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Triviality of hierarchical $O(N)$ model in four dimensions

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Abstract

The renormalization group transformation for the hierarchical $O(N)$ spin model in four dimensions is studied and convergence of the critical trajectory to the Gaussian fixed point is shown for a sufficiently large N.

1 Hierarchical $O(N)$ spin model

There is a long-standing conjecture that the continuum limit of the classical spin models in four dimensions will be Gaussian (the triviality of $O(N)$ spin models). We here consider the hierarchical version of this problem and describe the outline of the study in [1].

Let $N>1$ and $\Lambda>0$ be integers. The d dimensional hierarchical $O(N)$ spin model on the lattice $\mathcal{L}_\Lambda=\{0,1\}^\Lambda$ is defined as follows:

$$\phi_\theta=\phi_{\theta_\Lambda,\dots,\theta_1}\in\mathbb{R}^N\ ,\quad \theta=(\theta_\Lambda,\dots,\theta_1)\in\mathcal{L}_\Lambda\ , \tag{1}$$

$$H_\Lambda(\phi)=-\frac{1}{2}\sum_{n=1}^{\Lambda}\frac{1}{(2\omega)^n}\sum_{\theta_\Lambda,\dots,\theta_{n+1}=0,1}\left|\sum_{\theta_n,\dots,\theta_1=0,1}\phi_{\theta_\Lambda,\dots,\theta_1}\right|^2\ , \tag{2}$$

$$\langle F\rangle_{\Lambda,h_0^{(N)}}=\frac{1}{Z_{\Lambda,h_0^{(N)}}}\int d\phi F(\phi)\exp(-\beta H_\Lambda(\phi))\prod_{\theta\in\mathcal{L}_\Lambda}h_0^{(N)}(\phi_\theta), \tag{3}$$

$$Z_{\Lambda,h_0^{(N)}}=\int d\phi\exp(-\beta H_\Lambda(\phi))\prod_{\theta\in\mathcal{L}_\Lambda}h_0^{(N)}(\phi_\theta), \tag{4}$$

$$h_0^{(N)}(\mathbf{x})=\text{const.}\delta(|\mathbf{x}|-\sqrt{N}\alpha)\ ,\quad \mathbf{x}\in\mathbb{R}^N\ , \tag{5}$$

where

$$\beta=\frac{\omega-1}{2}\ ,\quad \omega=2^{2/d}\ ,\quad d>2\ , \tag{6}$$

$$\alpha>0\ . \tag{7}$$

2 RG transformation

Define *the block spins* ϕ' by

$$\phi'_\tau = \frac{1}{\sqrt{2\omega}} \sum_{\theta_1=0,1} \phi_{\tau\theta_1}\,, \quad \tau = (\tau_{\Lambda-1}, \ldots, \tau_1).$$

If there is a function $F'(\phi')$ of the block spins such that

$$F(\phi) = F'(\phi'),$$

then it holds that

$$\langle F\rangle_{\Lambda, h_0^{(N)}} = \langle F'\rangle_{\Lambda-1, \mathcal{R}h_0^{(N)}}\,,$$

where $\mathcal{R}$ is the mapping defined by

$$\mathcal{R}h(\mathbf{x}) = \text{const.} \exp(\frac{\beta}{2}|\mathbf{x}|^2) \int_{\mathbb{R}^N} h(\sqrt{\frac{\omega}{2}}\mathbf{x}+\mathbf{y}) h(\sqrt{\frac{\omega}{2}}\mathbf{x}-\mathbf{y})\, d\mathbf{y},\ \mathbf{x} \in \mathbb{R}^N. \tag{8}$$

Consider the renormalization group trajectory

$$h_n^{(N)} = \mathcal{R}^n h_0^{(N)}\,, \quad n \geq 0. \tag{9}$$

Note that

$$h_G(\mathbf{x}) = \text{const.} \exp(-\frac{1}{4}|\mathbf{x}|^2) \tag{10}$$

is *the trivial (Gaussian) fixed point* of $\mathcal{R}$.

3 Result

Let us state our result.

Theorem 3.1 *Let $d = 4$. For a sufficiently large N, there exists a positive constant α_N such that if $h_n^{(N)}, n \geq 0$, are defined by* (5) *and* (9) *with $\alpha = \alpha_N$, then the sequence of measures $h_n^{(N)}(\mathbf{x})d\mathbf{x}, n \geq 0$, weakly converges to the trivial fixed point measure $h_G(\mathbf{x})d\mathbf{x}$ as $n \to \infty$.*

Consequently, if we construct the continuum limit of the hierarchical $O(N)$ model in four dimensions by means of the critical trajectory in Theorem 3.1, the limit is inevitably Gaussian. The analogous fact will be shown for $d > 4$ dimensions by weaker bounds. See also [2] for the case of the hierarchical Ising model ($N = 1$) in four dimensions.

The proof of Theorem 3.1 is decomposed into three parts:

1. **$O(N)$ trajectory in the weak coupling regime**

   We obtain a criterion for the trajectory (9) to converge to h_G assuming that the trajectory has entered a vicinity of h_G. (Proposition 4.1)

2. $O(\infty)$ **trajectory**

   We explicitly calculate the $O(\infty)$ trajectories, i.e., the trajectories corresponding to $N = \infty$, and derive the asymptotic behavior of trajectories near the critical point. (Proposition 4.2)

3. **From** $O(N)$ **trajectory to** $O(\infty)$ **trajectory**

   We show that an $O(N)$ trajectory converges to an $O(\infty)$ trajectory as $N \to \infty$. Consequently, we can find the critical $O(N)$ trajectory in the vicinity of the critical $O(\infty)$ trajectory for a sufficiently large N. (Proposition 4.3)

4 Outline of the proof

The proof of Theorem 3.1 is based on *the characteristic function method* developed in [2].

4.1 Characteristic functions

We consider characteristic functions of effective measures

$$\hat{h}_n^{(N)}(\boldsymbol{\xi}) = \mathcal{F}h_n^{(N)}(\boldsymbol{\xi}) = \int_{\mathbb{R}^N} e^{\sqrt{-1}(\boldsymbol{\xi},\mathbf{x})} h_n^{(N)}(\mathbf{x})d\mathbf{x}\,, \quad n = 0, 1, 2, \cdots,$$

and write the renormalization group transformation for $\hat{h}_n^{(N)}$ as

$$\hat{h}_n^{(N)} = \mathcal{F}\mathcal{R}\mathcal{F}^{-1}\hat{h}_{n-1}^{(N)} = \mathcal{T}\mathcal{S}\hat{h}_{n-1}^{(N)}\,, \tag{11}$$

where

$$\mathcal{S}g(\boldsymbol{\xi}) = g(\frac{1}{\sqrt{2\omega}}\boldsymbol{\xi})^2, \tag{12}$$

$$\mathcal{T}g(\boldsymbol{\xi}) = \text{const.}\ \exp(-\frac{\beta}{2}\triangle)g(\boldsymbol{\xi})\,. \tag{13}$$

In the above, $\triangle$ denotes the N dimensional Laplacian and the constant is chosen so that

$$\mathcal{T}g\,(\mathbf{0}) = 1$$

holds. Since $\hat{h}_n^{(N)}$ has spherical symmetry, we shall often write

$$\hat{h}_n^{(N)}(\boldsymbol{\xi}) = \hat{h}_n^{(N)}(\xi),$$

where $\xi = |\boldsymbol{\xi}|$. Note that the mapping $\mathcal{T}\mathcal{S}$ has the trivial fixed point $\hat{h}_G(\xi) = \exp(-\xi^2)$.

4.2 The Lee–Yang property

The reason why we use the characteristic function is the fact that the 'potential' of the characteristic function has the remarkable positivity due to the Lee–Yang property.

Let us introduce a *potential* $V_n^{(N)}(\xi)$ and its Taylor coefficients $\mu_{k,n}^{(N)}$ by

$$\hat{h}_n^{(N)}(\xi) = e^{-V_n^{(N)}(\xi)} , \tag{14}$$

$$V_n^{(N)}(\xi) = \sum_{k=1}^{\infty} \mu_{k,n}^{(N)} \xi^k \tag{15}$$

for $n \geq 0$. (Note that $\hat{h}_n^{(N)}(0) = 1$, i.e. $V_n^{(N)}(0) = 0$.) The coefficient $\mu_{k,n}^{(N)}$ is called *a truncated correlation.* Since $\hat{h}_n^{(N)}(\xi)$ is even, $\mu_{k,n}^{(N)}$ vanishes if k is odd.

As is well-known, the hierarchical model has *the Lee-Yang property* for any $N \geq 1$: $\hat{h}_n^{(N)}(\xi)$ has only real zeros. (See e.g. [3].) As a result, the truncated correlations have the bound [5]:

$$0 \leq k\mu_{2k,n}^{(N)} \leq (2\mu_{4,n}^{(N)})^{k/2}, \ k \geq 3, \ n \geq 0. \tag{16}$$

This implies the following:

1. The Taylor expansion in the right hand side of (15) has a nonzero radius of convergence;

2. It suffices to prove $\lim_{n\to\infty} \mu_{4,n}^{(N)} = 0$ in order to ensure $\lim_{n\to\infty} \mu_{2k,n}^{(N)} = 0$ for all $k \geq 2$, which implies weak convergence of the trajectory to a Gaussian measure.

Next we introduce *the scaled potential* $v_n^{(N)}(\eta)$ and its Taylor expansion by

$$v_n^{(N)}(\eta) = \frac{1}{N} V_n^{(N)}(\sqrt{N}\eta) = \sum_{k=1}^{\infty} \nu_{k,n}^{(N)} \eta^k, \quad n \geq 0. \tag{17}$$

In other words, we scale the truncated correlation $\mu_{k,n}^{(N)}$ as

$$\nu_{k,n}^{(N)} = N^{k/2-1} \mu_{k,n}^{(N)}, \quad k \geq 1, \ n \geq 0.$$

Then, $\nu_{k,n}^{(N)}$ turns out to be $\mathcal{O}(1)$ with respect to N. We refer to $\nu_{k,n}^{(N)}$ as *a scaled truncated correlation.* In particular, for the trivial fixed point measure $h_G(\mathbf{x})$, the scaled potential is given by

$$v_G(\eta) = \eta^2 . \tag{18}$$

4.3 Differential equations for potentials

In view of (12) and (13), we consider the following equation:

$$\frac{\partial}{\partial t} \hat{h}_n^{(N)}(t, \boldsymbol{\xi}) = -\Delta \hat{h}_n^{(N)}(t, \boldsymbol{\xi}), \quad n \geq 1, t \in [0, \beta/2] , \tag{19}$$

or, equivalently

$$\frac{\partial}{\partial t} \hat{h}_n^{(N)}(t, \xi) = -\frac{\partial^2}{\partial \xi^2} \hat{h}_n^{(N)}(t, \xi) - \frac{N-1}{\xi} \frac{\partial}{\partial \xi} \hat{h}_n^{(N)}(t, \xi), \quad n \geq 1, t \in [0, \beta/2] ,$$

with the initial condition

$$\hat{h}_n^{(N)}(0,\xi) = \hat{h}_{n-1}^{(N)}(\frac{1}{\sqrt{2\omega}}\xi)^2 , \quad n \geq 1 .$$

Then, we have

$$\hat{h}_n^{(N)}(\xi) = \frac{\hat{h}_n^{(N)}(\frac{\beta}{2},\xi)}{\hat{h}_n^{(N)}(\frac{\beta}{2},0)}, \quad n \geq 1 .$$

We also define the t-dependent scaled potential and its expansion by

$$v_n^{(N)}(t,\eta) = -\frac{1}{N}\log \hat{h}_n^{(N)}(t,\sqrt{N}\eta) = \sum_{k=1}^{\infty} \nu_{k,n}^{(N)}(t)\eta^k , \quad n \geq 1, t \in [0,\beta/2] .$$

Then, the potentials $v_n^{(N)}(t,\eta), n \geq 1$, obey

$$\frac{\partial}{\partial t}v_n^{(N)}(t,\eta) = \left(\frac{\partial}{\partial \eta}v_n^{(N)}(t,\eta)\right)^2 - (1-\frac{1}{N})\frac{1}{\eta}\frac{\partial}{\partial \eta}v_n^{(N)}(t,\eta) - \frac{1}{N}\frac{\partial^2}{\partial \eta^2}v_n^{(N)}(t,\eta) , \tag{20}$$

$$v_n^{(N)}(0,\eta) = 2v_{n-1}^{(N)}(\frac{1}{\sqrt{2\omega}}\eta) , \tag{21}$$

$$v_n^{(N)}(\eta) = v_n^{(N)}(\frac{\beta}{2},\eta) - v_n^{(N)}(\frac{\beta}{2},0), \tag{22}$$

and the Taylor coefficients $\nu_{2j,n}^{(N)}(t), j \geq 1, n \geq 1$, obey

$$\frac{d}{dt}\nu_{2j,n}^{(N)}(t) = \sum_{\substack{m+\ell=2j+2 \\ m,\ell \geq 2}} m\ell \nu_{m,n}^{(N)}(t)\nu_{\ell,n}^{(N)}(t) - (2j+2)(1+\frac{2j}{N})\nu_{2j+2,n}^{(N)}(t) , \tag{23}$$

$$\nu_{2j,n}^{(N)}(0) = \frac{2}{(2\omega)^j}\nu_{2j,n-1}^{(N)} , \tag{24}$$

$$\nu_{2j,n}^{(N)} = \nu_{2j,n}^{(N)}(\frac{\beta}{2}) . \tag{25}$$

In particular for $j = 1,2,3,4$, the equation (23) gives

$$\frac{d}{dt}\nu_{2,n}^{(N)}(t) = 4\nu_{2,n}^{(N)}(t)^2 - 4(1+\frac{2}{N})\nu_{4,n}^{(N)}(t) , \tag{26}$$

$$\frac{d}{dt}\nu_{4,n}^{(N)}(t) = 16\nu_{2,n}^{(N)}(t)\nu_{4,n}^{(N)}(t) - 6(1+\frac{4}{N})\nu_{6,n}^{(N)}(t) , \tag{27}$$

$$\frac{d}{dt}\nu_{6,n}^{(N)}(t) = 24\nu_{2,n}^{(N)}(t)\nu_{6,n}^{(N)}(t) + 16\nu_{4,n}^{(N)}(t)^2 - 8(1+\frac{6}{N})\nu_{8,n}^{(N)}(t) , \tag{28}$$

$$\frac{d}{dt}\nu_{8,n}^{(N)}(t) = 32\nu_{2,n}^{(N)}(t)\nu_{8,n}^{(N)}(t) + 48\nu_{4,n}^{(N)}(t)\nu_{6,n}^{(N)}(t) - 10(1+\frac{8}{N})\nu_{10,n}^{(N)}(t) . \tag{29}$$

Note that $\nu_{2j,n}^{(N)}(t)$ has the positivity due to the Lee–Yang property

$$\nu_{2j,n}^{(N)}(t) \geq 0 , \quad j \geq 1, n \geq 1 , \tag{30}$$

since $\nu_{2j,n}^{(N)}(t)$ is regarded as a scaled truncated correlation for a hierarchical model with t–dependence.

4.4 Weak coupling regime

The positivity (30) implies that we have upper bounds of the solutions by dropping the last negative contributions in the right hand sides of (23). Furthermore, we can derive lower bounds by substituting the upper bounds in the last terms.

We perform the analysis described above in the weak coupling regime, i.e., in the vicinity of the fixed point (18). As is well-known, the quartic coefficient $\nu_{4,n}^{(N)}$ of the critical renormalization group trajectory has power decay in four dimensions, which can be seen by the second order perturbation. For this purpose, it suffices to bound $\nu_{2j,n}^{(N)}$ for $j = 1,2,3,4$ by using (26)–(29).

In order to state the result of the analysis in the weak coupling regime, we write $\nu_{k,n}^{(N)}, k = 2,6,8$, as follows:

$$\nu_{2,n}^{(N)} = 1 + \frac{1}{\sqrt{2}}(1+\frac{2}{N})\nu_{4,n}^{(N)} + \zeta_{2,n}^{(N)}\nu_{4,n}^{(N)^2} , \tag{31}$$

$$\nu_{6,n}^{(N)} = 4\nu_{4,n}^{(N)^2} + \zeta_{6,n}^{(N)}\nu_{4,n}^{(N)^3} , \tag{32}$$

$$\nu_{8,n}^{(N)} = \zeta_{8,n}^{(N)}\nu_{4,n}^{(N)^3} , \tag{33}$$

where $\nu_{4,n}^{(N)}$ is assumed to be small. In fact, we have:

Proposition 4.1 *Suppose that there exist a positive integer n_1 and positive constants $\alpha_\pm$ ($\alpha_- < \alpha_+$) such that*

1. *it holds that*

$$\zeta_{2,n_1}^{(N)} = \zeta , \quad \text{if } \alpha = \alpha_+ , \tag{34}$$

$$\zeta_{2,n_1}^{(N)} = -\zeta , \quad \text{if } \alpha = \alpha_- , \tag{35}$$

2. *for $\alpha \in [\alpha_-, \alpha_+]$, the following conditions are satisfied:*

$$|\zeta_{2,n_1}^{(N)}| \le \zeta ,$$

$$\nu_{4,n_1}^{(N)} \le \epsilon ,$$

$$|\zeta_{6,n_1}^{(N)}|\nu_{4,n_1}^{(N)} \le \epsilon_0 ,$$

$$|\zeta_{8,n_1}^{(N)}|\nu_{4,n_1}^{(N)} \le \epsilon_1 ,$$

where $\zeta, \epsilon, \epsilon_0$ and ϵ_1 are positive constants.

Then, there exists a value $\alpha_N \in [\alpha_-, \alpha_+]$ such that

$$\lim_{n\to\infty} \nu_{2,n}^{(N)} = 1 , \tag{36}$$

$$\lim_{n\to\infty} \nu_{4,n}^{(N)} = 0 \tag{37}$$

hold at $\alpha = \alpha_N$.

In the statement of Proposition 4.1, the condition 2 means that we are in the weak coupling regime, whereas the condition 1 enables us to perform the Bleher–Sinai argument ensuring existence of the critical mass parameter.

4.5 Analysis of $O(\infty)$ trajectory

Next we formally put $N = \infty$ in (20). Namely, we consider the equation

$$\frac{\partial}{\partial t}v_n^{(\infty)}(t,\eta) = \left(\frac{\partial}{\partial \eta}v_n^{(\infty)}(t,\eta)\right)^2 - \frac{1}{\eta}\frac{\partial}{\partial \eta}v_n^{(\infty)}(t,\eta) \tag{38}$$

with

$$v_n^{(\infty)}(0,\eta) = 2v_{n-1}^{(\infty)}(\frac{1}{\sqrt{2\omega}}\eta)\ , \tag{39}$$

$$v_n^{(\infty)}(\eta) = v_n^{(\infty)}(\frac{\beta}{2},\eta) - v_n^{(\infty)}(\frac{\beta}{2},0)\ , \tag{40}$$

where the initial point is chosen as follows:

$$v_0^{(\infty)}(\eta) = \lim_{N\to\infty} v_0^{(N)}(\eta) = \int_0^\eta \frac{2\alpha^2\eta}{1+\sqrt{1-4\alpha^2\eta^2}}d\eta\ . \tag{41}$$

The solution is referred to as *the* $O(\infty)$ *trajectory.* We however have to be aware that we have no spin system corresponding to the $O(\infty)$ trajectory.

In order to solve (38)–(41), we define functions $u_n(t,x)$ and $u_n(x)$ by

$$u_n(t,\eta^2) = v_n^{(\infty)}(t,\eta)\ ,\ n \geq 1\ , \tag{42}$$

$$u_n(\eta^2) = v_n^{(\infty)}(\eta)\ ,\ n \geq 0\ , \tag{43}$$

respectively. Then, (38)–(41) become

$$\frac{\partial}{\partial t}u_n(t,x) = 4x\left(\frac{\partial}{\partial x}u_n(t,x)\right)^2 - 2\frac{\partial}{\partial x}u_n(t,x)\ , \tag{44}$$

$$u_n(0,x) = 2u_{n-1}(\frac{x}{2\omega})\ , \tag{45}$$

$$u_n(x) = u_n(\frac{\beta}{2},x) - u_n(\frac{\beta}{2},0)\ , \tag{46}$$

$$u_0(x) = \int_0^x \frac{\alpha^2}{1+\sqrt{1-4\alpha^2 y}}dy\ , \tag{47}$$

where $n \geq 1$. Furthermore we denote the inverse of $p = u_n'(x)$ by $x = w_n(p)$ and the inverse of $p = u_n'(t,x)$ by $x = w_n(t,p)$ for each t. Then, $w_n(p)$ and $w_n(t,p)$ obey the following recursion relations:

$$\frac{\partial w_n}{\partial t}(t,p) = -4p^2\frac{\partial w_n}{\partial p}(t,p) - 8pw_n(t,p) + 2\ , \tag{48}$$

$$w_n(0,p) = 2\omega w_{n-1}(\omega p)\ , \tag{49}$$

$$w_n(p) = w_n(\frac{\beta}{2},p)\ , \tag{50}$$

$$w_0(p) = \frac{1}{2p} - \frac{\alpha^2}{4p^2}\ . \tag{51}$$

The system (48)–(51) is explicitly solved and we have

$$w_n(p) = \frac{1}{2p^2}\left(p + \frac{1}{2}\sum_{j=1}^{n}\frac{2^j}{\omega^j - 1 + p^{-1}} - \frac{1}{2}(\frac{2}{\omega})^n\alpha^2\right) , \quad n \geq 1, p > 0 , \tag{52}$$

$$w_n(t,p) = \frac{1}{2p^2}\left(p + \frac{1}{2}\sum_{j=1}^{n}\frac{2^j}{\omega^j - \omega + 4t + p^{-1}} - \frac{1}{2}(\frac{2}{\omega})^n\alpha^2\right) , \quad n \geq 1, t \geq 0, p > 0 . \tag{53}$$

Let us find the critical value of α. Introduce the variable s by

$$p = \frac{1}{1-s} \tag{54}$$

and regard s as a function of $x(= w_n(p))$. From (52), we see that

$$2xp^2 = \frac{1}{1-s} + \frac{1}{2}\sum_{j=1}^{n}\frac{2^j}{\omega^j - s} - \frac{1}{2}(\frac{2}{\omega})^n\alpha^2 . \tag{55}$$

Substituting

$$\frac{2^j}{\omega^j - s} = \left(\frac{2}{\omega}\right)^j + \left(\frac{2}{\omega^2}\right)^j s + \left(\frac{2}{\omega^3}\right)^j \frac{s^2}{1 - \frac{s}{\omega^j}} , \quad j \geq 0 ,$$

into (55), we obtain

$$s = \delta_n(2xp^2 + \gamma_n - R_n(s)) ,$$

where

$$\gamma_n = \frac{\omega - 1}{2 - \omega} + (\frac{1}{2}\alpha^2 - \frac{1}{2-\omega})\left(\frac{2}{\omega}\right)^n ,$$

$$\delta_n^{-1} = 1 + \frac{1}{2}\sum_{j=1}^{n}\left(\frac{2}{\omega^2}\right)^j ,$$

$$R_n(s) = \frac{s^2}{1-s} + \frac{1}{2}\sum_{j=1}^{n}\left(\frac{2}{\omega^3}\right)^j \frac{s^2}{1 - \omega^{-j}s} .$$

This shows that the critical value of α is $\sqrt{2+\sqrt{2}}$ and the critical trajectory tends to the trivial fixed point (18) (in the above notation we have $s = 0$) as $n \to \infty$, since $\delta_n \to 0$ and $R_n(s)$ is convergent.

Now, consider the Taylor expansion

$$v_n^{(\infty)}(\eta) = \sum_{j=1}^{\infty} \nu_{2j,n}^{(\infty)}\eta^{2j} , \quad n \geq 0 . \tag{56}$$

Based on the above analysis, we can deduce asymptotic behavior of the Taylor coefficients near the critical point. Let us write $\nu_{k,n}^{(\infty)}$, $k = 2, 6, 8$, $n \geq 0$, as:

$$\nu_{2,n}^{(\infty)} = 1 + \frac{1}{\sqrt{2}}\nu_{4,n}^{(\infty)} + \zeta_{2,n}^{(\infty)}\nu_{4,n}^{(\infty)2} , \tag{57}$$

$$\nu_{6,n}^{(\infty)} = 4\nu_{4,n}^{(\infty)2} + \zeta_{6,n}^{(\infty)}\nu_{4,n}^{(\infty)3} , \tag{58}$$

$$\nu_{8,n}^{(\infty)} = \zeta_{8,n}^{(\infty)}\nu_{4,n}^{(\infty)3} . \tag{59}$$

Then we have the following proposition.

Proposition 4.2 *There exist a positive integer n_1 and positive constants α_{++} and α_{--} ($\alpha_{++} > \alpha_{--}$) such that*

1. *it holds that*

$$\zeta_{2,n_1}^{(\infty)} \geq 2\zeta \ , \quad \text{at } \alpha = \alpha_{++}, \tag{60}$$
$$\zeta_{2,n_1}^{(\infty)} \leq -2\zeta \ , \quad \text{at } \alpha = \alpha_{--}, \tag{61}$$

2. *for $\alpha \in [\alpha_{--}, \alpha_{++}]$, the following conditions are satisfied:*

$$0 < \nu_{4,n_1}^{(\infty)} \leq \frac{1}{2}\epsilon \ , \tag{62}$$
$$|\zeta_{6,n_1}^{(\infty)}|\nu_{4,n_1}^{(\infty)} \leq \frac{1}{2}\epsilon_0 \ , \tag{63}$$
$$|\zeta_{8,n_1}^{(\infty)}|\nu_{4,n_1}^{(\infty)} \leq \frac{1}{2}\epsilon_1 \ . \tag{64}$$

In the above, $\zeta, \epsilon, \epsilon_0$ and ϵ_1 are the same constants as in Proposition 4.1.

4.6 From $O(N)$ trajectory to $O(\infty)$ trajectory

Finally we show that the $O(N)$ trajectory is approximated by the $O(\infty)$ trajectory.

Proposition 4.3 *For each $j = 1, 2, \cdots$, and for each $n = 0, 1, 2, \cdots$, it holds that*

$$\lim_{N \to \infty} \nu_{2j,n}^{(N)} = \nu_{2j,n}^{(\infty)} \ . \tag{65}$$

The convergence is uniform in α on any compact subset of $(0, \infty)$.

This fact is by no means trivial, because (20) is a singular perturbation of (38), to which the standard theory of differential equations does not apply: (19) is a diffusion equation in the inverse direction of time. Proposition 4.3 is shown by means of $1/N$ expansion developed by Kupiainen [4].

In order to apply Kupiainen's argument, we have to establish *the reflection positivity* [6] for our model. For $l = 1, 2, \cdots, \Lambda$, we define the reflection ρ_l on the lattice $\mathcal{L}_\Lambda$ by

$$(\rho_l\theta)_k = \begin{cases} \theta_k, & k \neq l \ , \\ 1 - \theta_k, & k = l \ , \end{cases} \qquad \theta \in \mathcal{L}_\Lambda \ .$$

Then, the measure $\langle \cdot \rangle_{\Lambda, h_0^{(N)}}$ has reflection positivity with respect to $\rho_l, l = 1, 2, \cdots, \Lambda$. Furthermore, since the reflection planes for $\rho_l, l = 1, 2, \cdots, \Lambda$, separate the 2^Λ points in $\mathcal{L}_\Lambda$ from each other, we have *the chessboard bound* [6].

The $1/N$ expansion yields the existence of the limit in the left hand side of (65). The fact that the limit coincides with the right hand side of (65) is shown by using the differential equations ((23) with $N = \infty$) for $\lim_{N \to \infty} \nu_{2j,n}^{(N)}(t)$.

4.7 Proof of Theorem 3.1

Theorem 3.1 follows from Proposition 4.1,Proposition 4.2 and Proposition 4.3.

We first use Proposition 4.2 and fix the integer n_1. Then using Proposition 4.3 for $n = n_1$ and $j \leq 4$, we see, for a sufficiently large N, that

$$\zeta_{2,n_1}^{(N)} \geq \zeta \ , \quad \text{at } \alpha = \alpha_{++},$$
$$\zeta_{2,n_1}^{(N)} \leq -\zeta \ , \quad \text{at } \alpha = \alpha_{--},$$

and that, for $\alpha \in [\alpha_{--}, \alpha_{++}]$,

$$0 < \nu_{4,n_1}^{(N)} \leq \epsilon \ ,$$
$$|\zeta_{6,n_1}^{(N)}|\nu_{4,n_1}^{(N)} \leq \epsilon_0 \ ,$$
$$|\zeta_{8,n_1}^{(N)}|\nu_{4,n_1}^{(N)} \leq \epsilon_1 \ .$$

Since $\zeta_{2,n_1}^{(N)}$ is continuous with respect to $\alpha \in [\alpha_{--}, \alpha_{++}]$, we can choose a subinterval $[\alpha_-, \alpha_+] \subset [\alpha_{--}, \alpha_{++}]$ so that the assumptions of Proposition 4.1 are satisfied. Theorem 3.1 follows from (36) and (37) by virtue of (16).

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