数理解析研究所講究録 1412 巻 2005 年 59-66

On Hyperfunction Solutions to Fuchsian hyperbolic Systems

Susumu YAMAZAKI (山崎 晋)
Department of General Education, College of Science and Technology
(日本大学理工学部一般教育)

Introduction

Fuchsian partial differential operator was defined by Baouendi-Goulaouic [B-G]. This includes non-characteristic type as a special case, and Cauchy-Kovalevskaja type theorem (namely, unique solvability for Cauchy problem) was proved in [B-G] under the conditions of characteristic exponents. After that, Tahara [T] treated a Fuchsian Volevič system and proved Cauchy-Kovalevskaja type theorem in the complex domain under the conditions of characteristic exponents. Further, as an application he obtained Cauchy-Kovalevskaja type theorem for this system in the framework of hyperfunctions under the hyperbolicity condition. On the other hand, Laurent-Monteiro Fernandes [L-MF 1] extended notion of Fuchsian type to a general system of differential equation (that is, coherent left $\mathscr{D}_X$-Module, here and in what follows, we shall write *Module* with a capital letter, instead of *sheaf of left modules*) which includes Fuchsian Volevič system, and proved Cauchy-Kovalevskaja type theorem in the complex domain in general setting (that is, without conditions of characteristic exponents). As for the uniqueness of hyperfunction solution for Cauchy problem, Oaku [O 1] and Oaku-Yamazaki [O-Y] extended the uniqueness result to Fuchsian system. Hence in this paper, we shall prove the solvability theorem for general Fuchsian hyperbolic system in the framework of hyperfunctions (that is, hyperfunctions with a real analytic parameter, or mild hyperfunctions) without the conditions of characteristic exponents. To this end, in addition to Cauchy-Kovalevskaja type theorem due to Laurent-Monteiro Fernandes [L-MF 1], we use the theory of *microsupports* due to Kashiwara-Schapira (see [K-S]). This theory enable us to prove our desired result, in fact, our key theorem (Theorem 2.2) is only an exercise of this theory, and from this, we easily deduce Cauchy-Kovalevskaja type theorem for general Fuchsian hyperbolic system in the framework of hyperfunctions.

1 Preliminaries

In this section, we shall fix the notation and recall known results used in later sections. General references are made to Kashiwara-Schapira [K-S].

We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of all the integers, real numbers and complex numbers respectively. Moreover we set $\mathbb{N} := \{n \in \mathbb{Z};\, n \geqslant 1\} \subset \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{R}_{>0} := \{r \in \mathbb{R};\, r > 0\}$.

In this paper, all the manifolds are assumed to be paracompact. If $\tau\colon E \to Z$ is a vector bundle over a manifold Z, then we set $\dot{E} := E \setminus Z$ and $\dot{\tau}$ the restriction of τ to $\dot{E}$. Let M be an $(n+1)$-dimensional real analytic manifold and N a one-codimensional closed real analytic submanifold of M. We denote by $f\colon N \to M$ the canonical embedding. Let X and Y be complexifications of M and N respectively such that Y is a closed submanifold of X and that $Y \cap M = N$. We also denote by $f\colon Y \to X$ the canonical embedding with same notation f. By local coordinates $(z, \tau) = (x + \sqrt{-1}\,y, t + \sqrt{-1}\,s)$ of X around each point of N, we have locally the following relation:

$$
\begin{tikzcd}
N = \mathbb{R}^n_x \times \{0\} \arrow[r, tail, "f"] \arrow[d, tail] & M = \mathbb{R}^n_x \times \mathbb{R}_t \arrow[d, tail] \\
Y = \mathbb{C}^n_z \times \{0\} \arrow[r, tail, "f"] & X = \mathbb{C}^n_z \times \mathbb{C}_\tau
\end{tikzcd}
\tag{1.1}
$$

The embedding f induces a natural embedding $f'\colon T_N Y \hookrightarrow T_M X$ and by this mapping we regard $T_N Y$ as a closed submanifold of $T_M X$. Further, f induces mappings:

$$
\begin{tikzcd}
N & \sqrt{-1}\,T^*_N M \arrow[l, two heads, "\pi"'] \arrow[d, tail, "i"] \arrow[dr, phantom, ""] & N \arrow[l, tail] \arrow[r, tail, "f"] \arrow[d, tail] \arrow[dr, phantom, "\square"] & M \arrow[d, tail, "i_M"] \\
T^*_N Y \arrow[u, phantom, ""] \arrow[d, "\pi_N"] & N \underset{M}{\times} T^*_M X \arrow[l, two heads, "f_d"'] \arrow[d, "\pi"] \arrow[r, equal] & N \underset{M}{\times} T^*_M X \arrow[r, tail, "f_\pi"] \arrow[d, "\pi"] \arrow[dr, phantom, "\square"] & T^*_M X \arrow[d, "\pi_M"] \\
N \arrow[r, equal] & N \arrow[r, equal] & N \arrow[r, tail, "f"] & M
\end{tikzcd}
\tag{1.2}
$$

Here π_N, π_M and π are canonical projections, i_N, i_M and i are zero-section embeddings, and $\square$ means the square is *Cartesian.*

We write $M \setminus N = \Omega_+ \sqcup \Omega_-$, where each $\Omega_\pm$ is an open subset and $\partial\Omega_\pm = N$. We set $M_+ := \Omega_+ \cup N$. By local coordinates, we can write

$$\Omega_+ = \{(x,t) \in M;\, t > 0\} \subset M_+ = \{(x,t) \in M;\, t \geqslant 0\}.$$

We remark that a natural morphism $\mathbb{C}_{M_+} \to \mathbb{C}_N$ gives natural morphisms

$$\boldsymbol{R}\mathcal{H}om_{\mathbb{C}_M}(\mathbb{C}_N, \mathbb{C}_M) = \omega_{N/M} \to \boldsymbol{R}\mathcal{H}om_{\mathbb{C}_M}(\mathbb{C}_{M_+}, \mathbb{C}_M) = \mathbb{C}_{\Omega_+} \to \mathbb{C}_M\,.$$

Here $\omega_{N/M}$ denotes the *relative dualizing complex.* As usual, we denote by $\nu_\bullet(*)$ and $\mu_\bullet(*)$ be specialization and microlocalization functors respectively. Further, $\mu hom\,(*,*)$ denotes μhom bifunctor. We denote by $\mathbf{D}^{\mathrm{b}}(X)$ the derived category of sheaves of $\mathbb{C}$-linear spaces with bounded cohomologies. Let F be an object of $\mathbf{D}^{\mathrm{b}}(X)$. Then, by Kashiwara-Schapira [K-S, Chapter IV], we obtain morphisms:

$$\begin{aligned}\boldsymbol{R}f_{d!}(f_\pi^{-1}\mu_M(F)\otimes\omega_{N/M}) &\to \boldsymbol{R}f_{d!}(f_\pi^{-1}\mu\mathcal{h}om(\mathbb{C}_{\Omega_+},F)\otimes\omega_{N/M})\\ &\to \mu_N(f^{-1}F\otimes\omega_{Y/X}).\end{aligned}\tag{1.3}$$

2 Near-Hyperbolicity Condition

Let F be an object of $\mathbf{D}^{\mathrm{b}}(X)$. We denote by $\mathrm{SS}(F)$ the *microsupport* of F due to Kashiwara-Schapira (see [K-S]). $\mathrm{SS}(F)$ is a closed conic involutive subset of T^*X and described as follows: Let (w) be local coordinates of X and $(w_0;\zeta_0)$ a point of T^*X. Then $(w_0;\zeta_0)\notin\mathrm{SS}(F)$ if and only if the following condition holds: There exist an open neighborhood U of w_0 in X and a proper convex closed cone $\gamma\subset X$ satisfying $\zeta_0\in\mathrm{Int}\,\gamma^{\circ a}\cup\{0\}$ such that

$$\boldsymbol{R}\Gamma(H_\varepsilon\cap(x+\gamma);F)\xrightarrow{\sim}\boldsymbol{R}\Gamma(L_\varepsilon\cap(x+\gamma);F)$$

holds for any $w\in U$ and any sufficiently small $\varepsilon>0$. Here $\mathrm{Int}\,A$ denotes the *interior* of A, $\gamma^{\circ a}:=\bigcap_{\zeta\in\gamma}\{w\in X;\ \mathrm{Re}\langle w,\zeta\rangle\leqslant 0\}$ and

$$L_\varepsilon:=\{w\in X;\ \mathrm{Re}\langle w-w_0,\zeta_0\rangle=-\varepsilon\}\subset H_\varepsilon:=\{w\in X;\ \mathrm{Re}\langle w-w_0,\zeta_0\rangle\geqslant-\varepsilon\}.$$

Next, we shall recall the definition of the near-hyperbolicity condition due to Laurent-Monteiro Fernandes [L-MF 2, Definition 1.3.1]:

2.1 Definition. Let F be an object of $\mathbf{D}^{\mathrm{b}}(X)$. We say F is *near-hyperbolic* at $x_0\in N$ in $\pm dt$-codirection if there exist positive constants C and ε_1 such that

$$\begin{aligned}&\mathrm{SS}(F)\cap\{(z,\tau;z^*,\tau^*)\in T^*X;\ |z-x_0|<\varepsilon_1,\ |\tau|<\varepsilon_1,\ t\neq 0\}\\ &\quad\subset\{(z,\tau;z^*,\tau^*)\in T^*X;\ |t^*|\leqslant C\big(|y^*|(|y|+|s|)+|x^*|\big)\}\end{aligned}$$

holds by local coordinates of X in (1.1) and the following associated coordinates of T^*X:

$$(z,\tau;z^*,\tau^*)=(x+\sqrt{-1}\,y,t+\sqrt{-1}\,s;x^*+\sqrt{-1}\,y^*,t^*+\sqrt{-1}\,s^*).$$

Our first main result is:

2.2 Theorem. *Let F be an object of $\mathbf{D}^{\mathrm{b}}(X)$. Assume that F is near-hyperbolic at $x_0\in N$ in $\pm dt$-codirection. Then, the morphisms in* (1.3) *induce isomorphisms for any $p^*\in T^*_NY\cap\pi_N^{-1}(x_0)$:*

$$\begin{aligned}\boldsymbol{R}f_{d!}(f_\pi^{-1}\mu_M(F)\otimes\omega_{N/M})_{p^*} &\xrightarrow{\sim}\boldsymbol{R}f_{d!}(f_\pi^{-1}\mu\mathcal{h}om(\mathbb{C}_{\Omega_+},F)\otimes\omega_{N/M})_{p^*}\\ &\xrightarrow{\sim}\mu_N(f^{-1}F\otimes\omega_{Y/X})_{p^*}.\end{aligned}$$

Sketch of Proof. Let $\nu_{\Omega_+}(F)$ be the inverse Fourier-Sato transform of $\mu hom(\mathbb{C}_{\Omega_+}, F)$. Since the Fourier-Sato transformation gives an equivalence, to prove the theorem, we may prove

$$\nu_M(F)|_{T_NY\cap\tau_N^{-1}(x_0)} \to \nu_{\Omega_+}(F)|_{T_NY\cap\tau_N^{-1}(x_0)} \to \nu_N(f^{-1}F)|_{T_NY\cap\tau_N^{-1}(x_0)}$$

are isomorphisms. By virtue of the microsupport theory, we can use the same argument as in [B-S, Lemme 3.2]. □

3 Microfunction with a Real Analytic Parameter.

Recall the diagram (1.2). As usual, we denote by $\mathscr{O}_X$, $\mathscr{A}_M := \mathscr{O}_X|_M$, $\mathscr{B}_M$ and $\mathscr{C}_M$ the sheaves of *holomorphic functions* on X, of *real analytic functions* on M, of *hyperfunctions* on M and of *microfunctions* on T^*_MX respectively.

3.1 Definition. We set

$$\mathscr{C}^A_{N|M} := \boldsymbol{R}f_{d!}\, f_\pi^{-1}\mathscr{C}_M\,, \quad \mathscr{B}^A_{N|M} := i_N^{-1}\mathscr{C}^A_{N|M} = \boldsymbol{R}\pi_{N*}\mathscr{C}^A_{N|M}\,.$$

It is known that $f_{d!}\, f_\pi^{-1}\mathscr{C}_M$ is the sheaf of *microfunction with a real analytic parameter* t, and $f_{d!}\, f_\pi^{-1}\mathscr{C}_M|_N$ is the sheaf of *hyperfunction with a real analytic parameter* t:

3.2 Proposition (cf. [S], [S-K-K, Chapter I, Theorem 2.2.6]).

(1) $\mathscr{C}^A_{N|M}$ *is concentrated in degree zero and conically soft*; *that is, the direct image of* $\mathscr{C}^A_{N|M}|_{\dot{T}^*_NY}$ *on* $\dot{T}^*_NY/\mathbb{R}_{>0}$ *is a soft sheaf.*

(2) *There exists the following exact sequence*:

$$0 \to \mathscr{A}_M|_N \to \mathscr{B}^A_{N|M} \to \dot{\pi}_{N*}\mathscr{C}^A_{N|M} \to 0.$$

(3) *There exists the following exact sequence*:

$$0 \to \mathscr{B}^A_{N|M} \to \mathscr{B}_M|_N \to \dot{\pi}_*(\mathscr{C}_M|_{\sqrt{-1}\dot{T}^*_NM}) \to 0.$$

We recall that the morphism in (1.3) induces *restriction morphisms*:

$$\mathscr{C}^A_{N|M} \to \mathscr{C}_N\,, \quad \mathscr{B}^A_{N|M} \to \mathscr{B}_N\,.$$

In order to study microlocal boundary value problems, Kataoka [Kt] defined the sheaf $\mathring{\mathscr{C}}_{N|M_+}$ of *mild microfunctions* on T^*_NY, and $\mathring{\mathscr{B}}_{N|M_+} := \mathring{\mathscr{C}}_{N|M_+}|_N$ is called the sheaf of *mild hyperfunctions.* Note that by Schapira-Zampieri [Sc-Z]

$$\mathring{\mathscr{C}}_{N|M_+} = \boldsymbol{R}f_{d!}\, f_\pi^{-1}\mu hom(\mathbb{C}_{\Omega_+}, \mathscr{O}_X) \otimes or_{M/X}[n+1]$$

holds. Here $or_{M/X}$ denotes the *relative orientation sheaf.* $\mathring{\mathscr{C}}_{N|M_+}$ is conically soft, and there exists an exact sequence:

$$0 \to \mathscr{A}_M|_N \to \mathring{\mathscr{B}}_{N|M_+} \to \dot{\pi}_{N*}\mathring{\mathscr{C}}_{N|M_+} \to 0.$$

Further by (1.3), the restriction morphism $\mathscr{C}^A_{N|M} \to \mathscr{C}_N$ factorizes through the *boundary value morphism* $\mathring{\mathscr{C}}_{N|M_+} \to \mathscr{C}_N$.

4 Cauchy and Boundary Value Problems for Fuchsian Hyperbolic Systems

First, we recall the definition of Fuchsian differential operators in the sense of Baouendi-Goulaouic [B-G].

4.1 Definition. Let us take local coordinates in (1.1). Then we say that P is a *Fuchsian differential operator* of weight (k, m) in the sense of Baouendi-Goulaouic [B-G] if P can be written in the following form

$$P(z,\tau,\partial_z,\partial_\tau) = \tau^k\partial_\tau^m + \sum_{j=1}^{k} P_j(z,\tau,\partial_z)\,\tau^{k-j}\partial_\tau^{m-j} + \sum_{j=k}^{m} P_j(z,\tau,\partial_z)\,\partial_\tau^{m-j}.$$

Here $\operatorname{ord} P_j \leqslant j$ $(0 \leqslant j \leqslant m)$, and $P_j(z,0,\partial_z) \in \mathscr{O}_Y$ $(1 \leqslant j \leqslant k)$.

Note that a Fuchsian differential operator of weight (m, m) is nothing but *an operator with regular singularity along Y in a weak sense* due to Kashiwara-Oshima [K-O].

Let $\mathscr{M}$ be a $\mathscr{D}_X$-Module. The *inverse image* in the sense of $\mathscr{D}$-Module is defined by

$$\boldsymbol{D}f^*\mathscr{M} := \mathscr{D}_{Y\to X} \overset{L}{\underset{f^{-1}\mathscr{D}_X}{\otimes}} f^{-1}\mathscr{M} \in \operatorname{Ob}\mathbf{D}^{\mathrm{b}}(\mathscr{D}_Y).$$

Here $\mathscr{D}_{Y\to X} := \mathscr{O}_Y \underset{f^{-1}\mathscr{O}_X}{\otimes} f^{-1}\mathscr{D}_X$ is the *transfer bi-Module* as usual. Further we set $\mathscr{M}_Y := \mathcal{H}^0\boldsymbol{D}f^*\mathscr{M} = \mathscr{O}_Y \underset{f^{-1}\mathscr{D}_X}{\otimes} f^{-1}\mathscr{M}$.

Next, let $\mathscr{M}$ be a *Fuchsian system along* Y in the sense of Laurent-Monteiro Fernandes [L-MF 1]. Since precise definition of Fuchsian system is complicated, we do not recall it here. We remark that $\mathscr{M}$ is Fuchsian along Y if and only if there exists locally an epimorphism $\bigoplus_{i=1}^{m} \mathscr{D}_X/\mathscr{D}_X P_i \twoheadrightarrow \mathscr{M}$, where each differential operators P_i is an operator with regular singularity along Y in a weak sense.

4.2 Remark. (1) Let $\mathscr{M}$ be a coherent $\mathscr{D}_X|_Y$-Module for which Y is non-characteristic. Then $\mathscr{M}$ is Fuchsian. More generally, any regular-specializable system is Fuchsian.
(2) Let $\mathscr{M}$ be a Fuchsian system along Y. Then:

- (i) By Laurent-Schapira [L-S, Théorème 3.3], all the cohomologies of $\boldsymbol{D}f^*\mathscr{M}$ are coherent $\mathscr{D}_Y$-Module.
- (ii) Laurent-Monteiro Fernandes [L-MF 1, Théorème 3.2.2] proved that there exists the following isomorphism (that is, Cauchy-Kovalevskaja type theorem):

$$f^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{O}_X) \simeq \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^*\mathscr{M},\mathscr{O}_Y).$$

4.3 Definition. Let $\mathscr{M}$ be a coherent $\mathscr{D}_X|_Y$-Module. Then we say $\mathscr{M}$ is *near-hyperbolic* at $x_0 \in N$ in $\pm dt$ codirection if $\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{O}_X)$ is near-hyperbolic in the sense of Definition 2.1. We remark that $\mathrm{SS}(\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{O}_X)) = \operatorname{char}(\mathscr{M})$. Here $\operatorname{char}(\mathscr{M})$ is the characteristic variety of $\mathscr{M}$.

4.4 Example. (1) Let P be a Fuchsian differential operator of weight (k, m) in the sense of Baouendi-Goulaouic [B-G]. Then $\mathscr{D}_X/\mathscr{D}_X P$ is Fuchsian along Y. Moreover, assume that P is Fuchsian hyperbolic in the sense of Tahara [T]; that is, the principal symbol is written as $\sigma_m(P)(z,\tau;z^*,\tau) = \tau^k p(z,\tau;z^*,\tau^*)$. Here $p(z,\tau;z^*,\tau^*)$ satisfies the following condition:

$$(4.1)\qquad \begin{cases} \text{If } (x,t;x^*) \text{ are real, then all the roots } \tau_j^*(x,t;x^*) \text{ of the equation} \\ p(x,t;x^*,\tau^*) = 0 \text{ with respect to } \tau^* \text{ are real.} \end{cases}$$

Then $\mathscr{D}_X/\mathscr{D}_X P$ is near-hyperbolic (see [L-MF 2, Lemma 1.3.2]).
(2) Let $P = \vartheta - A(z,\tau,\partial_z)$ be a *Fuchsian Volevič system* of size m due to Tahara [T]; that is,

(i) $A(z,\tau,\partial_z) = (A_{ij}(z,\tau,\partial_z))_{i,j=1}^m$ is a matrix of size m whose components are in $\mathscr{D}_X$ with $[A_{ij},\tau] = 0$;

(ii) There exists $\{n_i\}_{i=1}^m \subset \mathbb{Z}$ such that $\operatorname{ord} A_{ij}(z,\tau,\partial_z) \leqslant n_i - n_j + 1$ and $A_{ij}(z,0,\partial_z) \in \mathscr{O}_Y$ for any $1 \leqslant i,j \leqslant m$.

Set $\sigma(A)(z,\tau;z^*) := (\sigma_{n_i-n_j+1}(A_{ij})(z,\tau;z^*))_{i,j=1}^m$. Then

$$\operatorname{char}(\mathscr{D}_X^m/\mathscr{D}_X^m P) = \{(z,\tau;z^*,\tau^*) \in T^*X;\ \det(\tau\tau^* - \sigma(A)(z,\tau;z^*)) = 0\},$$

and we can prove that $\mathscr{D}_X^m/\mathscr{D}_X^m P$ is Fuchsian along Y. Moreover assume that P is Fuchsian hyperbolic in the sense of Tahara [T]; that is,

$$\det(\tau\tau^* - \sigma(A)(z,\tau;z^*)) = \tau^m p(z,\tau;z^*,\tau^*),$$

and $p(z,\tau;z^*,\tau^*)$ satisfies the condition (4.1). Then $\mathscr{D}_X^m/\mathscr{D}_X^m P$ is near-hyperbolic.

By Theorem 2.2 and Cauchy-Kovalevskaja type theorem, we obtain:

4.5 Theorem. *Let $\mathscr{M}$ be a Fuchsian system along Y. Assume that $\mathscr{M}$ is near-hyperbolic at $x_0 \in N$ in $\pm dt$-codirection. Then for any $p^* \in T_N^*Y \cap \pi_N^{-1}(x_0)$, the morphisms in* (1.3) *induce isomorphisms*

$$\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{C}_{N|M}^A)_{p^*} \simeq \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathring{\mathscr{C}}_{N|M_+})_{p^*} \simeq \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^*\mathscr{M},\mathscr{C}_N)_{p^*}.$$

In particular, the morphisms in (1.3) *induce isomorphisms*

$$\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{B}_{N|M}^A)_{x_0} \simeq \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathring{\mathscr{B}}_{N|M_+})_{x_0} \simeq \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^*\mathscr{M},\mathscr{B}_N)_{x_0}.$$

4.6 Remark. Oaku-Yamazaki [O-Y] showed that for any Fuchsian system $\mathscr{M}$ along Y, two morphisms

$$\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{C}_{N|M}^A)_{p^*} \rightarrowtail \mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathring{\mathscr{C}}_{N|M_+})_{p^*} \rightarrowtail \mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y,\mathscr{C}_N)_{p^*}$$

are always injective without the near-hyperbolicity condition. Precisely speaking, we always assumed that $\operatorname{codim}_M N \geqslant 2$ in [O-Y]. However, the same proof also works even in the case where $\operatorname{codim}_M N = 1$; Oaku ([O 1], [O 2]) defined the sheaf $\mathscr{C}_{N|M_+}^F$ of

F-mild microfunctions on T_N^*Y, and $\mathscr{B}^F_{N|M_+} := \mathscr{C}^F_{N|M_+}|_N$ is called the sheaf of *F-mild hyperfunctions* ([O 1], [O 2], cf. [O-Y]). As is mentioned above, we can apply the methods in [O-Y] of the higher-codimensional case to the one-codimensional case to prove the following: there exist natural morphisms $\mathscr{C}^A_{N|M} \rightarrowtail \mathring{\mathscr{C}}_{N|M_+} \rightarrowtail \mathscr{C}^F_{N|M_+} \to \mathscr{C}_N$ such that the composition coincides with $\mathscr{C}^A_{N|M} \to \mathscr{C}_N$, and these induce monomorphisms:

$$\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}^A_{N|M}) \rightarrowtail \mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathring{\mathscr{C}}_{N|M_+}) \rightarrowtail \mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}^F_{N|M_+}) \rightarrowtail \mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{C}_N)$$

for any Fuchsian system $\mathscr{M}$ along Y. Hence, under the near-hyperbolic condition, we obtain isomorphisms:

$$\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}^F_{N|M_+}) \xrightarrow{\sim} \mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{C}_N),$$
$$\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}^F_{N|M_+}) \xrightarrow{\sim} \mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{B}_N).$$

Our conjecture is: if $\mathscr{M}$ is a Fuchsian system along Y and satisfies near-hyperbolicity condition, then the following hold:

$$\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}^F_{N|M_+}) \xrightarrow{\sim} \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{C}_N).$$

References

[B-G] Baouendi, M. S. and Goulaouic, C., *Cauchy problems with characteristic initial hypersurface*, Comm. Pure Appl. Math. **26** (1973), 455–475.

[B-S] Bony, J.-M. et Schapira, P., *Solutions hyperfonctions du problème de Cauchy*, Hyperfunctions and Pseudo-Differential Equations, Proceedings Katata 1971 (Komatsu, H., ed.), Lecture Notes in Math. **287**, Springer, Berlin–Heidelberg–New York, 1973, pp. 82–98.

[K-O] Kashiwara, M. and Oshima, T., *Systems of differential equations with regular singularities and their boundary value problems*, Ann. of Math. **106** (1977), 145–200.

[K-S] Kashiwara, M. and Schapira, P., *Sheaves on Manifolds*, Grundlehren Math. Wiss. **292**, Springer, Berlin–Heidelberg–New York, 1990.

[Kt] Kataoka, K., *Micro-local theory of boundary value problems, I–II*, J. Fac. Sci. Univ. Tokyo Sect. IA **27** (1980), 355–399; ibid. **28** (1981), 31–56.

[L-MF 1] Laurent, Y. et Monteiro Fernandes, T., *Systèmes différentiels fuchsiens le long d'une sous-variété*, Publ. Res. Inst. Math. Sci. **24** (1981), 397–431.

[L-MF 2] ———, *Topological boundary values and regular $\mathscr{D}$-modules*, Duke Math. J. **93** (1998), 207–230.

[L-S] Laurent, Y. et Schapira, P., *Images inverse des modules différentiels*, Compositio Math. **61** (1987), 229–251.

[O 1] Oaku, T., *F-mild hyperfunctions and Fuchsian partial differential equations*, Group Representation and Systems of Differential Equations, Proceedings Tokyo 1982 (Okamoto, K., ed.), Advanced Studies in Pure Math. **4**, Kinokuniya, Tokyo; North-Holland, Amsterdam–New York–Oxford, 1984, pp. 223–242.

[O 2] ———, *Microlocal boundary value problem for Fuchsian operators, I*, J. Fac. Sci. Univ. Tokyo, Sect. IA **32** (1985), 287–317.

[O-Y] Oaku, T. and Yamazaki, S., *Higher-codimensional boundary value problems and F-mild microfunctions*, Publ. Res. Inst. Math. Sci. **34** (1998), 383–437.

[S] Sato, M., *Theory of hyperfunctions. I–II*, J. Fac. Sci. Univ. Tokyo Sect. IA **8** (1959–1960), 139–193 and 387–436.

[S-K-K] Sato, M., Kawai, T. and Kashiwara, M., *Microfunctions and pseudo-differential equations*, Hyperfunctions and Pseudo-Differential Equations, Proceedings Katata 1971 (Komatsu, H., ed.), Lecture Notes in Math. **287**, Springer, Berlin–Heidelberg–New York, 1973, pp. 265–529.

[Sc-Z] Schapira, P. and Zampieri, G., *Microfunctions at the boundary and mild microfunctions*, Publ. Res. Inst. Math. Sci. **24** (1988), 495–503.

[T] Tahara, H., *Fuchsian type equations and Fuchsian hyperbolic equations*, Japan J. Math. (N.S.) **5** (1979), 245–347.