80

A viewpoint on positivity of pseudodifferential operators from the Wick calculus

Nicolas Lerner (レルナー ニコラス)

Université de Rennes (レンヌ大学)

and Yoshinori Morimoto (森本 芳則)

Kyoto University (京都大学)

Graduate School of Human and Environmental Studies (大学院人間・環境学研究科)

27 August, 2003

1 Introduction

In this note we give a simple proof of Fefferman-Phong inequality by means of Wick calculus, instead of FBI operator discussed by Tataru[T]. Recently, the Wick calculus has been used by the first author in [L2, L3], treating irregular symbols which appear in the local solvability problem for pseudo-differential operators of principal type. As a generalization of the product formula given there, Ando and the second author [AM] have given a full expansion formula as follows (similar to that of Weyl pseudo-differential operators):

$$a^{Wick}b^{Wick} = (ab)^{Wick} - \frac{1}{2}\left(a' \cdot b' - \frac{1}{i}\{a,b\}\right)^{Wick} + \cdots$$

(1.1)
$$+ \frac{(-1)^k}{2^k\,k!}\left(\left(\left(\sum_{j=1}^{2n}\partial_{X_j}\partial_{Z_j} + \frac{H_{X_j}}{i}\partial_{Z_j}\right)^k a(X)b(Z)\right)\Bigg|_{Z=X}\right)^{Wick} + \cdots,$$

where, for $a(x,\xi) = a(X), (X \in \mathbb{R}^n_x \times \mathbb{R}^n_\xi)$, we define $a^{Wick} = a^{Wick}(x,D)$ on $L^2(\mathbb{R}^n)$ by

(1.2)
$$a^{Wick}(x,D)u(x) = (W^*a^\mu W u)(x) \qquad for \quad u \in L^2(\mathbb{R}^n).$$

Here $(Wu)(Y) = (Wu)(y,\eta)$ is a windowed Fourier transform of $u \in L^2(\mathbb{R}^n)$ defined by

$$(Wu)(Y) = \int_{R^n} \overline{g^Y(x)}u(x)dx \ , g^Y(x) = e^{ix\eta}g(x-y),$$

with a Gauss function $g = (4\pi^3)^{-n/4}\exp(-|x|^2/2)$,

a^μ is the multiplication operator by $a(Y)$ on $L^2(\mathbb{R}^{2n})$ and W^* is the adjoint operator of W. The formal expansion formula (1.1) seems to be known for polynomial symbols since the Wick calculus is classical and has a long history(cf., [S]). In fact, it is not difficult to obtain (1.1) in formal arguments because the Wick operators can be converted to the Weyl pseudo-differential operators and one may apply the product formula (in p.155 of [H]) of Weyl calculus *. However our interest here is to estimate the remainder term in the frame of Wick operators, yielding a variant of Bony's proof [B] of the Fefferman-Phong inequality.

Though our methods can be applied to the expansion formula of any order, for the brevity we confine ourselves to the one of first order in what follows:

---

*see the last paragraph of Section 2.

Proposition 1.1. *If $a(X), b(X)$ and those derivatives belong to L^∞ then we have*

$$a^{Wick}b^{Wick} = \left(ab - \frac{1}{2}a' \cdot b' + \frac{1}{2i}\{a,b\}\right)^{Wick} + R_2,$$

where the remainder term R_2 is an operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ satisfying two different estimates:

$$(1.3) \qquad \|R_2\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq C\|a\|_{L^\infty}\left(\sum_{|\beta|=2}\|b^{(\beta)}\|_{L^\infty}\right)$$

or

$$
(1.4) \qquad
\begin{aligned}
\|R_2\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq C\Bigg(&\sum_{|\alpha|=|\beta|=2}\|a^{(\alpha)}\|_{L^\infty}\|b^{(\beta)}\|_{L^\infty} \\
&+ \sum_{|\beta|=2,|\alpha|=|\gamma|=1}\|\left(a^{(\alpha)}b^{(\beta)}\right)^{(\gamma)}\|_{L^\infty} + \sum_{|\alpha|=2,|\gamma|=2}\|\left(ab^{(\alpha)}\right)^{(\gamma)}\|_{L^\infty}\Bigg),
\end{aligned}
$$

provided that all terms on the right hand side of (1.3) or (1.4) are well-defined . Here $a^{(\alpha)}(X) = \partial_X^\alpha a(X)$. Furthermore, we have

$$b^{Wick}a^{Wick} = \left(ab - \frac{1}{2}a' \cdot b' - \frac{1}{2i}\{a,b\}\right)^{Wick} + \tilde{R}_2,$$

where the remainder term $\tilde{R}_2$ has the same estimates as (1.3) or (1.4).

It should be noted that estimates (1.3) and (1.4) are not symmetric with respect to a and b. It is now well-known (ex., [L1]) that the sharp Gårding inequality follows directly from the Wick calculus because the Wick operator approximates the pseudodifferential operators (see Proposition 2.1 in Section 2). Here, by means of Proposition 1.1 we can prove:

Theorem 1.2.([FP], Theorem 18.6.8 of [H], [B]). *Let $0 \leq \delta < \rho \leq 1$. Assume that $a(x,\xi) \geq 0$ and*

$$(1.5) \qquad |\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \leq C < \xi >^{(\rho+\delta)(|\beta|-|\alpha|)/2} \quad \text{for } 4 \leq |\alpha| + |\beta|$$

$$(1.6) \qquad |\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \leq C < \xi >^{2(\rho-\delta)+\delta|\beta|-\rho|\alpha|} \quad \text{for } |\alpha| + |\beta| < 4.$$

Then there exists a constant $C' > 0$ such that

$$\text{Re } (a(x,D)u,u) \geq -C'\|u\|^2 \quad \text{for } u \in \mathcal{S}.$$

We remark that the condition (1.5) is satisfied if $a(x,\xi)$ belongs to $S_{\rho,\delta}^{2(\rho-\delta)}$ because

$$2(\rho-\delta) + \delta|\beta| - \rho|\alpha| = \frac{\rho-\delta}{2}(4 - |\alpha+\beta|) + \frac{\rho+\delta}{2}(|\beta|-|\alpha|).$$

This generalization of Fefferman-Phong inequality and further investigation was given by [B]. It should be noted that (1.5) is required only up to finite order of α, β (see (3.3) and (3.9) in Section 3). For the proof of Theorem 1.1, addition to Proposition 1.1 we need the usual Littlewood-Paley decomposition and the following lemma:

Lemma 1.3.([T], cf. [G], Lemma18.6.9 of [H]). *Let $a(X)$ be a non-negative $C^{3,1}$ function defined in $\mathbb{R}^d$ such that $\sup\limits_{|\alpha|=4} ||a^{(\alpha)}||_{L^\infty} \leq 1$. Then there exist an $M \in \mathbb{N}$ and a $C > 0$ depending only on the dimension d such that*

$$a(X) = \sum_{j=1}^{M} b_j(X)^2$$

where $b_j(X)$ are $C^{1,1}$ functions satisfying

(1.7) $$\sum_{|\alpha|=2} ||b_j^{(\alpha)}||_{L^\infty} + \sum_{|\alpha|=2,|\beta|=|\gamma|=1} ||\left(b_j^{(\alpha)}b_j^{(\beta)}\right)^{(\gamma)}||_{L^\infty} \leq C.$$

Note that $b_j^{(\beta)}$ with $|\beta| = 1$ is a Lipschitz function and the lemma is claiming that the first derivative in the distribution sense of $b_j^{(\alpha)}b_j^{(\beta)}$ with $|\alpha| = 2, |\beta| = 1$ is in fact L^∞.

We remark that the fact $b_j \in C^{1,1}$ is optimal in the case $d \geq 4$ ([BBCP]) though it looks like a function belonging to $C^{2,1}$ in the proof in [T] under rescaling arguments [†] . Since b_j only satisfies (1.7) we shall apply carefully Proposition 1.1 and its proof to the proof of Theorem 1.2.

2 Proof of Proposition 1.1

The formula in the proposition with the remainder term R_2 satisfying (1.3) is nothing but (2.4) of [AM] in the case of $N = 2$ (cf., Proposition 2.3 of [L2]). So we shall prove the second estimate (1.4) for R_2. If we define the operator Σ_Y as

(2.1) $$(\Sigma_Y u)(x) = (Wu)(Y)g^Y(x) \qquad for \quad u \in L^2(\mathbb{R}^n).$$

Then it follows from (1.2) that for $a \in L^\infty(\mathbb{R}^n)$ we have

(2.2) $$a^{Wick} = \int_{R^{2n}} a(Y)\Sigma_Y dY.$$

Since Σ_Y is a Weyl pseudo-differential operator with a symbol $p_Y(X) = \pi^{-n}e^{-|X-Y|^2}$ for each fixed Y (see Lemma 2.9 of [AM], cf.,(2.2) of [L2]) it follows from (2.2) that $a^{Wick}(x, D) = b^w(x, D)$, with its Weyl symbol

(2.3) $$b(x,\xi) = b(X) = \pi^{-n} \int_{\mathbb{R}^{2n}} a(X + Y)e^{-|Y|^2}dY.$$

Furthermore we have (see Lemma 2.10 of [AM])

(2.4) $$\Sigma_Y\Sigma_Y = (2\pi)^{-n}\Sigma_Y \qquad on \quad L^2(\mathbb{R}^n),$$

(2.5) $$||\Sigma_Y\Sigma_Z||_{\mathcal{L}(L^2(R^n))} \leq (2\pi)^{-2n}e^{-\frac{1}{4}|Y-Z|^2}.$$

By means of (2.2) we have

$$a^{Wick}b^{Wick} = \iint_{\mathbb{R}_Y^{2n}\times\mathbb{R}_Z^{2n}} a(Y)b(Z)\Sigma_Y\Sigma_Z dY dZ.$$

[†] In particular the product $b'b'''$ is not meaningful but requiring $(b'b'')' - b''b'' \in L^\infty$ makes sense for $b = b_j \in C^{1,1}$.

Using the Taylor formula $\quad b(Z) = \sum_{|\alpha| \leq 1} \dfrac{b^{(\alpha)}(Y)(Z-Y)^\alpha}{\alpha!} + b_2(Y, Z)$ with

$$b_2(Y, Z) = 2 \sum_{|\alpha|=2} \int_0^1 (1-\theta) b^{(\alpha)}(Y + \theta(Z-Y)) d\theta \frac{(Z-Y)^\alpha}{\alpha!}$$

we have

$$a^{Wick} b^{Wick} = \sum_{\ell=0}^1 \sum_{|\alpha|=\ell} \Omega_\alpha + R_2^0,$$

where

$$\Omega_\alpha = \frac{1}{\alpha!} \iint_{\mathbb{R}_Y^{2n} \times \mathbb{R}_Z^{2n}} a(Y) b^{(\alpha)}(Y)(Z-Y)^\alpha \Sigma_Y \Sigma_Z dY dZ,$$

$$R_2^0 = \iint_{\mathbb{R}_Y^{2n} \times \mathbb{R}_Z^{2n}} a(Y) b_2(Y, Z) \Sigma_Y \Sigma_Z dY dZ$$

If $\sigma(\Omega_\alpha)$ denotes the Weyl symbol of Ω_α, we have the formula

$$(2.6) \qquad \sigma(\Omega_\alpha) = \frac{\pi^{-n}}{2^{|\alpha|}\alpha!} \int_{R_Y^{2n}} a(X+Y) b^{(\alpha)}(X+Y) \left(\sum_{q=0}^\infty \frac{(2\Delta_Z)^q}{q!} Z^\alpha \Bigg|_{Z=\partial_Y + H_Y/i} \right) e^{-|Y|^2} dY,$$

as in the same way in p.134 in [AM]. If $|\alpha| = 1$ then we have the only term with $q = 0$, that is,

$$\sigma(\Omega_\alpha) = \frac{\pi^{-n}}{2} \int_{R_Y^{2n}} a(X+Y) b^{(\alpha)}(X+Y)(\partial_Y + H_Y/i)^\alpha e^{-|Y|^2} dY$$

$$= \frac{\pi^{-n}}{2} \int_{R_Y^{2n}} \left\{ (-(\partial_Y + H_Y/i))^\alpha \left(a(X+Y) b^{(\alpha)}(X+Y) \right) \right\} e^{-|Y|^2} dY,$$

where we have used the integration by parts in the last equality. In view of (2.3) we have

$$(2.7) \qquad \sum_{|\alpha|=1} \Omega_\alpha = \left(-\frac{1}{2} a' \cdot b' + \frac{1}{2i}\{a, b\} - \frac{1}{2} a \Delta_X b \right)^{Wick}$$

because $\sum_{|\alpha|=1} H_Y^\alpha \partial^\alpha b = 0$. We shall calculate R_2^0, whose principal part cancels the third term of the right hand side of (2.7). Using the Taylor formula for $a(Y)$ at $Y + \theta(Z - Y)$, we have

$$\begin{aligned} R_2^0 &= 2 \sum_{|\alpha|=2} \Bigg[\iint_{\mathbb{R}_Y^{2n} \times \mathbb{R}_Z^{2n}} \int_0^1 (1-\theta) \{a(Y + \theta(Z-Y)) \\ &\quad - \sum_{|\beta|=1} a^{(\beta)}(Y + \theta(Z-Y)) \theta(Z-Y)^\beta \\ &\quad + 2 \sum_{|\beta|=2} \int_0^1 (1-\tilde\theta) a^{(\beta)}(Y + (1-\tilde\theta)\theta(Z-Y)) \frac{\theta^2 (Z-Y)^\beta}{\beta!} d\tilde\theta \} \\ &\quad \times b^{(\alpha)}(Y + \theta(Z-Y)) d\theta \frac{(Z-Y)^\alpha}{\alpha!} \Sigma_Y \Sigma_Z dY dZ \Bigg] \\ &:= J_1 + J_2 + J_3. \end{aligned}$$

Since we have

$$J_3 = \sum_{|\alpha|,|\beta|=2} \frac{4}{\alpha!\beta!} \int_0^1 \int_0^1 d\theta d\tilde\theta (1-\theta)(1-\tilde\theta)\theta^2 \iint_{\mathbb{R}_Y^{2n} \times \mathbb{R}_Z^{2n}}$$

$$\times a^{(\beta)}(Y + (1-\tilde\theta)\theta(Z-Y))b^{(\alpha)}(Y+\theta(Z-Y))(Z-Y)^{\alpha+\beta}\Sigma_Y\Sigma_Z dYdZ$$

by means of (2.5) and Cotlar's lemma we get as in (2.19) of [L2]

$$(2.8) \qquad \|J_3\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le C \sum_{|\alpha|,|\beta|=2} \|a^{(\beta)}\|_{L^\infty}\|b^{(\alpha)}\|_{L^\infty}.$$

As to the term J_2 we use the Taylor formula for $\left(a^{(\beta)}b^{(\alpha)}\right)(Y+\theta(Z-Y))$ with $|\alpha|=2$ and $|\beta|=1$ at Y. Then

$$J_2 = - \sum_{|\alpha|=2,|\beta|=1} \frac{2}{\alpha!\beta!} \int_0^1 \theta(1-\theta)d\theta \left\{ \iint_{\mathbb{R}_Y^{2n} \times \mathbb{R}_Z^{2n}} \left((a^{(\beta)}b^{(\alpha)})(Y)\right)(Z-Y)^{\beta+\alpha}\Sigma_Y\Sigma_Z dYdZ \right.$$

$$\left. + \sum_{|\gamma|=1} \int_0^1 dt \iint_{\mathbb{R}_Y^{2n} \times \mathbb{R}_Z^{2n}} \left((a^{(\beta)}b^{(\alpha)})^{(\gamma)}(Y)\right)(Y+t\theta(Z-Y))\theta(Z-Y)^{\beta+\alpha+\gamma}\Sigma_Y\Sigma_Z dYdZ \right\}$$

$$= J_2^{(1)} + J_2^{(2)}.$$

By the same way as in (2.6) we have

$$\sigma(J_2^{(1)}) = \sum_{|\alpha|=2,|\beta|=1} \frac{-\pi^{-n}}{32^3\alpha!\beta!} \int_{R_Y^{2n}} (a^{(\beta)}b^{(\alpha)})(X+Y) \cdot \left(\sum_{q=0}^1 \frac{(2\Delta_Z)^q}{q!} Z^{\beta+\alpha}\bigg|_{Z=\partial_Y+H_Y/i} \right) e^{-|Y|^2}dY.$$

Note that the term between the last parentheses is the sum of differential operators with order 1 or 3. Use one derivative for the integration by parts. Then, by the same method as in (3.19) of [L2] we have

$$(2.9) \qquad \|J_2^{(1)}\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le C \sum_{|\gamma|=1,|\alpha|=2,|\beta|=1} \left\| \left(a^{(\beta)}b^{(\alpha)}\right)^{(\gamma)} \right\|_{L^\infty}.$$

For $J_2^{(2)}$ we have the same estimate by the same way as in (2.19) of [L2]. Now we estimate J_1 by using the Taylor formula for $ab^{(\alpha)}(Y+\theta(Z-Y))$ at Y again. We have

$$J_1 = \sum_{|\alpha|=2} \iint_{\mathbb{R}_Y^{2n} \times \mathbb{R}_Z^{2n}} a(Y)b^{(\alpha)}(Y)\frac{(Z-Y)^\alpha}{\alpha!}\Sigma_Y\Sigma_Z dYdZ$$

$$+ \sum_{|\alpha|=2,|\gamma|=1} \iint_{\mathbb{R}_Y^{2n} \times \mathbb{R}_Z^{2n}} (ab^{(\alpha)})^{(\gamma)}(Y)\frac{(Z-Y)^{\alpha+\gamma}}{3\alpha!}\Sigma_Y\Sigma_Z dYdZ$$

$$+2 \sum_{|\alpha|=2,|\gamma|=2} \int_0^1 \int_0^1 (1-\theta)\theta^2(1-t)d\theta dt$$

$$\times \iint_{\mathbb{R}_Y^{2n} \times \mathbb{R}_Z^{2n}} (ab^{(\alpha)})^{(\gamma)}(Y+t\theta(Z-Y))\frac{(Z-Y)^{\alpha+\gamma}}{\alpha!\gamma!}\Sigma_Y\Sigma_Z dYdZ$$

The operator norm of both last two terms on the right hand side are estimated by

$$(2.10) \qquad \sum_{|\alpha|=2,|\gamma|=2} || \left(ab^{(\alpha)}\right)^{(\gamma)} ||_{L^\infty}$$

with a constant factor. Writing the first term $\sum \Omega_\alpha$, we use the formula (2.6) with $|\alpha| = 2$. Then we have

$$(2.11) \qquad \sum_{|\alpha|=2} \Omega_\alpha = \frac{1}{2^2} \sum_{|\alpha|=2} \left((\partial_X + \frac{H_X}{i})^\alpha (ab^{(\alpha)}) \right)^{Wick} + \frac{1}{2}(a\Delta b)^{Wick},$$

where the right hand side is ordered according to $q = 0, 1$ in (2.6). In view of (2.7) we can see that R_2 satisfies the desired estimate (1.4). $\Box$

In the rest of this section we shall precise the comments for (1.1) stated in Introduction. We recall that the formula (2.3) and Taylor's formula

$$a(X + Y) = \sum_{|\alpha|\le\ell-1} a^{(\alpha)}(X)Y^\alpha/\alpha! + a_\ell(X,Y)$$

with

$$a_\ell(X,Y) = \ell \sum_{|\alpha|=\ell} \int_0^1 (1-\theta)^{\ell-1} a^{(\alpha)}(X+\theta Y)d\theta Y^\alpha/\alpha!,$$

yield the following (see the proof of Corollary 2.4 of [AM]);

Proposition 2.1. (cf. [S]) *Let $\ell > 0$ be an even integer and let $a(X)$ satisfy $a^{(\alpha)}(X) \in L^\infty$ for $|\alpha| \le \ell + 2N$. Then we have*

$$a^{Wick} = \left(\sum_{k=0}^{\ell/2-1} \frac{1}{k!} \left(\frac{\Delta_X}{4}\right)^k a \right)^w + r_\ell^w,$$

where $||r_\ell^w||_{\mathcal{L}(L^2(\mathbb{R}^n))} \le C_\ell \sum_{\ell\le|\alpha|\le\ell+2N} ||a^{(\alpha)}||_{L^\infty}$ for a constant $C_\ell > 0$ depending only on ℓ.

Making ℓ tend to ∞ we get $a^{Wick} = (e^{\frac{\Delta_X}{4}} a)^w$ and moreover $a^w = (e^{\frac{-\Delta_X}{4}} a)^{Wick}$ *formally, though* both are true for polynomial $a(X)$ (cf., [S]). Admitting those formula we obtain

$$a^{Wick}b^{Wick} = (e^{\frac{\Delta_X}{4}} a)^w (e^{\frac{\Delta_X}{4}} b)^w$$

$$= (e^{-\frac{1}{2i}H_X\cdot\partial_Z}(e^{\frac{\Delta_X}{4}} a(X)e^{\frac{\Delta_Z}{4}} b(Z))|_{Z=X})^w$$

$$= (e^{-\frac{\Delta_X}{4}} \left(e^{-\frac{1}{2i}H_X\cdot\partial_Z}(e^{\frac{\Delta_X}{4}} a(X)e^{\frac{\Delta_Z}{4}} b(Z))|_{Z=X} \right))^{Wick},$$

where the second equality follows from the product formula of Weyl calculus in p.155 of [H]. Noting that

$$-\Delta(f(X)g(X)) = -(\Delta_X + 2\partial_X \cdot \partial_Z + \Delta_Z)f(X)g(Z)|_{Z=X}$$

we formally get (1.1) because

$$e^{-\frac{1}{4}(\Delta_X + 2\partial_X\cdot\partial_Z + \Delta_Z) - \frac{1}{2i}H_X\cdot\partial_Z + \frac{\Delta_X}{4} + \frac{\Delta_Z}{4}} = e^{-\frac{1}{2}(\partial_X\cdot\partial_Z + \frac{H_X}{i}\cdot\partial_Z)}.$$

3 Proof of Theorem 1.2

Take the Littlewood-Paley decomposition

$$(3.1) \qquad \chi(\xi)^2 + \sum_{j=1}^{\infty} \varphi_j^2(\xi) = 1, \quad \varphi_j(\xi) = \varphi(2^{-j}|\xi|),$$

where $\chi(\xi) \in C_0^{\infty}$ and $\varphi \in C_0^{\infty}([1,2])$. Choose $\phi(x), \psi(x) \in C_0^{\infty}([1/3,3])$ such that $\varphi \subset\subset \phi \subset\subset \psi$, where $\phi \subset\subset \psi$ means that $\psi = 1$ on supp ϕ. Define ϕ_j and ψ_j by the same as φ_j. Set $a_j(x, \xi) = a(x, \xi)\phi_j(\xi)$ and set $\tilde{a}_j(x, \xi) = a_j(T_j^{-1}x, T_j\xi)$ with $T_j = 2^{j(\rho+\delta)/2}$. Then for the proof of Theorem 1.2 it suffices to show

$$(3.2) \qquad (\tilde{a}_j^w(x, D)u, u) \geq -C\|u\|^2 \text{ for } u \in \mathcal{S},$$

where $\tilde{a}_j^w(x, D)$ denotes the pseudodifferential operator of Weyl calculus defined by

$$\tilde{a}_j^w(x, D)u = (2\pi)^{-n} \iint \tilde{a}_j\left(\frac{x+y}{2}, \xi\right) e^{i(x-y)\cdot\xi} u(y)\,dy\,d\xi$$

In fact, it follows from (3.1) (with $\varphi_0 = \chi$) that

$$\text{Re } (a(x, D)u, u) = \sum_{j=0}^{\infty} \text{Re } (a(x, D)u, \varphi_j(D)^2 u)$$

$$= \sum_{j=0}^{\infty} \{\text{Re } (\varphi_j a_j(x, D)\psi_j u, \varphi_j u) + \text{Re } (\varphi_j a(x, D)(1 - \psi_j)u, \varphi_j u)\}$$

Write $\varphi_j(D)a(x, D)(1 - \psi_j) = 2^{-j(1-\delta)}\Phi_j(D)a(x, D)(1 - \psi_j(D))$ and note $\Phi_j(\xi) \in S_{1,0}^{1-\delta}$. The symbol of pseudodifferential operator $r_j(x, D) := \Phi_j(D)a(x, D)(1 - \psi_j(D))$ is given by

$$3 \sum_{|\alpha|=3} \int_0^1 \frac{(1-\theta)^2}{\alpha!} \left(\mathrm{O_S} - \iint e^{-iy\cdot\eta}(\partial_\xi^\alpha \Phi_j)(\xi + \theta\eta)(D_x^\alpha a)(x + y, \xi)(1 - \psi_j(\xi))\frac{dy\,d\eta}{(2\pi)^n}\right) d\theta.$$

Here $\mathrm{O_S}$ denotes the oscillatory integral (see [K]). In view of (1.5) and (1.6) we see that $D_x^\alpha a(x, \xi) \in S_{(\rho+\delta)/2,(\rho+\delta)/2}^{2\rho+\delta}$ and $(\partial_\xi^\alpha \Phi_j)(\xi) \in S_{1,0}^{-2-\delta}$. It follows from the integration by parts (see Theorem 3.1 of [K]) that $r_j(x, \xi)$ belongs to a bounded set of $S_{(\rho+\delta)/2,(\rho+\delta)/2}^{2(\rho-1)}$ uniformly with respect to j. By Calderón-Vaillancourt theorem we have

$$\left|\sum_{j=0}^{\infty} \text{Re } (\varphi_j a(x, D)(1 - \psi_j)u, \varphi_j u)\right| \leq C\|u\| \sum_{j=0}^{\infty} 2^{-(1-\delta)j}\|\varphi_j u\| \leq C'\|u\|^2.$$

If N is the smallest integer satisfying $N > n/2$ then the L^2 boundedness of $r_j(x, D)$ follows from the boundedness of derivatives of $r_j(x, \xi)$ up to Nth order. To this end, we need only (1.5) for α, β with

$$(3.3) \qquad |\alpha|, |\beta| \leq \frac{(N+4)(1 + (\rho + \delta)/2)}{2(1 - (\rho + \delta)/2)},$$

(see the proof of Theorem 3.1 of [K]). If we write $\tilde{a}_j(X) = \tilde{a}_j(x, \xi) = a_j(T_j^{-1}x, T_j\xi)$ with $T_j = 2^{j(\rho+\delta)/2}$ then it follows from (1.5) and (1.6) that

$$(3.4) \qquad |\partial_X^\alpha \tilde{a}_j(X)| \leq C_\alpha \quad \text{for } 4 \leq |\alpha|$$

(3.5)
$$|\partial_X^\alpha \tilde{a}_j(X)| \le C_\alpha T_j^{\frac{\rho-\delta}{\rho+\delta}(4-|\alpha|)} \quad \text{for } |\alpha| \le 3$$

If $\tilde{\varphi}_j(\xi) = \check{\varphi}_j(T_j\xi)$ then we have

(3.6)
$$|\partial_\xi^\alpha \tilde{\varphi}_j(\xi)| \le C_\alpha T_j^{-(2/(\rho+\delta)-1)|\alpha|}.$$

If we set $(Tu)(x) = u(Tx)$ for $u \in \mathcal{S}$, then we have

(3.7)
$$a_j(x,D)u = T_j \tilde{a}_j(x,D) T_j^{-1}u.$$

Noting this formula we shall consider

$$\text{Re } (\tilde{\varphi}_j \tilde{a}_j(x,D)\tilde{\psi}_j u, \tilde{\varphi}_j u)$$

instead of those removed tilders. If one write $\tilde{a}_j(x,D) = \tilde{b}_j^w(x,D)$ then it follows from Theorem 18.5.10 of [H]

$$\tilde{b}_j(x,\xi) = \tilde{a}_j(x,\xi) + \frac{i}{2}\sum_{j=1}\partial_{x_j}\partial_{\xi_j}\tilde{a}_j(x,\xi)$$

$$+\pi^{-n}\sum_{|\alpha|=2}\int_0^1 \frac{(1-\theta)}{2^2\alpha!}\left(\text{Os}-\iint e^{-iy\cdot\eta}(D_x^\alpha\partial_\xi^\alpha\tilde{a}_j)(x+\theta y,\xi+\eta)dyd\eta\right)d\theta$$

$$:= \tilde{a}_j(x,\xi) + i\tilde{c}_j(x,\xi) + \tilde{r}_j(x,\xi).$$

Note that $\tilde{r}_j(x,\xi)$ belongs to a bounded set of $S_{0,0}^0$ uniformly with respect to j, and moreover $\tilde{\varphi}_j^2(D)\tilde{c}_j^w(x,D)$ is equal to a selfadjoint operator $(\tilde{c}_j\tilde{\varphi}_j^2)^w(x,D)$ modulo L^2 bounded operator whose norm is independent of j [‡]. Hence there exists a constant $C > 0$ independent of j such that

$$\left|\text{Re } (\tilde{\varphi}_j\tilde{a}_j(x,D)\tilde{\psi}_j u, \tilde{\varphi}_j u) - \text{Re } (\tilde{\varphi}_j\tilde{a}_j^w(x,D)\tilde{\psi}_j u, \tilde{\varphi}_j u)\right| \le C(\|\tilde{\psi}_j u\|\|\tilde{\varphi}_j u\| + \|\tilde{\psi}_j u\|^2).$$

Since the sum of the right hand side with respect to j is estimated above by $\|u\|^2$ with a constant factor, we consider

$$\text{Re } (\tilde{\varphi}_j\tilde{a}_j^w(x,D)\tilde{\psi}_j u, \tilde{\varphi}_j u) = (\tilde{a}_j^w(x,D)\tilde{\varphi}_j u, \tilde{\varphi}_j u) + \text{Re } ([\tilde{\varphi}_j, \tilde{a}_j^w(x,D)]\tilde{\psi}_j u, \tilde{\varphi}_j u).$$

It follows from Theorem 18.5.4 of [H] that the symbol of $[\tilde{\varphi}_j, \tilde{a}_j^w(x,D)]$ is equal to

$$\frac{1}{i}\{\tilde{\varphi}_j, \tilde{a}_j\} + \frac{3}{2^3}\sum_{|\alpha|=3}\int_0^1 \frac{(1-\theta)^2}{\alpha!}\left(\text{Os}-\iiiint e^{-i(y\cdot\zeta-z\cdot\eta)}\times\right.$$

$$\left.\times(\partial_\xi^\alpha\tilde{\varphi}_j)(\xi+\theta\eta)(D_x^\alpha\tilde{a}_j)(x+z,\xi+\zeta) + (\partial_\xi^\alpha\tilde{\varphi}_j)(\xi+\zeta)(D_x^\alpha\tilde{a}_j)(x+\theta y,\xi+\theta\eta)\frac{dyd\eta dzd\zeta}{(\pi)^{2n}}\right)d\theta.$$

It follows from (3.4)-(3.6) that the second term belongs to a bounded set of $S_{0,0}^0$ uniformly with respect to j. Since $(\{\tilde{\varphi}_j, \tilde{a}_j\})^w(x,D)$ is selfadjoint, there exists a constant C independent of j such that

(3.8)
$$\left|\text{Re } ([\tilde{\varphi}_j, \tilde{a}_j^w(x,D)]\tilde{\psi}_j u, \tilde{\varphi}_j u)\right| \le C(\|\tilde{\psi}_j u\|\|\tilde{\varphi}_j u\| + \|\tilde{\psi}_j u\|^2).$$

Since the sum of the right hand side is estimated by $\|u\|^2$ with a constant factor, in view of (3.7) we see that (3.2) is enough for the proof of Theorem 1.2.

[‡]Both facts and (3.8) below follow only from estimates (1.5) with α, β satisfying (3.3) at most.

For the proof of (3.2) we shall write simply $a(X)$ instead of $\tilde{a}_j(X)$. In what follows we need only the fact that $a(X) \geq 0$ and it satisfies

$$(3.9) \qquad |\partial_X^\alpha a(X)| \leq 1 \text{ for } 4 \leq |\alpha| \leq 4 + 2N \qquad (\text{cf., (3.3)}),$$

because of a suitable normalization by a constant factor. It follows from Proposition 3.1 (see also Corollary 2.4 of [AM]) that we have for a constant $C > 0$

$$(3.10) \qquad \left\| \left(a - \frac{\Delta_X}{4} a \right)^{Wick} - a^w \right\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq C .$$

because of the Calderón-Vaillancourt theorem and (3.9). Therefore it suffices to show

$$(3.11) \qquad \left(\left(a - \frac{\Delta_X}{4} a \right)^{Wick} u, u \right) \geq -C' \|u\|^2.$$

By using Lemma 1.3 we shall show

$$\left(a - \frac{\Delta_X}{4} a \right)^{Wick} \equiv \sum_{j=1}^{M} \left(b_j - \frac{\Delta_X}{4} b_j \right)^{Wick} \left(b_j - \frac{\Delta_X}{4} b_j \right)^{Wick}$$

modulo L^2 bounded operator. This formula clearly yields (3.11). Since $b_j^{(\alpha)} \in L^\infty$ ($|\alpha| = 2$) it follows that

$$\left(b_j - \frac{\Delta_X}{4} b_j \right)^{Wick} \left(b_j - \frac{\Delta_X}{4} b_j \right)^{Wick} \equiv b_j^{Wick} b_j^{Wick} - \left(\frac{\Delta_X}{4} b_j \right)^{Wick} b_j^{Wick} - b_j^{Wick} \left(\frac{\Delta_X}{4} b_j \right)^{Wick}$$

We use Proposition 1.1 with (1.3), but a little modified form as follows:

$$a^{Wick} b^{Wick} = (ab)^{Wick} + \sum_{|\alpha|=1} \frac{1}{2} \left(-\partial_Y + H_Y/i \right)^\alpha (ab^{(\alpha)})^{Wick} + R_2,$$

which is obvious by (2.7). We have

$$\left(\frac{\Delta_X}{4} b_j \right)^{Wick} b_j^{Wick} = \left(b_j \frac{\Delta_X}{4} b_j \right)^{Wick} + R_1,$$

where the operator norm R_1 is estimated by

$$\sum_{|\alpha|=|\beta|=1} \| \left((\frac{\Delta_X}{4} b_j) b_j^{(\beta)} \right)^{(\alpha)} \|_{L^\infty} + \sum_{|\gamma|=2} \| \frac{\Delta_X}{4} b_j \|_{L^\infty} \| b_j^{(\gamma)} \|_{L^\infty}$$

with a constant factor, whose two terms are bounded by means of (1.7). We have the similar formula for $b_j^{Wick} \left(\frac{\Delta_X}{4} b_j \right)^{Wick}$. Hence

$$-\left(\frac{\Delta_X}{4} b_j \right)^{Wick} b_j^{Wick} - b_j^{Wick} \left(\frac{\Delta_X}{4} b_j \right)^{Wick} \equiv -2 \left(b_j \frac{\Delta_X}{4} b_j \right)^{Wick}.$$

Now we consider $b_j^{Wick} b_j^{Wick}$ by using the estimate (1.4) and its proof. It follows that

$$b_j^{Wick} b_j^{Wick} = (b_j^2)^{Wick} - \frac{1}{2} (b_j' \cdot b_j')^{Wick} + R_2^j.$$

The remainder term R_2^j is composed of terms estimated in (2.8)-(2.11), by setting $a = b = b_j$. Those coming from (2.8) and (2.9) are bounded from the condition (1.7). We must estimate remainder terms coming from (2.9) and the first term of the right hand side of (2.11) as the sum of R_2^j with respect to j. Note that for $\alpha = \alpha_1 + \alpha_2$ with $|\alpha_j| = 1$ we have

$$a^{(\alpha)} - 2 \sum_{j=1}^{M} b_j^{(\alpha_1)} b_j^{(\alpha_2)} = 2 \sum_{j=1}^{M} b_j b_j^{(\alpha)}.$$

The left hand side is continuous and its second derivatives in the distribution sense belong to L^∞ by means of (1.7). Therefore

$$\left(\sum_{j=1}^{M} b_j b_j^{(\alpha)} \right)^{(\beta)} \in L^\infty \text{ for } \alpha, \beta \text{ with } |\alpha| = |\beta| = 2.$$

If the integration by parts in the arguments preceding (2.10) and (2.11) is done after summing up R_2^j with respect to j, we see that

$$\sum_{j}^{M} b_j^{Wick} b_j^{Wick} \equiv \sum_{j=1}^{M} (b_j^2)^{Wick} - \frac{1}{2} (b_j' \cdot b_j')^{Wick}.$$

Finally we have

$$\sum_{j=1}^{M} \left(b_j - \frac{\Delta_X}{4} b_j \right)^{Wick} \left(b_j - \frac{\Delta_X}{4} b_j \right)^{Wick} \equiv \left(\sum_{j=1}^{M} b_j^2 - \frac{1}{2} b_j' \cdot b_j' - 2b_j \frac{\Delta_X}{4} b_j \right)^{Wick}$$

$$= \left(a - \frac{\Delta_X}{4} a \right)^{Wick}.$$

References

[AM] H. Ando and Y. Morimoto, Wick calculus and the Cauchy problem for some dispersive equations, Osaka J. Math., **39-1** (2002), 123-147.

[BBCP] J.M. Bony, F. Broglia, F. Colombini and L. Pernazza, private communications.

[B] J.M. Bony, Sur l'inégalité de Feffereman-Phong,Seminaire Équations aux Dérivées Partielles, 1998-1999, École Polytech. Palaiseau, 199, Exp. No. III, 16.

[FP] C. Fefferman and D.H. Phong, On positivity of pseudo-differential operators, Proc. Nat. Acad. Sci.,**75** (1978), 4673-4674.

[G] Pengfei Guan, C^2 apriori estimates for degenerate Monge-Ampère equations, Duke Math. J.,**86** (1997), 323-346.

[H] L. Hörmander, The analysis of Linear Partial Differential Operators III, Springer-Verlag, Berlin, Heiderberg, New York, Tokyo (1985).

[K] H. Kumano-go, Pseudo-differential operators, MIT Press, 1982

[L1] N. Lerner, Coherent states and evolution equations, General theory of partial differential equations and microlocal analysis, editors Qi Min-you, L.Rodino, Pitman Research notes 349, Longman, 1995.

[L2] N. Lerner, The Wick calculus of pseudo-differential operators and energy estimates, "New trends in microlocal analysis" (J.M. Bony and M. Morimoto, eds.), Springer-Verlag, Berlin, Heiderberg, New York, Tokyo (1996), 23-37.

[L3] N. Lerner, Energy methods via coherent states and advanced pseudo-differential calculus. Multidimensional complex analysis and partial differential equations (São Carlos, 1995), 177–201, Contemp. Math., 205, Amer. Math. Soc., Providence, RI, 1997.

[S] M. Shubin, Pseudo-differential operators and spectral theory, Springer-Verlag, Berlin, second edition, 2001.

[T] D. Tataru, On the Fefferman-Phong inequality and related problems, Communication in P. D. E.**27**(2002), 2101-2138.

Nicolas Lerner
Irmar, Campus de Beaulieu
Université Rennes 1
35042 Rennes Cedex
France
e-mil: nicolas.lerner@univ-rennes1.fr

Yoshinori Morimoto
Graduate School of Human and Environmental Studies
Kyoto University
606-8501 Kyoto
Japan
e-mail: morimoto@math.h.kyoto-u.ac.jp