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Singular Cauchy problems for nonlinear equations

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Abstract

We consider singular Cauchy problems for quasilinear equations of second order, and show that the solution is holomorphic outside of two characteristic hypersurfaces. The characteristic hypersurfaces themselves may have singularities, and the solution is described in terms of monoidal transformation.

1 Introduction

In this article we consider singular Cauchy problems for quasilinear equations of second order. We investigate a particular phenomenon of quasilinear problems.

Let $x = (x_1, x') = (x_1, x_2, x'') \in X = \mathbf{C}^n$, and $D = \partial/\partial x$. We define $Z = \{x_1 = x_2 = 0\} \subset Y = \{x_1 = 0\} \subset X$. We consider a quasilinear operator

$$Fu = \sum_{|\alpha|=2} F_\alpha(x, u, \nabla u) D^\alpha u + f(x, u, \nabla u)$$

in a neighborhood $\omega \subset \mathbf{C}^n$ of the origin. We consider the following Cauchy problem:

$$(1) \qquad Fu = 0, \quad u(0, x') = u_0(x'), \quad D_1 u(0, x') = u_1(x').$$

Let $\omega_Y = \omega \cap Y$, $\omega_Z = \omega \cap Z$. We assume that the initial values are holomorphic on the universal covering space $\mathcal{R}(\omega_Y \setminus Z)$ of $\omega_Y \setminus Z$. We assume the following conditions.

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First, Y is a noncharacteristic hypersurface, and F_α do not depend on ∇u:

$$(2) \qquad \begin{cases} F_{(2,0,\cdots,0)} = 1, \\ F_\alpha = F_\alpha(x,u) \in \mathcal{O}_{\mathbf{C}^{n+1},(0,u^\circ)}, \\ f(x,u,p) \in \mathcal{O}_{\mathbf{C}^{2n+1},(0,u^\circ,p^\circ)}. \end{cases}$$

Here $\mathcal{O}$ denotes the sheaf of holomorphic functions, and $u^\circ \in \mathbf{C}$, $p^\circ \in \mathbf{C}^n$ are fixed points which we shall naturally determine from the initial values later. Let us define the principal symbol $\sigma_2(F)(x,u;\xi)$ by

$$\sigma_2(F)(x,u;\xi) = \sum_{|\alpha|=2} F_\alpha(x,u)\xi^\alpha.$$

We next assume that the characteristic roots are polynomials of ξ:

$$(3) \qquad \begin{cases} \sigma_2(F)(x,u;\xi) = \lambda_1(x,u;\xi)\lambda_2(x,u;\xi), \\ \lambda_i(x,u;\xi) = \sum\limits_{1\le j\le n} \lambda_{ij}(x,u)\xi_j \in \mathcal{O}_{\mathbf{C}^{2n+1},(0,u^\circ,0)}, \\ \lambda_{i1} = 1. \end{cases}$$

We also assume that the characteristic roots are separate in the direction $\xi = (0,1,0,\cdots,0)$:

$$(4) \qquad \lambda_{12}(0,u^\circ) \neq \lambda_{22}(0,u^\circ).$$

Finally we assume that the initial values satisfy

$$(5) \qquad |D^{\alpha'}u_j(x')| \le \exists M, \quad \text{if } x' \in \omega_Y \setminus Z,\ j+|\alpha'| \le 2.$$

Under these assumptions, we want to solve (1), and study the propagation of the singularities. Roughly speaking, we can solve (1) outside of two characteristic hypersurfaces, but we must take some difficulties into account.

Remark. Let $j+|\alpha'| \le 1$. We have

$$D^{\alpha'}u_j(x') = \int_\varepsilon^{x_2} D^{\alpha'}D_2u_j(\tau,x'')d\tau + D^{\alpha'}u_j(\varepsilon,x'')$$

for small $\varepsilon > 0$. Here we can let $x_2 \to 0$, and we can define

$$D^{\alpha'}u_j(0,x'') = \lim_{x_2\to 0} D^{\alpha'}u_j(x') \in \mathcal{O}(\omega_Z).$$

Then we have $|D^{\alpha'}u_j(x') - D^{\alpha'}u_j(0,x'')| \leq \exists M|x_2|$. We define $u^\circ = u(0)$, $p^\circ = \nabla u(0)$ in this sense.

Let us briefly review the results for linear problems and semilinear problems. If Fu is linear, there are many papers studying this problem. We only refer to [1, 6], where the reader can find further references. In this case the characteristic roots λ_1, λ_2 are independent of u, and we can define the characteristic functions $\varphi_1(x), \varphi_2(x)$ by

$$\sum_{1\leq j\leq n} \lambda_{ij}(x)D_j\varphi_i(x) = 0, \quad \varphi_i(0,x') = x_2,$$

and the characteristic hypersufaces Z_1, Z_2 through Z by $Z_i = \{x \in \omega;\ \varphi_i(x) = 0\}$. Under the above assumptions (2)–(5) applied for a linear operator, it is known that there exists a unique solution $u(x) \in \mathcal{O}(\mathcal{R}(\omega \setminus Z_1)) + \mathcal{O}(\mathcal{R}(\omega \setminus Z_2))$, shrinking ω if necessary.

Semilinear problems were studied by E. Leichtnam [3]. In this case the principal part is linear, and we can define Z_1, Z_2 in the same way as linear equations. We need to solve the problem in a function space which is closed under nonlinear calculation. Since the above function space $\mathcal{O}(\mathcal{R}(\omega \setminus Z_1)) + \mathcal{O}(\mathcal{R}(\omega \setminus Z_2))$ does not enjoy this property, we consider $\mathcal{O}(\mathcal{R}(\omega \setminus Z_1 \setminus Z_2))$ instead. Leichtnam proved that there exists a unique solution $u(x) \in \mathcal{O}(\mathcal{R}(\omega \setminus Z_1 \setminus Z_2))$.

Let us consider quasilinear problems. This case contains essential differences from the above cases, and we need a completely new method. Let us point out two differences between quasilinear problems and (semi)linear problems.

The first difference is as follows. In the above cases,

(a) We first determine the characteristic hypersurfaces $\underline{Z_1, Z_2}$ from the principal symbol as above.

(b) We next find a solution outside of $\underline{Z_1, Z_2}$.

However, in quasilinear problems we have

(c) The principal symbol depends on the solution,

in addition. Therefore (a)–(c) constitute a circular reasoning, and we cannot determine none of Z_1, Z_2, u in this way.

Secondly, in quasilinear problems the characteristic functions φ_i themselves may be singular. Of course they are holomorphic functions in (semi)linear problems. We shall later give examples which explain this.

To overcome these two difficulties, we use monoidal transformation ${}_Z\widetilde{X}$ of X with center Z. Monoidal transformation was introduced in [2] in order to consider a linear equation $Fu = g(x)$ for $g(x) \in \mathcal{O}(\mathcal{R}(\omega \setminus Z_1 \setminus Z_2))$, and the generalization to higher order equations. It was used to simplify the geometry of $\omega \setminus Z_1 \setminus Z_2$ in [2]. We shall employ this method to resolve the above situation.

Remark. A. Nabaji and C. Wagschal [4, 5] considered similar problems for quasilinear equations. However, he did not consider general Cauchy problems. In general the solution should be singular along two hypersurfaces Z_1, Z_2. However, sometimes the solution may be singular along one of Z_1, Z_2 alone, which was studied in [4, 5]. In such a case the two difficulties mentioned above do not appear. For example, if the solution is singular along both Z_1 and Z_2, then these two singularities cause interference one another, and the characteristic functions φ_1, φ_2 may become singular. This phenomenon is our main interest, which was not treated by [4, 5].

2 Main result

In order to introduce monoidal transformation, we prepare some notations. We define the linearized characteristic roots λ_i° by $\lambda_i^\circ(\xi_1, \xi_2) = \xi_1 + \lambda_{i2}(0, u^\circ)\xi_2$, and the linearized characteristic functions y_1, y_2 by $y_i = x_2 - \lambda_i^\circ x_1$. Let $y = (y_1, y_2, y'') = (y_1, y_2, x'')$ and $\omega_i = \{x \in \omega;\ |y_i| > \varepsilon|(y_1, y_2)|\}$ for a small $\varepsilon > 0$. From (4) we have $dy_1 \wedge dy_2 \neq 0$ and $Z = \{y_1 = y_2 = 0\}$.

Remark. We explain our basic strategy. Roughly speaking, λ_i° is an approximation of λ_i, and y_i is a characteristic function corresponding to λ_i°. As we have seen, we cannot determine the characteristic function φ_i in the usual way. But we can expect that y_i is an approximation of φ_i, as long as λ_i° is not widely different from λ_i. Therefore we can also expect that the characteristic hypersurface Z_i (which we cannot determine immediately) should be contained in $Z_i' = \omega \setminus \omega_i$, whatever the solution may be. In other words, we cannot determine Z_i themselves, but we can determine their conoidal neighborhoods Z_i' in the sense of monoidal transformation.

Let us see how we can solve (1) by means of this notion. Omitting the universal covering, we want to solve (1) on $\omega \setminus Z_1 \setminus Z_2$. However, if $Z_i \subset Z_i'$ is true, we have

$$\omega \setminus Z_1 \setminus Z_2 = (\omega \setminus Z_1 \setminus Z_2') \cup (\omega \setminus Z_1' \setminus Z_2)$$

(See Figure 1).

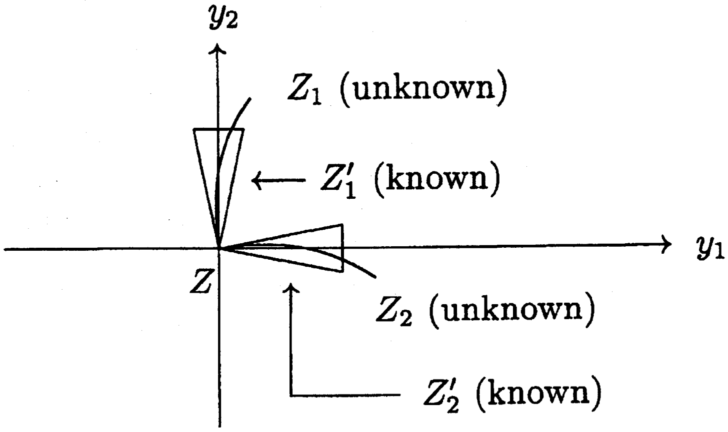

Figure 1: monoidal transformation

Therefore we may solve (1) on $X_1 = \omega \setminus Z_1 \setminus Z_2' = \omega_2 \setminus Z_1$ and $X_2 = \omega \setminus Z_1' \setminus Z_2 = \omega_1 \setminus Z_2$ individually. In fact if $u^i(x) \in \mathcal{O}(X_i)$ satisfy (1), we have $u^1 = u^2$ on $X_1 \cap X_2$, due to the uniqueness of the Cauchy problem on $X_1 \cap X_2$. Therefore we need to solve (1) on $X_1 \subset \omega_2$ for example, which we shall realize.

Now let us give the precise statement of the main result. Let $\pi_2 : \mathcal{R}(\omega_2) \longrightarrow \omega_2$ be the natural projection. Let $\tilde{y} \in \mathcal{R}(\omega_2)$ and $\pi_2(\tilde{y}) = y \in \omega_2$. Since we may regard $\mathcal{R}(\omega_2) \subset \mathbf{C} \times \mathcal{R}(\mathbf{C} \setminus \{0\}) \times \mathbf{C}^{n-2}$, we may write $\tilde{y} = (\tilde{y}_1, \cdots \tilde{y}_n) \in \mathbf{C} \times \mathcal{R}(\mathbf{C} \setminus \{0\}) \times \mathbf{C}^{n-2}$, therefore $\tilde{y}_j = y_j \in \mathbf{C}$ if $j \neq 2$. We define $|\tilde{y}_j| = |y_j|$ for every j. Let $\tilde{y}^\circ \in \mathcal{R}(\omega_2)$ be a point such that $\pi_2(\tilde{y}^\circ) = (0, r, 0, \cdots, 0) \in \omega_2$ with small $r > 0$. We define $\arg \tilde{y}_2$ for $\tilde{y} \in \mathcal{R}(\omega_2)$ in such a way that $\arg \tilde{y}_2^\circ = 0$ and

$y_2 = |\widetilde{y}_2| \exp(\sqrt{-1} \arg \widetilde{y}_2)$ for $y = \pi_2(\widetilde{y})$. Since $\widetilde{y}$ is determined by $y = \pi_2(\widetilde{y})$ and $\theta = \arg \widetilde{y}_2$, we denote $\widetilde{y}$ also by y^θ and sometimes simply by y, if confusion is not likely. Using the x variables, we denote the same point by x^θ or simply by x.

In the following theorem, we shall define a characteristic function $\varphi_1 = y_1 - \varphi_1'(y') \in \mathcal{O}(\mathcal{R}(\omega_2))$ corresponding to λ_1. Let

$$Z_1 = \{\widetilde{x} \in \mathcal{R}(\omega_2);\ y_1 = \varphi_1'(y')\}$$

and

$$\omega_2' = \mathcal{R}(\omega_2) \setminus Z_1.$$

Let $\pi_2' : \mathcal{R}(\omega_2') \longrightarrow \omega_2$ be the composition of the natural projections from $\mathcal{R}(\omega_2')$ to $\omega_2' \subset \mathcal{R}(\omega_2)$ and from $\mathcal{R}(\omega_2)$ to ω_2. If $\widetilde{y} \in \mathcal{R}(\omega_2')$, then we can define $\theta_1 = \arg(y_1 - \varphi_1'(y'))$ and $\theta_2 = \arg y_2$ as before. In this case $\widetilde{y} \in \mathcal{R}(\omega_2')$ is determined by $y = \pi_2'(\widetilde{y})$ and θ_1, θ_2, therefore we denote $\widetilde{y}$ also by y^{θ_1,θ_2} or by y. Using the x variables, we denote the same point by x^{θ_1,θ_2} or by x. Then we have the following

Theorem 1. *We assume* (2)–(5). *There exists a holomorphic function* $\varphi_1'(y')$ *on* $\mathcal{R}(\omega_2)$, *and for any small* $\varepsilon > 0$ *there exists a unique solution* $u(x)$ *of* (1) *on*

$$\omega_2'' = \{x^{\theta_1,\theta_2} \in \mathcal{R}(\omega_2');\ |\theta_1 - \theta_2| < 1/\varepsilon\},$$

shrinking ω *if necessary. Here* ω_2' *is defined as above. Furthermore,* Z_1 *is a characteristic hypersurface through* Z *corresponding to* λ_1 *(and to the present solution* u*), in the following sense: We can define* $u(x)|_{Z_1} = \lim_{y_1 \to \varphi_1'(y')} u(x)$ *and* $\varphi_1'(0, y'') = \lim_{y_2 \to 0} \varphi_1'(y')$, *and we have*

$$\{\lambda_1(x, u(x), D_x)(y_1 - \varphi_1'(y'))\}|_{Z_1} = 0,$$
$$\varphi_1'(0, y'') = 0.$$

3 Examples

Here we give two examples. The first one does not satisfy assumption (2), but in this example we can evidently see what happens in quasilinear problems. The second one satisfies all the assumptions.

Example 1. Let $n = 2$ and consider

$$(6)\qquad \begin{cases} Fu = D_1^2 u - \dfrac{D_1 u}{D_2 u} D_1 D_2 u = 0, \\ u(0, x_2) = x_2 + c x_2^q + (x_2 + c x_2^q)^q, \\ D_1 u(0, x_2) = 1 + q(x_2 + c x_2^q)^{q-1}, \end{cases}$$

for $2 < q \in \mathbf{R} \setminus \mathbf{Z}$. Here $F_{(1,1)}(x, u, p) = p_1/p_2$, which does not satisfy assumption (2). Note that the initial values belong to $\mathcal{O}(\mathcal{R}(\omega_Y \setminus Z))$ for small ω, because we have

$$\begin{aligned} (x_2 + c x_2^q)^q = &\ x_2^q (1 + c x_2^{q-1})^q \\ = &\ x_2^q \Big(1 + c q x_2^{q-1} + \frac{c^2 q(q-1)}{2} x_2^{2(q-1)} + \cdots \Big) \\ \in &\ \mathcal{O}(\mathcal{R}(\omega_Y \setminus Z)), \end{aligned}$$

for example. Let $h(x) = x_1 + x_2 + c x_2^q$. We can directly see that $u = h + h^q$ is a unique solution of (6).

Let us rewrite this in terms of our general result. The characteristic roots are

$$\lambda_1 = \xi_1 - p_1 \xi_2 / p_2, \quad \lambda_2 = \xi_1.$$

Here we modify the definition of λ_i° by $\lambda_i^\circ = \lambda_i |_{x=0, u=u^\circ, p=p^\circ}$, and from the initial values in (6) we have $u^\circ = 0, p^\circ = (1, 1)$. It follows that

$$\lambda_1^\circ = \xi_1 - \xi_2, \ \lambda_2^\circ = \xi_1, \ y_1 = x_2 + x_1, \ y_2 = x_2.$$

Therefore we have

$$\begin{aligned} \omega_1 &= \{x \in \omega;\ |x_2 + x_1| > \varepsilon |(x_2 + x_1, x_2)|\}, \\ \omega_2 &= \{x \in \omega;\ |x_2| > \varepsilon |(x_2 + x_1, x_2)|\}. \end{aligned}$$

Let us consider the above solution $u = h(x) + h(x)^q$ in $\mathcal{R}(\omega_2)$. In this domain x_2^q is a holomorphic function and we do not discuss of such a singularity. This solution u has a singularity along the hypersurface $Z_1 = \{x \in \mathcal{R}(\omega_2);\ x_1 + x_2 = \varphi_1'(x_2)\}$, where $\varphi_1' = -c x_2^q \in \mathcal{O}(\mathcal{R}(\omega_2))$. It follows that $u \in \mathcal{O}(\mathcal{R}(\mathcal{R}(\omega_2) \setminus Z_1))$.

We next consider in $\mathcal{R}(\omega_1)$. We have

$$\begin{aligned} &(x_1 + x_2 + c x_2^q)^q \\ = &\ (x_1 + x_2)^q \Big(1 + \frac{c x_2^q}{x_1 + x_2}\Big)^q \\ = &\ (x_1 + x_2)^q \Big\{1 + q \frac{c x_2^q}{x_1 + x_2} + \frac{q(q-1)}{2} \Big(\frac{c x_2^q}{x_1 + x_2}\Big)^2 + \cdots \Big\} \end{aligned}$$

which is convergent in $\mathcal{R}(\omega_1)$. Here $(x_1 + x_2)^q$ and $1/(x_1 + x_2)$ are holomorphic, and the solution has a singularity along the hypersurface

$$Z_2 = \{x \in \mathcal{R}(\omega_1);\ x_2 = \varphi_2'(x_1 + x_2)\},$$

where $\varphi_2' = 0$. This means $u \in \mathcal{O}(\mathcal{R}(\mathcal{R}(\omega_1) \setminus Z_2))$. In this way we can discuss the two singularities along Z_1, Z_2 separately by monoidal transformation.

Example 2. Let $n = 2$ and consider

$$(7) \qquad \begin{cases} Fu = D_1^2 u - \dfrac{1}{1+u} D_1 D_2 u + \dfrac{1}{1+u}(D_1 u)^2 = 0, \\ u(0, x_2) = 0, \\ D_1 u(0, x_2) = x_2^q, \end{cases}$$

for $1 < q \in \mathbf{R} \setminus \mathbf{Z}$. We have $u^\circ = 0$ and

$$\begin{array}{ll} \lambda_1 = \xi_1 + \xi_2/(1+u), & \lambda_2 = \xi_1, \\ \lambda_1^\circ = \xi_1 + \xi_2, & \lambda_2^\circ = \xi_1, \\ y_1 = x_2 - x_1, & y_2 = x_2. \end{array}$$

It follows that

$$\begin{aligned} \omega_1 &= \{x \in \omega;\ |x_2 - x_1| > \varepsilon |(x_2 - x_1, x_2)|\}, \\ \omega_2 &= \{x \in \omega;\ |x_2| > \varepsilon |(x_2 - x_1, x_2)|\}. \end{aligned}$$

In this case we cannot immediately obtain the solution, but after some calculation we can prove the following results. We define

$$\begin{aligned} Z_1 = &\ \{x \in \mathcal{R}(\omega_2);\ x_2 - x_1 = \varphi_1'(x_2)\}, \\ Z_2 = &\ \{x \in \mathcal{R}(\omega_1);\ x_2 = \varphi_2'(x_2 - x_1)\} \end{aligned}$$

where $\varphi_1' = x_2^{q+2}/(q+1)(q+2)$, $\varphi_2' = 0$. There exists $h(x) \in \mathcal{O}(\mathcal{R}(\mathcal{R}(\omega_2) \setminus Z_1))$ satisfying $|h| \leq 1/2$ such that we have a solution of (7) of the form

$$u(x) = \frac{(1 + h(x))(x_2 - x_1 - \varphi_1'(x_2))^{q+1} - x_2^{q+1}}{q+1}.$$

Let us consider in ω_1. As before, $(x_2 - x_1 - \varphi_1')^{q+1}$ is singular, x_2^{q+1} is regular in ω_1, and h is singular but small. Therefore u has a singularity along $Z_1 \subset \mathcal{R}(\omega_2)$, mainly caused by $(x_2 - x_1 - \varphi_1')^{q+1}$. Therefore we have $u(x) \in \mathcal{O}(\mathcal{R}(\mathcal{R}(\omega_2) \setminus Z_1))$. Similarly we can prove $u(x) \in \mathcal{O}(\mathcal{R}(\mathcal{R}(\omega_1) \setminus Z_2))$.

4 Sketch of the proof

By a holomorphic transformation around the origin, we may assume that $\lambda_{1j}(x, u_0(0, x'')) = 0$ for $2 \leq j \leq n$.

Let us determine $Z_1 \subset \mathcal{R}(\omega_2)$ and solve (1) in $\omega_2'' \subset \mathcal{R}(\mathcal{R}(\omega_2) \setminus Z_1)$. The principal part is divided into two characteristic components $\lambda_1(x, u, D_x)$ and $\lambda_2(x, u, D_x)$. Roughly speaking, $\lambda_i(x, u, D_x)$ corresponds to the characteristic hypersurface Z_i, but we have deleted a neighborhood Z_2' of Z_2 and we consider (1) in $\mathcal{R}(\omega_2) = \mathcal{R}(\omega \setminus Z_2')$. On the initial hypersurface Y the singularity existed on Z, therefore $\lambda_2(x, u, D_x)$ does not make such singularity propagate toward any directions in ω_2. In this sense, $\lambda_2(x, u, D_x)$ has nothing to do with the singularity propagation in this domain, and we can expect that the propagation is caused by $\lambda_1(x, u, D_x)$ alone, from $Z \subset Y$ into some direction Z_1. Applying the Hamilton-Jacobi method of first order equations to $\lambda_1(x, u, D_x)$, we can simultaneously determine the characteristic hypersurface Z_1 and the solution u.

Let $x = x(t)$ and $u = u(t)$ be expressed by complex parameters $t = (t_1, \cdots, t_n)$. We require that they satisfy the characteristic system defined by $\lambda_1(x, u, D_x) = \sum_{1 \leq j \leq n} \lambda_{1j}(x, u) D_{x_j}$:

$$(8) \qquad \begin{cases} D_{t_1} x_j(t) = & \lambda_{1j}(x(t), u(t)), \quad 1 \leq j \leq n, \\ x_1(0, t') = & 0, \\ x_j(0, t') = & t_j, \qquad\qquad 2 \leq j \leq n. \end{cases}$$

Since $\lambda_{11} = 1$, we have $x_1 = t_1$. We also need to rewrite Fu in t variables. Let $C_{ij}(t) = \partial x_i / \partial t_j - \delta_{ij}$ and $C(t) = (C_{ij}(t); 1 \leq i, j \leq n)$. We also require

$$(9) \qquad |C_{ij}| < 1/2n.$$

Then we have $\partial x / \partial t = I_n + C$. It follows that $\partial t / \partial x = (\partial x / \partial t)^{-1} = \sum_{0 \leq k < \infty} C^k = C' = (C'_{ij}; 1 \leq i, j \leq n)$ and $D_{x_j} = \sum_{1 \leq i \leq n} C'_{ij} D_{t_i}$. From now on, we regard C, C' as functions of $\nabla_t x = \partial x / \partial t$. We have

$$\begin{aligned} \lambda_1(x, u, D_x) = & \ D_{t_1}, \\ \lambda_2(x, u, D_x) = & \sum_{1 \leq j \leq n} \lambda_{2j}(x, u) D_{x_j} \\ = & \sum_{1 \leq i, j \leq n} \lambda_{2j}(x, u) C'_{ij}(\nabla_t x) D_{t_i}. \end{aligned}$$

Let

$$\begin{aligned} u = & \sum_{|\alpha|=2} F_\alpha(x,u,\nabla u)D^\alpha u + f(x,u,\nabla_x u) \\ = & \lambda_2(x,u,D_x)(\lambda_1(x,u,D_x)u) + f'(x,u,\nabla_x u). \end{aligned}$$

We regard

$$\nabla_x u = {}^t(D_{x_1}u,\cdots,D_{x_n}u) = {}^tC'(\nabla_t x)^t(D_{t_1}u,\cdots,D_{t_n}u),$$

and we have

$$\begin{aligned} Fu = & \sum_{1\le i,j\le n} \lambda_{2j}(x(t),u(t))C'_{ij}(\nabla_t x)D_{t_i}D_{t_1}u(t) \\ & +g(x,u,\nabla_t x,\nabla_t u) \\ = & \sum_{1\le i\le n} G_i(x,u,\nabla_t x)D_{t_i}D_{t_1}u(t) + g(x,u,\nabla_t x,\nabla_t u) \\ = & Gu(t), \end{aligned}$$

where $G_i(x,u,\nabla_t x) = \sum_{1\le j\le n} \lambda_{2j}(x(t),u(t))C'_{ij}(\nabla_t x)$. Therefore we need to solve

$$(10) \qquad Gu(t) = 0,\ u(0,t') = v_0(t'),\ D_{t_1}u(0,t') = v_1(t'),$$

where v_0, v_1 are naturally defined by u_0, u_1:

$$\begin{aligned} v_0(t') = & u_0(t'), \\ v_1(t') = & u_1(t') + \sum_{2\le j\le n} \lambda_{1j}(0,t',u_0(t'))D_{t_j}u_0(t'). \end{aligned}$$

We need to solve (8) and (10) under the assumption (9). We emphasize again that λ_2 is not an important operator, and we have transformed the important operator λ_1 into D_{t_1}. D_{t_1} alone propagates the singularity in the present domain. Therefore we can easily investigate the propagation of the singularity, using t variables. By an elementary calculation, we can prove the following fact.

Let $\lambda_{22}^\circ = \lambda_{22}(0,u^\circ)$. If $1 << a << 1/R$, we define

$$\begin{aligned} \Omega_0(R) = & \{t\in \mathbf{C}^n;\ |t_2| < a|t_2 - \lambda_{22}^\circ t_1|, \\ & |t_2 - \lambda_{22}^\circ t_1| < R,\ |t_j| < R\ (2\le j\le n)\}, \\ \Omega_1(a,R) = & \{t\in\Omega_0(R);\ t_2\neq 0\}. \end{aligned}$$

$\Omega_0(R)$ corresponds to ω_2, and $\{t_2 = 0\}$ corresponds to Z_1 in the base space, using t variables. Let $\pi_0 : \mathcal{R}(\Omega_0(R)) \longrightarrow \Omega_0(R)$ and $\pi_1 : \mathcal{R}(\Omega_1(a,R)) \longrightarrow \Omega_1(a,R)$ be the natural projections. An arbitrary point $\widetilde{t} \in \mathcal{R}(\Omega_0(R))$ is determined by $t = \pi_0(\widetilde{t})$ and $\theta = \arg(t_2 - \lambda_{22}^{\circ} t_1)$ as before, therefore we denote $\widetilde{t}$ by t^{θ} or t. An arbitrary point $\widetilde{t} \in \mathcal{R}(\Omega_1(R))$ is determined by $t = \pi_1(\widetilde{t})$ and $\theta_1 = \arg(t_2 - \lambda_{22}^{\circ} t_1)$, $\theta_2 = \arg t_2$, therefore we denote $\widetilde{t}$ by t^{θ_1,θ_2} or t. We finally define

$$\Omega_2(a,R) = \{t^{\theta_1,\theta_2} \in \mathcal{R}(\Omega_1(a,R));\ |\theta_1 - \theta_2| < a\}.$$

Then we have the following

Proposition 1. *Let a be a large number (It may be as large as we wish). Then choosing a small $R > 0$, there exists a unique solution $x(t), u(t)$ of* (8) *and* (10) *satisfying* (9).

In order to prove Theorem 1, we change the variables from t to x. Since $u(t)$ is defined for $t \in \Omega_2(a,R)$, it is determined in the image $\{x(t);\ t \in \Omega_2(a,R)\}$ of $\Omega_2(a,R)$, as a function of x. We can prove that ω_2'' is a subset of this image.

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