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NON EXISTENCE OF TWISTED WAVE EQUATIONS

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ABSTRACT. Let $X$ be a complex manifold, $V$ an involutive submanifold of its cotangent bundle, and $\Sigma$ a bicharacteristic leaf of $V$. A ring of twisted differential operators $\mathcal{A}$ on $X$ has a characteristic class in $H^4(X; d\mathcal{O}_X^\vee)$. To such a class we associate a class in $H^2(\Sigma; \mathbb{C}_\Sigma^\times)$ whose vanishing is necessary for the existence of an $\mathcal{A}$-module globally simple along $V$. As an application, we show that there are no generalized massless field equations with non trivial twist on Grassmann manifolds.

1. STATEMENT OF THE PROBLEM

The grassmannian $G$ of 2-dimensional planes in a 4-dimensional complex vector space $\mathbb{T}$ is a homogeneous space $G/H$, where $G = SL(4; \mathbb{C})$ and $H$ is the stabilizer of a point. Let $M = (\mathbb{R}^4, \| \cdot \|)$ be the Minkowski space, where $\| (x_0, x_1, x_2, x_3) \| = x_0^2 - x_1^2 - x_2^2 - x_3^2$. According to Penrose, consider the embedding

$\iota: M \to G$

$(x_0, x_1, x_2, x_3) \mapsto (x_0 + x_3, x_1 + ix_2, 1, 0), (x_1 - ix_2, x_0 - x_3, 0, 1)$

where $\langle v, w \rangle C$ denotes the vector space spanned by $v$ and $w$. The closure of $\iota(M)$ is an orbit of the totally real form $SU(2, 2) \subset SL(4; \mathbb{C})$, whose action corresponds to that of the Poincaré group on $M$. Thus $G$ is a conformal compactification of the complexified Minkowski space. In particular, the usual wave equation, as well as the other massless field equations, extend as differential operators acting between homogeneous bundles on $G$. Let us denote by $\mathcal{M}(m)$ the quasi-equivariant $D_G$-module corresponding to the massless field equation of helicity $m \in \mathbb{Z}/2$. As we will recall, the modules $\mathcal{M}(m)$ are "globally simple" along the characteristic variety $V$ of the wave equation.

Denote by $g = \mathfrak{s}l(4; \mathbb{C})$ the Lie algebra of $G = SL(4; \mathbb{C})$. By the Beilinson-Bernstein-Kashiwara correspondence, quasi-equivariant $D_G$-module are associated with $(g, H)$-modules with trivial twist. In order to deal with arbitrary twists, one has to consider quasi-equivariant modules over rings of twisted differential operators on $G$.

It thus arises a natural question: given a ring $\mathcal{A}$ of twisted differential operators on $G$, are there $\mathcal{A}$-modules globally simple along $V$?

This is a report on the talk given by the first named author at the meeting Recent Trends in Microlocal Analysis, RIMS, August 25–29, 2003, announcing results from a joint paper [4].
2. TWISTED SHEAVES AND DIFFERENTIAL OPERATORS

Let us briefly review the notions of twisted sheaves and twisted differential operators from [7, 1] (see also [2] for an exposition).

Let \( X \) be a complex manifold, \( \mathcal{C}_X \) the constant sheaf with stalk \( \mathbb{C} \) on \( X \), \( \mathcal{O}_X \) the structure sheaf on \( X \), and \( \mathcal{D}_X \) the ring of finite order differential operators on \( X \).

- A ring of twisted differential operators (a t.d.o. ring for short) is an \( \mathcal{O}_X \)-ring locally isomorphic to the ring \( \mathcal{D}_X \). They are classified by \( H^1(X; d\mathcal{O}_X) \), up to isomorphisms.

A basic example of t.d.o. ring is the ring

\[ \mathcal{D}_\mathcal{L} = \mathcal{L} \otimes_{\mathcal{O}} \mathcal{D}_X \otimes_{\mathcal{O}} \mathcal{L}^{-1} \]

of differential operators acting on a line bundle \( \mathcal{L} \). For \( \lambda \in \mathbb{C} \) one can also consider the t.d.o. ring \( \mathcal{D}_{\mathcal{L}^\lambda} \) defined as follows. If \( s \) is a nowhere vanishing local section of \( \mathcal{L} \), sections of \( \mathcal{D}_{\mathcal{L}^\lambda} \) are written as \( s^\lambda \otimes P \otimes s^{-\lambda} \), for \( P \in \mathcal{D}_X \). If \( t \) is another nowhere vanishing local section of \( \mathcal{L} \), then \( s^\lambda \otimes P \otimes s^{-\lambda} = t^\lambda \otimes Q \otimes t^{-\lambda} \) in \( \mathcal{D}_{\mathcal{L}^\lambda} \) if and only if \( Q = (s/t)^\lambda \cdot P \cdot (s/t)^{-\lambda} \) in \( \mathcal{D}_X \). This is independent from the choice of a branch for the ramified function \( (s/t)^\lambda \). It is also possible to give a meaning to \( \mathcal{L}^\lambda \) as a twisted sheaf, as follows.

Denote by \( \text{Mod}(\mathcal{C}_X) \) the category of sheaves of \( \mathbb{C} \)-vector spaces on \( X \), and by \( \text{MMod}(\mathcal{C}_X) \) the corresponding \( \mathbb{C} \)-stack, \( U \mapsto \text{Mod}(\mathcal{C}_U) \).

- A stack of twisted sheaves is a \( \mathbb{C} \)-stack \( \mathcal{G} \) on \( X \) locally \( \mathbb{C} \)-equivalent to the stack of sheaves \( \text{MMod}(\mathcal{C}_X) \). They are classified by \( H^2(X; \mathcal{C}_X^\times) \), up to \( \mathbb{C} \)-equivalences. Twisted sheaves are objects of \( \mathcal{G}(X) \).

For an open covering \( X = \bigcup_i U_i \), let \( c_{ijk} \in \mathcal{C}_X^\times(U_{ijk}) \) be a Čech cocycle for the characteristic class of \( \mathcal{G} \) in \( H^2(X; \mathcal{C}_X^\times) \). Twisted sheaves in \( \mathcal{G}(X) \) are described by a family of sheaves \( F_i \) on \( U_i \), and a family of isomorphisms \( \theta_{ij} : F_j|_{U_{ij}} \rightarrow F_i|_{U_{ij}} \), satisfying \( \theta_{ij} \circ \theta_{jk} = c_{ijk} \theta_{ik} \) on \( U_{ijk} \).

For \( B \) a sheaf of \( \mathbb{C} \)-algebras, let \( \text{Mod}(B; \mathcal{G}) \) be the category of \( B \)-modules in \( \mathcal{G} \).

- Twisted line bundles are object of \( \text{Mod}(\mathcal{O}_X; \mathcal{G}) \) locally isomorphic to \( \mathcal{O}_X \).

The twisted sheaf \( \mathcal{L}^\lambda \) is an example of a twisted line bundle. Its twist is described as follows. Let \( s_i \) be non vanishing sections of \( \mathcal{L} \) on \( U_i \). Then \( \mathcal{L}^\lambda \) belongs to a stack of twisted sheaves whose cocycle \( c_{ijk} \) describes the difference of determinations between the ramified functions \( (s_i/s_j)^\lambda(s_j/s_k)^\lambda \) and \( (s_i/s_k)^\lambda \).

To any t.d.o. ring \( \mathcal{A} \) one associates a stack of twisted sheaves \( \mathcal{G}_\mathcal{A} \) and a twisted line bundle \( \mathcal{O}_{\mathcal{A}} \in \text{Mod}(\mathcal{O}_X; \mathcal{G}_\mathcal{A}) \), such that

\[ \mathcal{A} \simeq \mathcal{O}_\mathcal{A} \otimes_{\mathcal{O}} \mathcal{D}_X \otimes_{\mathcal{O}} \mathcal{O}_\mathcal{A}^{-1}. \]
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The Riemann-Hilbert correspondence associates flat connections of rank 1 in $\text{Mod}(\mathcal{A})$ with local systems of rank 1 in $\mathcal{G}_A(X)$, by $\mathcal{M} \mapsto \text{Hom}_A(\mathcal{M}, \mathcal{O}_A)$. Recalling that $\mathcal{G}_A$ is globally $\mathbb{C}$-equivalent to the stack of sheaves $\text{Mod}((\mathbb{C}_X)$ if and only if there is a local systems of rank 1 in $\mathcal{G}_A(X)$, it follows

**Proposition 2.1.** Let $\mathcal{A}$ be a t.d.o. ring on $X$, and $\mathcal{M}$ a flat connection of rank 1 in $\text{Mod}(\mathcal{A})$. Then $\mathcal{G}_A$ is globally $\mathbb{C}$-equivalent to $\text{Mod}(\mathbb{C}_X)$.

We will make use of the exact sequence

$$H^1(X; \mathcal{O}_X^\times) \to H^1(X; d\mathcal{O}_X) \to H^2(X; \mathbb{C}_X^\times),$$

induced by the short exact sequence

$$1 \to \mathbb{C}_X^\times \to \mathcal{O}_X^\times \to d\mathcal{O}_X \to 0.$$

If $\mathcal{L}$ is a line bundle, and $\mathcal{A}$ a t.d.o. ring, one has $\gamma([\mathcal{L}]) = [D_{\mathcal{L}}]$, $\delta([\mathcal{A}]) = [\mathcal{G}_A]^{-1}$.

3. SYSTEMS WITH SIMPLE CHARACTERISTICS

Let us now recall some definitions and results on microdifferential operators, due to [11, 10]. See also [6, 8] for an exposition.

Let $X$ be a complex manifold, and $\pi: T^*X \to X$ its cotangent bundle. Denote by $\mathcal{E}_X$ the ring of microdifferential operators on $T^*X$, and by $F_m\mathcal{E}_X$ its subsheaf of microdifferential operators of order at most $m$.

Let $(x)$ be a system of local coordinates in $X$, and denote by $(x; \xi)$ the associated system of symplectic coordinates in $T^*X$. With this choice of coordinates, a microdifferential operator $P \in F_m\mathcal{E}$ is described by its total symbol $\{p_k(x; \xi)\}_{k \leq m}$, where $p_k \in \mathcal{O}_{T^*X}(k)$ is a function homogeneous of degree $k$. The principal symbol of order $m$, independent from the choice of coordinates, is given by

$$\sigma_m: F_m\mathcal{E} \to \mathcal{O}_{T^*X}(m)$$

$$P \mapsto p_m.$$

Denoting by $a: T^*X \to T^*X$ the antipodal map, the formal adjoint of $P$ is the operator $P^* \in a^{-1}F_m\mathcal{E}$ whose principal symbol $\{p^*_k(x; \xi)\}_{k \leq m}$ is given by

$$p^*_k(x; \xi) = \sum_{k=1-|\alpha|} \frac{(-1)^{|\alpha|}}{\alpha!} (\partial^\alpha_\xi \partial^\alpha_x p_l)(x; -\xi).$$

For $P \in F_m\mathcal{E}$, one has $P - (-1)^m P^* \in F_{m-1}\mathcal{E}$. The subprincipal symbol of $P$ in the coordinate system $(x)$ is defined by

$$\sigma'_{m-1}(P) = \frac{1}{2} \sigma_{m-1}(P - (-1)^m P^*)$$

$$= p_{m-1} - \frac{1}{2} \sum_i \partial_{y_i} \partial_{\xi_i} p_m.$$
Identifying $X$ with the zero-section of $T^*X$, we set $\dot{T}^*X = T^*X \setminus X$. In this paper, by submanifold of $T^*X$ we mean a smooth locally closed submanifold. A submanifold $V$ of $T^*X$ is conic if it is locally $\mathbb{C}^\infty$-invariant. The canonical 1-form induces a homogeneous symplectic structure on $T^*X$. A submanifold $V$ of $T^*X$ is involutive if for any pair $f, g$, of holomorphic functions vanishing on $V$, the Poisson bracket $\{f, g\}$ vanishes on $V$.

Let $V$ be a conic involutive submanifold of $\dot{T}^*X$. The ring $\mathcal{E}_V$ on $V$ is the subring of $\mathcal{E}_X$ generated by

$$\mathcal{J}_V = \{ P \in F_1 \mathcal{E}_X : \sigma_1(P)|_V = 0 \}.$$ 

The $V$-filtration is defined by $F_k^V \mathcal{E}_X = F_k \mathcal{E}_X \cdot \mathcal{E}_V = \mathcal{E}_V \cdot F_k \mathcal{E}_X$.

If $\mathcal{M}$ is an $\mathcal{E}_X$-module and $\mathcal{M}_0$ is an $F_0 \mathcal{E}_X$-submodule, we set $\mathcal{M}_k = (F_k \mathcal{E}_X) \mathcal{M}_0$.

**Definition 3.1.** One says that a coherent $\mathcal{E}_X$-module $\mathcal{M}$ is globally simple along $V$ if it admits a coherent sub-$F_0 \mathcal{E}_X$-module $\mathcal{M}_0$ such that $\mathcal{M} = \mathcal{E}_X \mathcal{M}_0$, $\mathcal{E}_V \mathcal{M}_0 \subset \mathcal{M}_0$, and $\mathcal{M}_0/\mathcal{M}_{-1}$ is a locally isomorphic to $\mathcal{O}_V(0)$. Such an $\mathcal{M}_0$ is called a V-lattice.

To $P \in \mathcal{J}_V$ one associates the operator $\mathcal{L}_V(P) \in F_1 D_{\Omega_V^{1/2}}$ defined as follows. For $\omega \in \Omega_V$ one sets

$$\mathcal{L}_V(P)(\omega^{1/2}/\sqrt{dx}) = \left( L_{H_{\sigma_1(P)}(\omega^{1/2})} + \sigma_0(P) \cdot \omega^{1/2} \right) / \sqrt{dx},$$

where $dx$ is the volume form associated with a chosen local coordinate system. The Hamiltonian vector field $H_{\sigma_1(P)}$ is tangent to $V$ since $\sigma_1(P)|_V = 0$. Recall that for $v \in \Theta_V$, one sets $L_v(\omega^{1/2}) = \frac{1}{2} \frac{L_v(\omega)}{\omega} \omega^{1/2}$. The operator $\mathcal{L}_V(\cdot)$ does not depend on the choice of coordinates, and extends as a ring morphism

$$\mathcal{L}_V : \mathcal{E}_V \rightarrow D_{\Omega_V^{1/2}}.$$  

(3.1)

Denote by $D_V(0)$ the subring of $D_V$ consisting of differential operators which commute with the Euler vector field on $V$, by $D_V^{bc}$ the subring of $D_V$ consisting of differential operators which commute with the functions $a \in \mathcal{O}_V$ constant along the bicharacteristic leaves of $V$, and set $D_V^{bc}(0) = D_V^{bc} \cap D_V(0)$.

The following result is essentially due to [10, 9].

**Theorem 3.2.** The ring morphism (3.1) induces a ring isomorphism

$$\mathcal{L}_V : \mathcal{E}_V/F_{-1} \mathcal{E}_V \rightarrow D_{\Omega_V^{1/2}}^{bc}(0).$$

(3.2)
4. Statement of the main result

Let $A$ be a t.d.o. ring on $X$, and $V$ a conic involutive submanifold of $T^*X$. All definitions and constructions of §3 extend to the twisted case. In particular, setting

$$
\mathcal{E}_A = \pi^{-1}\mathcal{O}_A \otimes_{x^{-1}\mathcal{O}} \mathcal{E}_X \otimes_{x^{-1}\mathcal{O}} \pi^{-1}\mathcal{O}_A^{-1},
$$

$$
\mathcal{E}_{V,A} = \pi^{-1}\mathcal{O}_A \otimes_{x^{-1}\mathcal{O}} \mathcal{E}_V \otimes_{x^{-1}\mathcal{O}} \pi^{-1}\mathcal{O}_A^{-1},
$$

Theorem 3.2 gives an isomorphism

\begin{equation}
L_V: \mathcal{E}_{V,A}/F_{-1}\mathcal{E}_A \cong D^{bic}_{\Omega^2_{V/X} \otimes_{\mathcal{O}} \pi^*\mathcal{O}_A}(0),
\end{equation}

where $\pi_V$ is the restriction to $V$ of $\pi: T^*X \rightarrow X$.

Let $j_{\Sigma}: \Sigma \rightarrow V$ be the embedding of a smooth bicharacteristic leaf of $V$. For $\mathcal{G}$ a twisted line bundle on $V$, we consider the restriction functor

$$
j_{\Sigma}^*: \text{Mod}(D^{bic}_{\mathcal{G}}(0)) \rightarrow \text{Mod}(D^{bic}_{j_{\Sigma}\mathcal{G}}(0)),
$$

and the pull-back

\begin{equation}
\pi_{\Sigma}^*: H^2(X; \mathbb{C}_X) \rightarrow H^2(\Sigma; \mathbb{C}_\Sigma).
\end{equation}

Recall the maps $\gamma$ and $\delta$ in (2.1).

**Theorem 4.1.** Let $V$ be a conic involutive submanifold of $T^*X$, and let $\Sigma$ be a smooth bicharacteristic leaf of $V$. Let $A$ be a t.d.o. ring on $V$, and let $\mathcal{M}$ be a globally simple $\mathcal{E}_A$-module along $V$. Then

$$
\delta(\pi_{\Sigma}^*[A]) = \delta(-\frac{1}{2} \cdot \gamma([\Omega_{\Sigma/X}])) \quad \text{in } H^2(\Sigma; \mathbb{C}_{\Sigma}).
$$

**Sketch of proof.** The proof follows the same lines as in [9, §I.5.2]. Let $\mathcal{M}_0$ be a $V$-lattice in $\mathcal{M}$. By (4.1), $\mathcal{M}_0/\mathcal{M}_{-1}$ is locally isomorphic to $\mathcal{O}_V(0)$ as $D^{bic}_{\Omega^2_{V/X} \otimes_{\mathcal{O}} \pi^*\mathcal{O}_A}(0)$-modules. Note that $j_{\Sigma}^*\Omega_V \simeq \Omega_{\Sigma}$. Then $j_{\Sigma}^*(\mathcal{M}_0/\mathcal{M}_{-1})$ is a $D^{bic}_{\Omega^2_{V/X} \otimes_{\mathcal{O}} \pi^*\mathcal{O}_A}(0)$-module which is locally isomorphic to $\mathcal{O}_{\Sigma}(0)$. The statement follows by Proposition 2.1. $\square$

5. Non existence of twisted wave equations

Let $T \simeq \mathbb{C}^4$ be a complex vector space, $\mathbb{P}$ the projective space of lines in $T$, and $G$ the Grassmannian of 2-dimensional subspaces in $T$. The Penrose correspondence (see [5]) is associated with the double fibration

\begin{equation}
P \leftarrow F \rightarrow G
\end{equation}

where $F = \{ (y, x) \in \mathbb{P} \times G; y \subset x \}$ is the incidence relation, and $f$, $g$ are the natural projections. The double fibration (5.1) induces the maps

$$
\hat{T}^*\mathbb{P} \leftarrow \hat{T}^*F(\mathbb{P} \times G) \rightarrow \hat{T}^*G
$$
where $T^{*}_{\mathbb{P}}(\mathbb{P} \times \mathcal{G}) \subset T^{*}(\mathbb{P} \times \mathcal{G})$ denotes the conormal bundle to $\mathbb{P}$, and $p$ and $q$ are the natural projections. Note that $p$ is smooth surjective, and $q$ is a closed embedding. Set

$$V = q(T^{*}_{\mathbb{P}}(\mathbb{P} \times \mathcal{G})).$$

Then $V$ is a closed conic regular involutive submanifold of $T^{*}\mathcal{G}$, and $q$ identifies the fibers of $p$ with the bicharacteristic leaves of $V$.

For $m \in \mathbb{Z}$ consider the line bundles $\mathcal{O}_{\mathbb{P}}(m)$, and set

$$\mathcal{M}_{(1+m/2)} = H^{0}(\mathbb{D}g_{*}\mathbb{D}f^{*}(\mathcal{D}_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}}(-m))),$$

where $\mathbb{D}g_{*}$ and $\mathbb{D}f^{*}$ denote the direct and inverse image in the derived categories of $\mathcal{D}$-module. As we recalled in the Introduction, according to Penrose $\mathcal{G}$ is a conformal compactification of the complexified Minkowski space, and the $\mathcal{D}_{\mathcal{G}}$-module $\mathcal{M}_{(1+m/2)}$ corresponds to the massless field equation of helicity $1 + m/2$.

By [3], $\mathcal{M}_{(1+m/2)}$ is globally simple along $V$.

**Theorem 5.1.** Let $\mathcal{A}$ be a t.d.o. ring on $\mathcal{G}$, and $\mathcal{M}$ an $\mathcal{A}$-module globally simple along $V$. Then $\delta[\mathcal{A}] = 1$ in $H^{2}(\mathcal{G}; \mathbb{C}_{\mathcal{G}}^{\mathcal{G}})$. In particular, $\text{Mod}(\mathcal{A})$ and $\text{Mod}(\mathcal{D}_{\mathcal{G}})$ are $\mathbb{C}$-equivalent.

**Proof.** By Theorem 4.1 it is enough to show that for a bicharacteristic leaf $\Sigma \subset V$, one has an isomorphism $\tau_{\Sigma}^{\sharp} : H^{1}(\mathcal{G}; d\mathcal{O}_{\mathcal{G}}) \rightarrow H^{1}(\Sigma; d\mathcal{O}_{\Sigma})$, and moreover $\delta(-\gamma((\Omega_{\mathcal{G}}/\Omega_{\Sigma})/2) = 1 \in H^{2}(\Sigma; \mathbb{C}_{\Sigma}^{\Sigma})$.

With the identification

$$T^{*}\mathcal{G} = \{(x; \xi); x \subset T, \xi \in \text{Hom}(T/x, x)\},$$

one has

$$V = \{(x; \xi); \text{rk}(\xi) = 1\},$$

where $\text{rk}(\xi)$ denotes the rank of the linear map $\xi$. There is an isomorphism

$$T^{*}\mathbb{P} = \{(y, z; \theta); y \subset z \subset T, \theta: T/z \rightarrow y\},$$

where $y$ is a line and $z$ is a hyperplane. The projection $q: V \rightarrow T^{*}\mathbb{P}$ is given by $q(x; \xi) = (\text{im} \xi, x + \ker \xi; \tilde{\xi})$, where $\tilde{\xi}$ satisfies $\xi \circ \ell = \xi$ for $\ell: T/x \rightarrow T/x + \ker \xi$ the natural map.

Recall that the bicharacteristic leaves of $V$ are the fibers of $q$. For $(y, z, \theta) \in T^{*}\mathbb{P}$, the bicharacteristic leaf $q^{-1}(y, z, \theta)$ of $V$ is given by

$$\Sigma = \{(x; \xi); y \subset x \subset z, \xi = \theta \circ \ell\},$$

where $\ell: T/x \rightarrow T/z$ is the natural map. Thus, $\Sigma$ is the projective space of lines in $z/y$. Hence the sequence

$$H^{1}(\Sigma; \mathcal{O}_{\Sigma}^{\Sigma}) \rightarrow H^{1}(\Sigma; d\mathcal{O}_{\Sigma}) \rightarrow H^{2}(\Sigma; \mathbb{C}_{\Sigma}^{\Sigma})$$

is isomorphic to the sequence of additive abelian groups

$$\mathbb{Z} \rightarrow \mathbb{C} \rightarrow \mathbb{C}/\mathbb{Z}.$$
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with $1 \in \mathbb{Z}$ corresponding to $[\mathcal{O}_{\Sigma}(-1)]$.

Denote by $\mathcal{O}_{G}(-1)$ the determinant of the tautological bundle on $G$. Then $H^1(G; d\mathcal{O}_G) \cong \mathbb{C}$ with generator $D_{\mathcal{O}_G(-1)}$. Since $\pi_{\Sigma}^* \mathcal{O}_G(-1) \cong \mathcal{O}_{\Sigma}(-1)$, it follows that

\[\pi_{\Sigma}^* : H^1(G; d\mathcal{O}_G) \sim H^1(\Sigma; d\mathcal{O}_{\Sigma}).\]

There are isomorphisms $\Omega_G \simeq \mathcal{O}_G(-4)$, and $\Omega_{\Sigma} \simeq \mathcal{O}_{\Sigma}(-2)$, so that $\pi_{\Sigma}^* \Omega_G \simeq \pi_{\Sigma}^* \mathcal{O}_G(-4) \simeq \mathcal{O}_{\Sigma}(-4)$, and $\Omega_{\Sigma/G} \simeq \mathcal{O}_{\Sigma}(2)$. It follows that

\[[\Omega_{\Sigma/G}] = 2 \quad \text{in} \quad \mathbb{Z} \simeq H^1(\Sigma; \mathcal{O}_{\Sigma}^2),\]

and therefore

\[\delta(-\gamma([\Omega_{\Sigma/G}])/2) = 0 \quad \text{in} \quad \mathbb{C}/\mathbb{Z} \simeq H^2(\Sigma; \mathbb{C}_{\Sigma}^\times).\]

By Theorem 4.1, it follows that $\delta(\pi_{\Sigma}^*[A]) = 1$. Hence $\delta([A]) = 1$, so that $\mathcal{G}_A$ is globally $C$-equivalent to $\mathcal{M}(G)$. In particular, $\mathcal{O}_A$ in an untwisted sheaf. The equivalence

\[
\text{Mod}(\mathcal{D}_G) \sim \rightarrow \text{Mod}(\mathcal{A})
\]

is given by $\mathcal{M} \mapsto \mathcal{O}_A \otimes_\mathcal{O} \mathcal{M}$. \qed

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