\sum_y p_{xy} = \frac{1}{4+m^2+\Gamma^{(ave)}(x,x)}\left(4+m^2+\Gamma^{(ave)}(x,x)\right.$$

$$-\frac{c}{N\beta}\left\{\sum_{y:|x-y|=1}(G^{(ave)}(x,x) - G^{(ave)}(x,y))\right.$$

$$\left.\left.+\chi_{\partial\Delta}(x)\chi_{\partial\Delta}(y)\sum_{y\in\partial\Delta}B_{\partial\Delta}(x,y)(G^{(ave)}(x,x) - G^{(ave)}(x,y))\right\}\right) \tag{3.50}$$

IV. THE O(N) SYMMETRIC SPIN MODEL

What we have shown in the previous section is that the averaged Green's functions $G_\Lambda^{(ave)}$ are decreasing fast, smooth and that ψ acts as differentiations. These facts mean that W_Λ in the expression of the partition function are small. This is a good news. On the other hand, this argument must be taken with a grain of salt since our analysis depends on cancellations of integrals due to complex impurities. Namely $|W(\Lambda, \Delta)|$ can be large though $\int W d\mu$ tends to zero. But this result can be justified by deforming contours of ψ_x in the integrals.

A. Smallness of $|W|$ and the Possibility of the Polymer Expansion

To prove that the free energy is analytic in $\beta \in [0, \infty)$, we use the cluster expansion to express thermodynamic quantities by convergent sums of finite volume quantities. Finite volume quantities are analytic in β. Then absolute convergences imply the analyticity of the thermodynamic quantities. In the present model, this would be ensured by the integrability of $\det^{-N/2}(1 + i\kappa G_\Delta \psi_\Delta)$ and convergence of polymer expansion of $\det(1 + W(\Delta_i, \Lambda_i))$. But we discuss this problem in the forthcoming papers, and the remaining part of this paper is devoted to some plausible arguments (some of them are, of course, rigorous).

First of all, we show that $W(\Delta, \Lambda)$ is a.e. finite uniformly in β with respect to $d\mu_0$. In fact, $W(\Delta, \Lambda)$ is similar to

$$W(\Delta, \Lambda) \equiv -(i\kappa)^2 U_\Delta(\psi_\Delta) G_\Delta^{-1/2} G_{\Delta,\Lambda} \psi_\Lambda G_\Lambda^{1/2} U_\Lambda(\psi_\Lambda) G_\Lambda^{-1/2} G_{\Lambda,\Delta} \psi_\Delta G_\Delta^{1/2} \qquad (4.1)$$

where

$$U_\Delta(\psi_\Delta) \equiv \frac{1}{1 + i\kappa G_\Delta^{1/2} \psi_\Delta G_\Delta^{1/2}} \qquad (4.2)$$

$$U_\Lambda(\psi_\Lambda) \equiv \frac{1}{1 + i\kappa G_\Lambda^{1/2} \psi_\Lambda G_\Lambda^{1/2}} \qquad (4.3)$$

are normal operators whose norms are less than or equal to 1. Set

$$X_{\Delta,\Lambda}(\psi_\Lambda) = G_\Delta^{-1/2} G_{\Delta,\Lambda} \psi_\Lambda G_\Lambda^{1/2}, \quad X_{\Lambda,\Delta}(\psi_\Delta) = G_\Lambda^{-1/2} G_{\Lambda,\Delta} \psi_\Delta G_\Delta^{1/2}. \qquad (4.4)$$

We show that X's are (component-wise) bounded uniformly in β with respect to $d\mu_0$ (or bounded if $\{\psi_i; i \in \Delta\}$ satisfy $|\psi_i| < c\beta^{-1/2}$ and $|\sum_{i \in \Delta} \psi_i| < c\beta^{-1}$):

Lemma 12 *The following bounds hold uniformly in $\beta > 0$:*

$$\|U_\Delta(\psi_\Delta)\| \leq 1, \tag{4.5}$$

$$\|U_\Lambda(\psi_\Lambda)\| \leq 1, \tag{4.6}$$

$$\int \mathrm{Tr} X^*_{\Lambda,\Delta}(\psi_\Delta) X_{\Lambda,\Delta}(\psi_\Delta) d\mu_0 \leq O(|\Delta|) \tag{4.7}$$

$$\int \mathrm{Tr} X^*_{\Delta,\Lambda}(\psi_\Lambda) X_{\Delta,\Lambda}(\psi_\Lambda) d\mu_0 \leq O(|\Lambda|) \tag{4.8}$$

where $|\Delta| < \beta$ is assumed in the last two inequalities.

This lemma is immediate because of our previous analysis. But the norm may grow like $|\Lambda|$ and we show that the norm of $W(\Delta, \Lambda)$ is bounded by $O(|\Delta|)$ uniformly in β by the localization.

B. $\int W^p d\mu$ **is small uniformly in** Λ

1. Structures of $\hat{G}_\Delta$

Now we assume that all $\{\psi\}$ are small, $\psi_\Delta(x) = 2^{-1/2}([G_\Delta^{\circ 2}]^{-1/2}\tilde\psi)(x)$, $|\tilde\psi(x)| < O(1)$. Let the spectral resolutions of G_Δ and $G_\Delta^{\circ 2}$ be given respectively by

$$G_\Delta = e_0 P_0 + \sum_{i=1}^{|\Delta|-1} e_i P_i, \quad G_\Delta^{\circ 2} = \hat{e}_0 \hat{P}_0 + \sum_{i=1}^{|\Delta|-1} \hat{e}_i \hat{P}_i, \tag{4.9}$$

where $\{e_0 > \cdots > e_{|\Lambda|-1}\}$ (resp. $\{\hat{e}_0 > \cdots > \hat{e}_{|\Lambda|-1}\}$) are the eigenvalues of G_Δ (resp. $G_\Delta^{\circ 2}$) and P_i (resp. $\hat{P}_i$) are the projections. Then

$$\hat{G}_\Delta \equiv [G_\Delta^{\circ 2}]^{1/2} = \hat{e}_0^{1/2} \hat{P}_0 + \sum_{i=1}^{|\Delta|-1} \hat{e}_i^{1/2} \hat{P}_i, \tag{4.10}$$

where $\hat{e}_0 = O(\beta^2 |\Delta|)$ (resp. $e_0 = O(\beta|\Delta|)$) is the largest eigenvalue of $G_\Delta^{\circ 2}$ (resp. G_Δ) and and $\hat{P}_0$ (resp. P_0) is the projection operator to the eigenspace. For simplicity, we set

$$\hat{G}_\Delta^{-1} = \frac{1}{\sqrt{2\beta}}(-\Delta)_\Delta^{1/2} + O(\beta^{-1}|\Delta|^{-1/2})\hat{P}_0,$$

$$G_\Delta^{1/2} = (-\Delta)_\Delta^{1/2} + O(|\Delta|^{1/2}\beta^{1/2})P_0$$

$$G_\Delta^{-1/2} = (-\Delta)_\Delta^{1/2} + O(|\Delta|^{-1/2}\beta^{-1/2})P_0$$

where we put $\sum_{i=1}^{|\Delta|-1}(e_i)^{1/2}P_i = (-\Delta)^{1/2}$ since P_i are the projections to the subspaces of $\{\psi_i; \sum \psi_i = 0\}$. (We are sorry for the abuse of notation.)

We put

$$\psi(x) = \frac{1}{\sqrt{2}}[\hat{G}_\Delta^{-1}\tilde{\psi}](x) \tag{4.11}$$

$$= \frac{1}{2\sqrt{\beta}}[(-\Delta)_\Delta^{1/2}\tilde{\psi}](x) + O\left(\frac{1}{\beta\sqrt{|\Delta|}}\right)[\hat{P}_{0,\Delta}\tilde{\psi}](x), \tag{4.12}$$

$$\hat{G}_\Delta = (G_\Delta^{\circ 2})^{1/2} \tag{4.13}$$

where $\hat{P}_0$ is the projection operator to the eigenspace of $G_\Delta^{\circ 2}$ of the largest eigenvalue $e_0 = O(|\Lambda|\beta^2)$ and

$$\hat{P}_0 \sim P_0 = \frac{1}{|\Delta|}\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} + O(\frac{1}{\beta|\Delta|}) \tag{4.14}$$

We remark that

$$W(\Delta, \Lambda)$$
$$= -\frac{1}{2N}U_\Delta(\Psi_\Delta)G_\Delta^{-1/2}G_{\Delta,\Lambda}[\sum_{\Delta_i \subset \Lambda}\hat{G}_{\Delta_i}^{-1}\tilde{\psi}_{\Delta_i}]\frac{1}{G_\Lambda^{-1} + i\kappa\psi_\Lambda}G_\Lambda^{-1}G_{\Lambda,\Delta}[\hat{G}_\Delta^{-1}\tilde{\psi}_\Delta]G_\Delta^{1/2} \tag{4.15}$$

are the matrices of size $|\Delta| \times |\Delta|$ which are functions of infinite variables $\tilde{\psi}(x)$ and, as we have shown, are bounded uniformly in β if Λ is finite.

We show that these matrices are finite almost everywhere uniformly in $\beta > 0$ and $\Lambda \subset Z^2$. To discuss this, we first approximate $U_\Delta(\psi_\Delta)$ by 1 since Δ is small, and replace

$$\frac{1}{G_\Lambda^{-1} + i\kappa\psi_\Lambda}G_\Lambda^{-1}G_{\Lambda,\Delta} \tag{4.16}$$

by

$$\int\frac{1}{G_\Lambda^{-1} + i\kappa\psi_\Lambda}G_\Lambda^{-1}G_{\Lambda,\Delta}d\mu_0 \sim G_\Lambda^{(m_{eff})}G_\Lambda^{-1}G_{\Lambda,\Delta}. \tag{4.17}$$

Since $G_\Lambda^{(m_{eff})}(x, y)$ is close to $G^{(m_{eff})}(x, y)$ for all $x, y \in \Lambda$ if Λ is large, we take $|\Lambda|^{1/2} \geq N\beta$ so that

$$G_\Lambda^{(ave)} \sim \chi_\Lambda G^{(m_{eff})}\chi_\Lambda \tag{4.18}$$

Then we have

$$[G_\Lambda^{(m_{eff})}G_\Lambda^{-1}G_{\Lambda,\Delta}](x, y) = \sum_{\zeta\in\partial\Lambda}G_\Lambda^{(m_{eff})}(x, \zeta)\nabla_n G(\zeta, y)$$

$$+ \sum_{\zeta,\xi\in\partial\Lambda}\frac{1}{2}B_{\partial\Lambda}(\zeta, \xi)(G_\Lambda^{(m_{eff})}(x, \zeta) - G_\Lambda^{(m_{eff})}(x, \xi))(G(y, \zeta) - G(y, \xi))$$

$$+ \sum_{\zeta\in\partial\Lambda}\delta_{\partial\Lambda}(\zeta)G_\Lambda^{(m_{eff})}(x, \zeta)G(\zeta, y)$$

Since $\delta_{\partial\Lambda}(x) \sim m$ for large Λ and $m_{eff}^2 = O((N\beta)^{-1})$, we see the last term of the above is rather small, and we see that the most leading term is the surface term

$$\sum_{\zeta \in \partial\Lambda} G_\Lambda^{(m_{eff})}(x,\zeta)\nabla_n G(\zeta,y) = \sum_{\zeta \in \Lambda}\sum_{\mu=1,2}(\nabla_\mu G_\Lambda^{(m_{eff})})(x,\zeta)(\nabla_\mu G)(\zeta,y)$$

$$\sim \int \frac{\sum(1-\cos p_\mu)}{4+m^2-2\sum\cos p_\mu}\bar{G}^{(ave)}(p)e^{ip(x-y)}\frac{d^2p}{(2\pi)^2}$$

$$= G^{(m_{eff})}(x,y)$$

for $x \in \Lambda$ and $y \in \Delta$.

We discussed that $G_\Lambda^{(ave)}(x,y)$ are close to $[\chi_\Lambda G^{(m_{eff})}\chi_\Lambda](x,y)$ if Λ are sufficiently large (if side-length of Λ is larger than $\sqrt{N}\beta$). Then we can first assume that $[G_\Lambda^{-1}+i\kappa\psi_\Lambda]^{-1}$ behaves like the restriction of $G^{(ave)}$ to Λ:

$$\chi_\Lambda G^{(ave)}\chi_\Lambda \sim \chi_\Lambda G^{(m_{eff})}\chi_\Lambda, \quad G^{(m_{eff})} = \frac{1}{-\Delta+m_{eff}^2}$$

2. $\int W^p d\mu$ is bounded uniformly in Λ

We first show that the averages of $W(\Delta,\Lambda)^p$ are small and tend to 0 as $N \to \infty$ uniformly in β. We substitute the previous expressions into (4.15) and have the decomposition $W = \sum_{k=1}^{16} W_k$, where under the previous simplifications,

$$W_1 = -\frac{1}{8\beta N}(-\Delta)_\Delta^{1/2}G_{\Delta,\Lambda}[(-\Delta)_\Lambda^{1/2}\tilde{\psi}_\Lambda]G_{\Lambda,\Delta}^{(m_{eff})}[(-\Delta)_\Delta^{1/2}\tilde{\psi}_\Delta](-\Delta)_\Delta^{1/2},$$

$$W_2 = -\frac{\sqrt{\beta|\Delta|}}{8\beta N}(-\Delta)_\Delta^{1/2}G_{\Delta,\Lambda}[(-\Delta)_\Lambda^{1/2}\tilde{\psi}_\Lambda]G_{\Lambda,\Delta}^{(m_{eff})}[(-\Delta)_\Delta^{1/2}\tilde{\psi}_\Delta]P_{0,\Delta}',$$

$$\cdots = \cdots$$

$$W_{16} = -\frac{1}{2\beta^2 N|\Delta|}P_{0,\Delta}'G_{\Delta,\Lambda}[P_{0,\Lambda}\tilde{\psi}_\Lambda]G_{\Lambda,\Delta}^{(m_{eff})}[P_{0,\Delta}\tilde{\psi}_\Delta]P_{0,\Delta}'$$

where we have used the following abbreviation:

$$(-\Delta)_\Delta = (-\Delta)_\Delta^{FBC}, \quad (-\Delta)_\Lambda = \sum_{\Delta_i \subset \Lambda}(-\Delta)_{\Delta_i}^{FBC}, \tag{4.19}$$

$$P_0 = P_{0,\Delta}, \quad P_{0,\Lambda} = \sum_{\Delta_i \subset \Lambda}P_{0,\Delta_i} \tag{4.20}$$

Lemma 13 *Again under the same approximation, $\int W(\Lambda,\Delta)^p(x,y)d\mu_0$ is finite and tends to 0 (as $N,\beta \to \infty$) where $d\mu_0 = \prod d\mu_\Delta(\tilde{\psi}_\Delta)$. More precisely*

$$\int W(x,y)^2 d\mu_0 \le \frac{const.}{\beta N|\Delta|^2} \to 0 \tag{4.21}$$

Proof. Put $W = W(\Delta, \Lambda)$ for simplicity, and we have

$$\int W(x,y)^2 d\mu_0 = \frac{1}{N^2} \sum_{x',y' \in \Delta} \sum_{x''',y''' \in \Delta} \left\{ \sum_{\Delta_i \subset \Lambda} \sum_{x'',y'' \in \Delta_i} [G_{\Delta_i}^{\circ 2}]^{-1}(x'',y'')[G_\Delta^{\circ 2}]^{-1}(x''',y''') \right.$$

$$\times G_\Delta^{-1/2}(x,x')G_{\Delta,\Lambda}(x',x'')G_{\Lambda,\Delta}^{(m_{eff})}(x'',x''')G_\Delta^{1/2}(x''',y)$$

$$\left. \times G_\Delta^{-1/2}(x,y')G_{\Delta,\Lambda}(y',y'')G_{\Lambda,\Delta}^{(m_{eff})}(y'',y''')G_\Delta^{1/2}(y''',y) \right\}$$

Since we can set (with some abuse of notation)

$$G_\Delta^{1/2} = (-\Delta)_\Delta^{1/2} + O(|\Delta|^{1/2}\beta^{1/2})P_0'$$

$$G_\Delta^{-1/2} = (-\Delta)_\Delta^{1/2} + O(|\Delta|^{-1/2}\beta^{-1/2})P_0'$$

and since both x' and x''' are in Δ of small size, we introduce three types of functions

$$f_1(x,x') = G_{\Delta,\Lambda}(x,x')G_{\Lambda,\Delta}^{(m_{eff})}(x',x), \tag{4.22}$$

$$f_2(x,x') = \sum_\zeta (-\Delta)_\Delta^{1/2}(x,\zeta)G_{\Delta,\Lambda}(\zeta,x')G_{\Lambda,\Delta}^{(m_{eff})}(x',x), \tag{4.23}$$

$$f_3(x,x') = \sum_{\zeta,\xi} (-\Delta)_\Delta^{1/2}(x,\zeta)G_{\Delta,\Lambda}(\zeta,x')(-\Delta)_\Delta^{1/2}(x,\xi)G_{\Lambda,\Delta}^{(m_{eff})}(x',\xi) \tag{4.24}$$

Their Fourier transforms $\tilde{f}_i$ ($i = 1, 2, 3$) are given by

$$\tilde{f}_1 = \int \frac{1}{[m^2 + 2\sum(1 - \cos k)][m_{eff}^2 + 2\sum(1 - \cos(p-k))]} \frac{d^2k}{(2\pi)^2}$$

$$\leq \chi_{<m_{eff}}(p)\frac{\beta}{m_{eff}^2} + \chi_{>m_{eff}}(p)\frac{\beta}{m_{eff}^2 + p^2}$$

$$\tilde{f}_2 = \int \frac{|k|}{[m^2 + 2\sum(1 - \cos k)][m_{eff}^2 + 2\sum(1 - \cos(p-k))]} \frac{d^2k}{(2\pi)^2}$$

$$\leq \chi_{<m_{eff}}(p)\frac{1}{m_{eff}^2} + \chi_{>m_{eff}}(p)|\log m_{eff}|\frac{|p|}{m_{eff}^2 + p^2},$$

$$\tilde{f}_3 = \int \frac{|k||p-k|}{[m^2 + 2\sum(1 - \cos k)][m_{eff}^2 + 2\sum(1 - \cos(p-k))]} \frac{d^2k}{(2\pi)^2}$$

$$\leq \chi_{<m_{eff}}(p)|\log m_{eff}| + \chi_{>m_{eff}}(p)\frac{1}{\sqrt{m_{eff}^2 + p^2}}.$$

where $m_{eff}^2 = c/N\beta$, $\chi_{<m_{eff}}$ (resp. $\chi_{>m_{eff}}$) is the characteristic function of $\{|p| < m_{eff}\}$ (resp. $\{|p| > m_{eff}\}$) and we have assumed $m < m_{eff} < 1$ without loss of generality. The second bound for $\tilde{f}_1(p) \geq 0$ comes from the estimate

$$\int \frac{1}{m^2 + 2\sum(1 - \cos(p_i - k_i))} \frac{d^2k}{(2\pi)^2} < -\text{const.} \log m < \text{const.}\beta$$

Note that $(\Delta)_\Delta \equiv (-\Delta)_\Delta^{FBC}$ is hermitian though there is a boundary effect, $(-\Delta)(x,x) = 4$ and then $\sum_{x',y''\in\Delta}(-\Delta)(x',y')e^{ip(x'-y')}$ yields the factor bounded by $(1-\cos p)$.

Thus we have

$$\sum_{x'} f_1^2(x,x') \leqq c\frac{\beta^2}{m_{eff}^2} \leqq cN\beta^3 \tag{4.25}$$

and

$$\sum_{\zeta,\xi} f_2(x,\zeta)(-\Delta)_\Delta(\zeta,\xi)f_2(\xi,x') \leqq c_1\frac{1}{m_{eff}^2} + c_2|\log m_{eff}|^3, \tag{4.26}$$

$$\sum_{\zeta,\xi} f_3(x,\zeta)(-\Delta)_\Delta(\zeta,\xi)f_3(\xi,x') \leqq c_1 m_{eff}^4|\log m_{eff}| + c_2 O(1) \tag{4.27}$$

We furthermore substitute

$$[G_\Delta^{\circ 2}]^{-1} = \frac{1}{2\beta}\left\{(-\Delta)_\Delta^{FBC} + (-\Delta)_{\partial\Delta}^{FBC}\right\} + O(\beta^{-2})$$

The we see that these are enough to conclude the conclusion.   Q.E.D.

These analysis implies that if we assume that ψ_Δ do not interact ψ's contained in the denominators, then their effects are bounded by $O(\beta^{-1})$.

3. Tadpole Contributions in $\int W^p d\mu$

In the previous integrals, we have neglected tadpole contributions in the integrals of W. In fact, they are the most important contributions and are larger than the non-tadpole contributions though they tend to 0 as $N \to \infty$. We set

$$W(\Delta, \Lambda)(x,y) = \kappa^2\left[G_\Delta^{-1/2}\psi_\Delta\frac{1}{G_\Delta^{-1} + i\kappa\psi_\Delta}G_\Delta^{-1}G_{\Delta,\Lambda}\right.$$
$$\left.\times \psi_\Lambda\frac{1}{G_\Lambda^{-1} + i\kappa\psi_\Lambda}G_\Lambda^{-1}G_{\Lambda,\Delta}G_\Delta^{1/2}\right](x,y) \tag{4.28}$$

and we use integration by parts. Let

$$d\mu_0 = \prod_\Delta\left\{\exp[- <\psi_\Delta, G_\Delta^{\circ 2}\psi_\Delta>]\prod_{x\in\Delta}d\psi(x)\right\} \tag{4.29}$$

(except for the normalization constant). Then by integration by parts, we have

$$\int \psi(\zeta)F(\psi)d\mu_0 = \frac{1}{2}\sum_{\zeta'\in\Delta}[G_\Delta^{\circ 2}]^{-1}(\zeta,\zeta')\int \frac{\partial}{\partial\psi(\zeta')}F(\psi)d\mu_0 \tag{4.30}$$

where $\zeta \in \Delta$ and

$$\frac{\partial}{\partial \psi(\zeta')} \frac{1}{G_\Lambda^{-1} + i\kappa\psi}(x,y) = -i\kappa \frac{1}{G_\Lambda^{-1} + i\kappa\psi}(x,\zeta') \frac{1}{G_\Lambda^{-1} + i\kappa\psi}(\zeta',y)$$

etc. Similarly we have

$$\int \psi(\zeta)\psi(\xi) \frac{1}{G_\Lambda^{-1} + i\kappa\psi}(x,y)d\mu_0$$

$$= \frac{1}{2}[G_\Delta^{o2}]^{-1}(\zeta,\xi) \int \frac{1}{G_\Lambda^{-1} + i\kappa\psi}(x,y)d\mu_0 + \frac{1}{4}\kappa^2 \sum_{\zeta',\xi'}[G_\Delta^{o2}]^{-1}(\zeta,\zeta')[G_\Delta^{o2}]^{-1}(\xi,\xi')$$

$$\times \int \frac{1}{G_\Lambda^{-1} + i\kappa\psi}(x,\zeta') \frac{1}{G_\Lambda^{-1} + i\kappa\psi}(\zeta',\xi') \frac{1}{G_\Lambda^{-1} + i\kappa\psi}(\xi',y)d\mu_0$$

$$+(\text{same as above with } \zeta' \to \xi')$$

The first term on the RHS in the above is the approximation which we just have argued in the previous section. The second and the third are the contraction terms (tadpoles diagrams) and the reminiscence of $\text{tr}(G\psi)^4$. Thus we have

$$\int W(x,y)d\mu_0 = -\frac{\kappa^4}{4} \sum_{\Delta_i \subset \Lambda} \sum_{x',x'',\zeta,\zeta'' \in \Delta} \sum_{\xi,\xi' \in \Delta_i} \sum_{x''' \in \Lambda}$$

$$\times [G_\Delta^{o2}]^{-1}(\zeta,\zeta')[G_\Delta^{o2}]^{-1}(\xi,\xi')$$

$$\times \int G_\Delta^{-1/2}(x,x') \frac{1}{G_\Delta^{-1} + i\kappa\psi_\Delta}(x',\zeta) \frac{1}{G_\Delta^{-1} + i\kappa\psi_\Delta}(\zeta,x'')(G_\Delta^{-1}G_{\Delta,\Lambda})(x'',\xi)$$

$$\times \frac{1}{G_\Lambda^{-1} + i\kappa\psi_\Lambda}(\xi,\xi') \frac{1}{G_\Lambda^{-1} + i\kappa\psi_\Lambda}(\xi',x''')(G_\Lambda^{-1}G_{\Lambda,\Delta})(x''',\zeta')G_\Delta^{1/2}(\zeta',y)d\mu_0 \quad (4.31)$$

Since the size of Δ is so small compared with β, we can put

$$\sum_{x' \in \Delta} G_\Delta^{-1/2}(x,x') \frac{1}{G_\Delta^{-1} + i\kappa\psi_\Delta}(x',\zeta) \frac{1}{G_\Delta^{-1} + i\kappa\psi_\Delta}(\zeta,x'') = G_\Delta^{1/2}(x,\zeta)G_\Delta(\zeta,x'')$$

Note that ψ_Δ and ψ_Λ are independent and that

$$\int \frac{1}{G_\Lambda^{-1} + i\kappa\psi_\Lambda}(\xi,\xi') \frac{1}{G_\Lambda^{-1} + i\kappa\psi_\Lambda}(\xi',x''')d\mu_0$$

$$= G^{(ave)}(\xi,\xi')G^{(ave)}(\xi',x''') + O(\beta^{-1})O(G^{(ave)}(\xi,\xi')G^{(ave)}(\xi',x''')) \quad (4.32)$$

and

$$\Gamma^{(ave)}(x,y) \equiv \frac{\kappa^2}{2}[G_\Delta^{o2}]^{-1}(x,y)G^{(ave)}(x,y)$$

$$= \Gamma_0^{(ave)}(x,y) + \Gamma_d^{(ave)}(x,y) + \delta\Gamma^{(ave)}(x,y) \quad (4.33)$$

where

$$\Gamma_0^{(ave)}(x,y) \;=\; \frac{G^{(ave)}(x,x)}{N\beta}(-\Delta)_\Delta^{(F)}(x,y) \tag{4.34}$$

$$\Gamma_d^{(ave)}(x,y) \;=\; \frac{1}{N\beta}(-\Delta)_\Delta^{(F)}(x,y)O(|x-y|)$$

$$\delta\Gamma^{(ave)}(x,y) \;=\; O\left(\frac{\log\beta}{N\beta^2|\Delta|}\right) \tag{4.35}$$

and $\Gamma_d^{(ave)}(x,y)$ is regarded as a diagonal matrix whose diagonal components are $1/(N\beta)$ since $(-\Delta)_\Delta^{(F)}(x,y) = 0$ unless $|x-y| \le 1$. This follows from

$$[G_\Delta^{\circ 2}]^{-1}(\zeta,\zeta') \;=\; \frac{1}{2\beta}G_\Delta^{-1}(\zeta,\zeta') + \frac{1}{\beta^2|\Delta|}P_{\Delta,0}(\zeta,\zeta'),$$

$$G_\Delta(\zeta,\zeta') \;=\; (-\Delta)_\Delta + O(\beta|\Delta|)P_\Delta'$$

see (4.9), namely $P_{\Delta,0}$ is the projection operator to the eigenspace of the largest eigenvalue $e_0 = O(\beta^2|\Delta|)$ of $G_\Delta^{\circ 2}$.

Similarly we define

$$\Gamma_\Delta(x,y) \;\equiv\; \frac{\kappa^2}{2}[G_\Delta^{\circ 2}]^{-1}(x,y)G(x,y)$$

$$= \Gamma_0(x,y) + \Gamma_d(x,y) + \delta\Gamma(x,y) \tag{4.36}$$

where

$$\Gamma_0(x,y) \;=\; \frac{1}{N}(-\Delta)_\Delta^{(FBC)}(x,y) \tag{4.37}$$

$$\Gamma_d(x,y) \;=\; \frac{1}{N\beta}(-\Delta)_\Delta^{(FBC)}(x,y)O(|x-y|)$$

$$\delta\Gamma(x,y) \;=\; O\left(\frac{\log\beta}{N\beta^2|\Delta|}\right) \tag{4.38}$$

and $\Gamma_d(x,y)$ is again regarded as a diagonal matrix whose diagonal components are $1/(N\beta)$ since $(-\Delta)_\Delta^{(FBC)}(x,y) = 0$ unless $|x-y| \le 1$.

Then we have

$$\int W(x,y)d\mu_0 \;=\; -\sum_{\zeta,\zeta'}\sum_{\eta,\eta'}\sum_\omega G_\Delta^{1/2}(x,\zeta)\Gamma_\Delta(\zeta,\zeta')G_{\Delta,\Lambda}(\zeta',\eta)$$

$$\times\Gamma^{(ave)}(\eta,\eta')G_{\Lambda,\Delta}^{(m_{eff})}(\eta',\omega)G_\Delta^{-1/2}(\omega,y)$$

where we can again put (for notational simplicity)

$$G_\Delta^{1/2}(\zeta,\zeta') \;=\; (-\Delta)_\Delta^{1/2} + O(\sqrt{\beta|\Delta|})P_{0,\Delta}',$$

$$G_\Delta^{-1/2}(\zeta,\zeta') \;=\; (-\Delta)_\Delta^{1/2} + O(\frac{1}{\sqrt{\beta|\Delta|}})P_{0,\Delta}'$$

see (4.9).

Thus it is easy to see that the largest contribution comes from $\Gamma_d^{(ave)}$ in $\Gamma^{(ave)}$ and Γ_d in Γ_Δ and it is easy to see that

$$\left| \int W(x,y) d\mu_0 \right| \leq \text{const.} \sum_{\zeta,\zeta' \in \Delta} \sum_{\omega \in \Delta}$$

$$\times \left\{ \sum_{\Delta_i \subset \Lambda} \sum_{\eta,\eta' \in \Delta_i} G_\Delta^{1/2}(x,\zeta) \Gamma_\Delta(\zeta,\zeta') G_{\Delta,\Lambda}(\zeta',\eta) \Gamma^{(ave)}(\eta,\eta') G_{\Lambda,\Delta}^{(m_{eff})}(\eta',\omega) G_\Delta^{-1/2}(\omega,y) \right\}$$

$$\leqq \text{const.} \frac{1}{N^2 \beta^2} \sum_{\zeta \in \Lambda} G_{\Delta,\Lambda}(x,\zeta) G_{\Lambda,\Delta}^{(m_{eff})}(\zeta,y)$$

$$\leqq \text{const.} \frac{1}{N^2 \beta^2} \frac{\beta}{m_{eff}} \sim \text{const.} \frac{1}{N}$$

where we have used $m_{eff}^2 > 1/\beta N$. This converges to zero as $N \to \infty$. For other contributions which comes from $(-\Delta)_\Delta^{1/2}$, G_Δ^{-1}, etc. also converge to quantities bounded by $(N\beta|\Delta|)^{-1}$. (We multiply these differential operators to the Green's functions, which improve the convergence.) This is left to the reader as an exercise.

V. WORKS IN FUTURE : POLYMER EXPANSIONS ETC.

To complete our discussion, we need to establish polymer-expansion for the present system by introducing paved sets $\Box_i$ which are collections of Δ_j and chosen large. Namely $\Lambda = \cup \Box_i$, $\Box_i = \cup_{k \in I_i} \Delta_k$, $\Box_i \cap \Box_j = \emptyset$, $i \neq j$. Then using the Feshbach-Krein formula [7], we have

$$\det(1 + i\kappa G_\Lambda \psi_\Lambda) = \prod_{i=1}^{n} \det(1 + i\kappa G_{\Box_i} \psi_{\Box_i}) \prod_{i=1}^{n-1} \det(1 + W(\Box_i, \Lambda_i))$$

where

$$\Lambda_i = \Lambda - \cup_{k=1}^{i} \Box_k \qquad (5.1)$$

and

$$W(\Box_i, \Lambda_i) = -(i\kappa)^2 \psi_{\Box_i} \frac{1}{1 + i\kappa G_{\Box_i} \psi_{\Box_i}} G_{\Box_i, \Lambda_i} \psi_{\Lambda_i} \frac{1}{1 + i\kappa G_{\Lambda_i} \psi_{\Lambda_i}} G_{\Lambda_i, \Box_i} \qquad (5.2)$$

Furthermore we decompose $\Box_i$ into smaller cubes $\Delta_{i,k}$ in the same way, that is

$$\det(1 + i\kappa G_{\Box_i} \psi_{\Box_i}) = \prod_k \det(1 + i\kappa G_{\Delta_{i,k}} \psi_{\Delta_{i,k}}) \det(1 + W(\Delta_{i,k}, \Box_{i,k}))$$

where $\square_i = \cup_k \Delta_{i,k}$, $\square_{i,k} = \square_i - \cup_{j=1}^k \Delta_{i,j}$ and putting $\Delta \equiv \Delta_{i,k}$, $\square \equiv \square_{i,k}$ for simplicity,

$$W(\Delta, \square) = -(i\kappa)^2 \psi_\Delta \frac{1}{1 + i\kappa G_\Delta \psi_\Delta} G_{\Delta,\square} \psi_\square \frac{1}{1 + i\kappa G_\square \psi_\square} G_{\square,\Delta} \qquad (5.3)$$

Or sandwiching them with $G_\square^{1/2}$ and $G_\square^{1/2}$, we rather consider

$$W(\square_i, \Lambda_i) = -(i\kappa)^2 G_\square^{1/2} \psi_{\square_i} \frac{1}{1 + i\kappa G_{\square_i} \psi_{\square_i}} G_{\square_i,\Lambda_i} \psi_{\Lambda_i} \frac{1}{1 + i\kappa G_{\Lambda_i} \psi_{\Lambda_i}} G_{\Lambda_i,\square_i} G_\square^{-1/2}$$

$$= -(i\kappa)^2 G_\square^{1/2} \psi_{\square_i} \frac{1}{G_{\square_i}^{-1} + i\kappa \psi_{\square_i}} G_{\square_i}^{-1} G_{\square_i,\Lambda_i} \psi_{\Lambda_i} \frac{1}{G_{\Lambda_i}^{-1} + i\kappa \psi_{\Lambda_i}} G_{\Lambda_i}^{-1} G_{\Lambda_i,\square_i} G_\square^{-1/2}$$

$$W(\Delta, \square) = -(i\kappa)^2 G_\Delta^{1/2} \psi_\Delta \frac{1}{1 + i\kappa G_\Delta \psi_\Delta} G_{\Delta,\square} \psi_\square \frac{1}{1 + i\kappa G_\square \psi_\square} G_{\square,\Delta} G_\Delta^{-1/2}$$

$$= -(i\kappa)^2 G_\Delta^{1/2} \psi_\Delta \frac{1}{G_\Delta^{-1} + i\kappa \psi_\Delta} G_\Delta^{-1} G_{\Delta,\square} \psi_\square \frac{1}{G_\square^{-1} + i\kappa \psi_\square} G_\square^{-1} G_{\square,\Delta} G_\Delta^{-1/2}$$

These new W functions have better properties since Δ's are embedded in paved sets $\square_i$. As before, for each $\Delta \subset \square$, we put

$$d\mu_\Delta = \det{}_2^{-N/2}(1 + i\kappa G_\Delta \psi_\Delta) \prod_{x \in \Delta} d\psi(x)$$

$$= \det{}_3^{-N/2}(1 + i\kappa G_\Delta \psi_\Delta) \exp[- < \psi_\Delta, G_\Delta^{\circ 2} \psi_\Delta >] \prod_{x \in \Delta} d\psi(x). \qquad (5.4)$$

We regard $W(\square_i, \Lambda_i)$ and $W(\Delta_k, \square_{i,k})$ as two types of corrections to the (almost Gaussian) measure $\prod d\mu_{\Delta_i}$. The works in this direction are now in progress, and will be published in the forthcoming paper [6].

Acknowledgments

[1] D. Brydges, J. Fröhlich and T. Spencer, The Random Walk Representation of Classical Spin Systems and Correlation Inequalities, Commun. Math. Phys.**83**: 123 (1982).

[2] T. Chen, Localization Length and Boltzmann Limit for the Anderson Model at Small Disorders in D=3, arXiv:math-phy/0305051

[3] A.C.D.Enter and S.B.Schlosman, First Order Transitions for n−vector Models in 2 and More Dimension, Phys. Rev. Letters. **89**: (2002)

[4] J. Fröhlich and T. Spencer, Absence of Diffusion in the Anderson Tight Binding Model for Large Disorder or Low Energy, Commun. Math. Phys.**88**: 151 (1983)

[5] K.R.Ito, Renormalization Recursion formulas and Flows of 2D $O(N)$ Spin Models, Jour. Stat. Phys., **107**: 821-856 (2002).

[6] K.R.Ito, Paper in Preparation (2004).

[7] K. R. Ito and H. Tamura, N Dependence of Critical Temperatures of 2D $O(N)$ Spin Models, Commun. Math. Phys., **202**: 127 (1999)

[8] K. R. Ito, F.Hiroshima and H. Tamura, Green's functions with Complex Random Potential in two Dimensions Derived from the 2D $O(N)$ Spin Model, Preprint (2004, July)

[9] C. Copper, Mass Generations in the Large N non-linear σ Model, Commun. Math. Phys., **202**: 89 (1999)

[10] E. Lieb and M. Loss, Analysis, Graduate Studies in Math. (Vol.14, American Math. Soc., 1994)

[11] A.Polyakov, Interactions of Goldstone Bosons in Two Dimensions, Phys. Lett,**59B**: 79 (1975).

[12] C. Shubin, W. Schlag, W. Wolf, Frequency Concentration and Localization Length for the Anderson Model at Small disorders, Journal d'analysys math., **88**, 173 (2003).

[13] B.Simon, The $P(\phi)_2$ Euclidean (Quantum) Field Theory, (Princeton Univ. Press, 1974)

[14] K. Wilson, Confinement of Quarks, Phys. Rev. D **10**: 2455 (1974) and Renormalization Groups and Critical Phenomena, Rev. Mod.Phys. **55**: 583 (1983)