<table>
<thead>
<tr>
<th>Title</th>
<th>GLOBAL $L^2$-BOUNDEDNESS THEOREMS FOR A CLASS OF FOURIER INTEGRAL OPERATORS AND THEIR APPLICATION (Recent Trends in Microlocal Analysis)</th>
</tr>
</thead>
<tbody>
<tr>
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GLOBAL $L^2$-BOUNDEDNESS THEOREMS FOR
A CLASS OF FOURIER INTEGRAL OPERATORS
AND THEIR APPLICATION

MITSURU SUGIMOTO

This article is based on the joint work with Michael Ruzhansky (Imperial College) which will appear in [13], [14], [15] and so on.

Fourier integral operators

We consider the following Fourier integral operator:

\[ T u(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\phi(x,y,\xi)} a(x,y,\xi) u(y) \, dy \, d\xi \]

(\(x \in \mathbb{R}^n\), where \(a(x,y,\xi)\) is an amplitude function and \(\phi(x,y,\xi)\) is a real phase function of the form

\[ \phi(x,y,\xi) = x \cdot \xi + \varphi(y,\xi). \]

Note that, by the equivalence of phase function theorem, Fourier integral operators with the local graph condition can always be written in this form locally.

Local $L^2$ mapping property of (1) has been established by Hörmander [9] and Eskin [7]. The aim of this article is to present global $L^2$-boundedness properties of operators (1).

When is $T$ globally $L^2$-bounded?

- **(Asada and Fujiwara [1])** Assume that all the derivatives of \(a(x,y,\xi)\) and all the derivatives of each entry of the matrix

\[ D(\phi) = \begin{pmatrix}
\partial_x \partial_y \phi & \partial_x \partial_\xi \phi \\
\partial_x \partial_\xi \phi & \partial_\xi \partial_\xi \phi
\end{pmatrix}
\]

are bounded. Also assume that \(|\det D(\phi)| \geq C > 0\). Then $T$ is $L^2(\mathbb{R}^n)$-bounded.

This result was used to construct the fundamental solution of Schrödinger equation in the way of Feynman's path integral.

(The result of Kumano-go [12] was used to construct the fundamental solution of hyperbolic equations, and it requires that

\[ J(y,\xi) = \phi(x,y,\xi) - (x - y) \cdot \xi \]

satisfies

\[ \left| \partial_y^\alpha \partial_\xi^\beta J(y,\xi) \right| \leq C_{\alpha\beta}(1 + |\xi|)^{1-|\beta|} \]

for all $\alpha$ and $\beta$.)

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However, there one had to make a quite restrictive and not always natural assumption on the boundedness of $\partial_\xi \partial_\xi \phi$, which fails in many important cases.

The case we have in mind is

$$\phi(x, y, \xi) = x \cdot \xi - y \cdot \psi(\xi),$$

where $\psi(\xi)$ is a smooth function of growth order 1. If we take $\psi(\xi) = \xi$, then we have $\phi(x, y, \xi) = x \cdot \xi - y \cdot \xi$, and the operator $T$ defined by it is a pseudo-differential operator.

We cannot use Asada-Fujiwara's result with our example (2), because the boundedness of the entries of $\partial_\xi \partial_\xi \phi$ fails generally. (We cannot use Kumano-go's either by the same reason.)

Why is the phase function (2) important?

Because it is used to represent a canonical transformations. In fact, if we take $a(x, y, \xi) = 1$, we have

$$Tu(x) = F^{-1}[(Fu)(\psi(\xi))](x)$$

hence

$$T \cdot \sigma(D) = (\sigma \circ \psi)(D) \cdot T.$$

Especially, for a positive and homogeneous function $p(\xi) \in C^\infty(\mathbb{R}^n \setminus 0)$ of degree 1, we have the relation

$$T \cdot (-\Delta) \cdot T^{-1} = p(D)^2$$

if we take

$$\psi(\xi) = p(\xi) \frac{\nabla p(\xi)}{|\nabla p(\xi)|}$$

and assume that the hypersurface

$$\Sigma = \{\xi; p(\xi) = 1\}$$

has non-vanishing Gaussian curvature.

The curvature condition on $\Sigma$ means that the Gauss map

$$\frac{\nabla p}{|\nabla p|}: \Sigma \to S^{n-1}$$

is a global diffeomorphism and its Jacobian never vanishes. (See Kobayashi and Nomizu [11].) Hence, we can construct the inverse $C^\infty$-map $\psi^{-1}(\xi)$ of $\psi(\xi)$ defined by (5). On account of (3), the inverse $T^{-1}$ can be given by replacing $\psi$ by $\psi^{-1}$.

The $L^2$-property of the Laplacian $-\Delta$ is well known in various situations. Our objective is to know the $L^2$-property of the operator $T$, so that we can extract the $L^2$-property of the operator $p(D)^2$ from that of the Laplacian.
Main result

The following is our main result, which is expected to have many applications. For \( m \in \mathbb{R} \), we set
\[
\langle x \rangle^m = (1 + |x|^2)^{m/2}.
\]
Let \( L^2_m(\mathbb{R}^n) \) be the set of functions \( f \) such that the norm
\[
\|f\|_{L^2_m(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |\langle x \rangle^m f(x)|^2 \, dx \right)^{1/2}
\]
is finite.

**Theorem 1.** Let \( \phi(x, y, \xi) = x \cdot \xi + \varphi(y, \xi) \). Assume that
\[
|\det \partial_y \partial_\xi \varphi(y, \xi)| \geq C > 0,
\]
and all the derivatives of entries of \( \partial_y \partial_\xi \varphi \) are bounded. Also assume that
\[
|\partial^\alpha \varphi(y, \xi)| \leq C_\alpha \langle y \rangle^{1-|\alpha|} \quad \text{for all } |\alpha| \geq 1,
\]
\[
|\partial^\alpha_\xi \partial^\beta \varphi(x, y, \xi)| \leq C_{\alpha \beta} \langle x \rangle^{-|\alpha|}
\]
for all \( \alpha, \beta, \) and \( \gamma \). Then \( T \) is bounded on \( L^2_m(\mathbb{R}^n) \) for any \( m \in \mathbb{R} \).

Theorem 1 says that, if amplitude functions \( a(x, y, \xi) \) have some decaying properties with respect to \( x \), we do not need the boundedness of \( \partial_\xi \partial_\phi \) for the \( L^2 \)-boundedness, as required in Asada-Fujiwara [1], and we can have weighted estimates, as well.

The same is true when both phase and amplitude functions have some decaying properties with respect to \( y \).

**Theorem 2.** Let \( \phi(x, y, \xi) = x \cdot \xi + \varphi(y, \xi) \). Assume that
\[
|\det \partial_y \partial_\xi \varphi(y, \xi)| \geq C > 0.
\]
Also assume that
\[
|\partial^\alpha_y \partial^\beta_\xi \varphi(y, \xi)| \leq C_\alpha \langle y \rangle^{1-|\alpha|} \quad \text{for all } |\alpha|, |\beta| \geq 1,
\]
\[
|\partial^\alpha_y \partial^\beta_\xi \partial^\gamma \varphi(x, y, \xi)| \leq C_{\alpha \beta \gamma} \langle x \rangle^{-|\alpha|}
\]
for all \( \alpha, \beta, \) and \( \gamma \). Then \( T \) is bounded on \( L^2_m(\mathbb{R}^n) \) for any \( m \in \mathbb{R} \).

If the amplitude \( a(x, y, \xi) \) is independent of the variable \( x \) or \( y \), the decaying property can be automatically satisfied. Furthermore, we can reduce the regularity assumptions for amplitude and phase functions in this case.

**Theorem 3.** Let \( \phi(x, y, \xi) = x \cdot \xi + \varphi(y, \xi) \) and \( a(x, y, \xi) = a(x, \xi) \). Assume that
\[
|\det \partial_y \partial_\xi \varphi(y, \xi)| \geq C > 0
\]
and each entry \( h(y, \xi) \) of \( \partial_y \partial_\xi \varphi(y, \xi) \) satisfies
\[
|\partial^\alpha_y h(y, \xi)| \leq C_\alpha, \quad |\partial^\beta_\xi h(y, \xi)| \leq C_\beta
\]
for \( |\alpha|, |\beta| \leq 2n + 1 \). Also assume
\[
\partial^\alpha_y \partial^\beta_\xi a(x, \xi) \in L^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)
\]
for one of the followings:

(i) $\alpha, \beta \in \{0, 1\}^n$,  
(ii) $|\alpha|, |\beta| \leq \lfloor n/2 \rfloor + 1$,  
(iii) $|\alpha| \leq \lfloor n/2 \rfloor + 1, \beta \in \{0, 1\}^n$,  
(iv) $\alpha \in \{0, 1\}^n, |\beta| \leq \lfloor n/2 \rfloor + 1$.

Then $T$ is $L^2(\mathbb{R}^n)$-bounded.

Theorem 3 with $\varphi(y, \xi) = -y \cdot \xi$ is a refined version of known results on the $L^2$-boundedness of pseudo-differential operators with non-regular symbols: (i) with $\alpha, \beta \in \{0, 1, 2, 3\}^n$ is due to Calderón and Vaillancourt [3], (ii) is due to Cordes [6], and conditions (iii) with $|\alpha| \leq \lfloor n/2 \rfloor + 1, \beta \in \{0, 1, 2\}^n$, is due to Coifman and Meyer [5].

**Theorem 4.** Let $\phi(x, y, \xi) = x \cdot \xi + \varphi(y, \xi)$ and $a(x, y, \xi) = a(y, \xi)$. Assume that

$$|\partial_y^\alpha \partial_\xi^\beta a(y, \xi)| \leq C_{\alpha\beta},$$

for $|\alpha|, |\beta| \leq 2n + 1$. Also assume that

$$|\det \partial_y \partial_\xi \varphi(y, \xi)| \geq C > 0$$

and each entry $h(y, \xi)$ of $\partial_y \partial_\xi \varphi(y, \xi)$ satisfies

$$|\partial_y^\alpha h(y, \xi)| \leq C_\alpha, \quad |\partial_\xi^\beta h(y, \xi)| \leq C_\beta$$

for $|\alpha|, |\beta| \leq 2n + 1$. Then the operator $T$ is $L^2(\mathbb{R}^n)$-bounded.

**An example of how to use our results**

Kato and Yajima [10] showed that the classical Schrödinger equation

$$\{ (i\partial_t + \Delta_x)u(t, x) = 0, \quad u(0, x) = g(x) \}$$

has the global smoothing estimate

$$||\langle x\rangle^{-1}\langle D\rangle^{1/2}u||_{L^2(\mathbb{R}_t \times \mathbb{R}^n)} \leq C||g||_{L^2(\mathbb{R}_x^n)},$$

where $n \geq 3$.

From this fact, we can extract a similar estimate for generalized Schrödinger equations

$$\{ (i\partial_t - p(D)^2)u(t, x) = 0, \quad u(0, x) = g(x) \}.$$

**Assumption.** $p(\xi) \in C^\infty(\mathbb{R}^n \setminus 0)$ is homogeneous of order 1, $p(\xi) > 0$, and the hypersurface $\Sigma = \{\xi; p(\xi) = 1\}$ has non-vanishing Gaussian curvature.

Remember that we have the relation

$$T^{-1} \cdot p(D)^2 = (-\Delta) \cdot T^{-1}.$$
by (4). Operating $T^{-1}$ from the left hand side of equation (7), we have, by this relation,

\[
\begin{cases}
(i\partial_t - \Delta)T^{-1}u(t,x) = 0, \\
T^{-1}u(0,x) = T^{-1}g(x).
\end{cases}
\]

Hence, from (6) and Theorem 1, we obtain the following conclusion:

**Theorem 5.** Suppose $n \geq 3$. Under the assumption above, the solution $u(t,x)$ to generalized Schrödinger equation (7) has the same global smoothing estimate (6) as the classical one has.

**Remark 1.** Walther [16] consider the case of radially symmetric $p(\xi)^2$. Theorem 5 says that we can treat more general case.

### Smoothing effect with a structure

By using the idea above, we can have more refined global smoothing estimates. In order to state them, we introduce some notations:

- Classical orbit determined by $p(D)^2$:

\[
\begin{cases}
\dot{x}(t) = \nabla_\xi p^2(\xi(t)), & \dot{\xi}(t) = 0 \\
x(0) = 0, & \xi(0) = k.
\end{cases}
\]

(8)

- The set of the path of all classical orbits:

\[
\Gamma_p = \{(x(t),\xi(t)); \text{sol. of (8), } t \geq 0, k \in \mathbb{R}^n \setminus 0\}
\]

\[
= \{(t\nabla p(\xi), \xi); \xi \in \mathbb{R}^n \setminus 0, t > 0\}.
\]

- Notation:

\[
\sigma(x,\xi) \sim \langle x \rangle^a|\xi|^b
\]

\[\iff\]

\[
\begin{cases}
\sigma(x,\xi) \in C^\infty(\mathbb{R}^n_x \times (\mathbb{R}^n_\xi \setminus 0)), \\
\sigma(\lambda x,\xi) = \lambda^a\sigma(x,\xi); (\lambda > 1, |x| \gg 1), \\
\sigma(x,\lambda \xi) = \lambda^b\sigma(x,\xi); (\lambda > 0).
\end{cases}
\]

**Theorem 6.** Suppose $n \geq 2$. Assume

\[
\sigma(x,\xi) = 0 \text{ on } \Gamma_p, \quad \sigma(x,\xi) \sim \langle x \rangle^{-1/2}|\xi|^{1/2}.
\]

Then the solution $u$ to equation (7) satisfies

\[
\|\sigma(X,D)u\|_{L^2(\mathbb{R}_t \times \mathbb{R}^n_x)} \leq C\|g\|_{L^2(\mathbb{R}^n_x)}.
\]
Remark 2. Without the structure condition $\sigma(x, \xi) = 0$ on $\Gamma_p$, we have the estimate in Theorem 6 for the followings:

\[
\begin{align*}
\cdot \sigma(x, \xi) &= \langle x \rangle^{-s} |\xi|^{1/2} \quad (s > 1/2) \quad \text{(Ben-Artzi and Klainerman [2])} \\
\cdot \sigma(x, \xi) &= |x|^\alpha |\xi|^{\alpha} \quad (0 < \alpha < 1/2) \quad \text{(Kato and Yajima [10])}
\end{align*}
\]

We have a similar result for inhomogeneous equations

\[
\begin{cases}
(i \partial_t - p(D)^2) u(t, x) = f(t, x) \\
u(0, x) = 0.
\end{cases}
\]

Theorem 7. Suppose $n \geq 2$. Assume

\[
\begin{align*}
\sigma(x, \xi) &\geq 0, \quad \sigma(x, \xi) = 0 \text{ on } \Gamma_p \\
\sigma(x, \xi) &\sim \langle x \rangle^{1/2} |\xi|.
\end{align*}
\]

Then the solution $u$ to (9) satisfies the estimate

\[
\|\sigma(X, D_x)u\|_{L^2(\mathbb{R} \times \mathbb{R}^n)} \leq C \|\langle x \rangle^{3/2} f\|_{L^2(\mathbb{R} \times \mathbb{R}^n)}.
\]

Combining Theorems 6 and 7, we have an estimate for the equation

\[
\begin{cases}
(i \partial_t - p(D)^2) u(t, x) = f(t, x) \\
u(0, x) = g(x).
\end{cases}
\]

Corollary 8. Suppose $n \geq 2$ and $s, \tilde{s} \geq 0$. Assume

\[
\begin{align*}
\sigma(x, \xi) &\geq 0, \quad \sigma(x, \xi) = 0 \text{ on } \Gamma_p \\
\sigma(x, \xi) &\sim |\xi|.
\end{align*}
\]

Then the solution $u$ to (10) satisfies the estimate

\[
\|\langle x \rangle^{1/2} \sigma(X, D_x)u\|_{H^s(\mathbb{H}^n)} \leq C \|\langle x \rangle^{2s+\tilde{s}+1/2} g\|_{L^2(\mathbb{R}^n)} + C \|\langle x \rangle^{3/2} f\|_{H^s(\mathbb{H}^n)}.
\]
Finally, we refer to further applications. We consider the following nonlinear Schrödinger equation:

\[
\begin{aligned}
(i\partial_t + \Delta_x) u(t, x) &= |\nabla u(t, x)|^N \\
u(0, x) &= g(x), \quad t \in \mathbb{R}, \ x \in \mathbb{R}^n.
\end{aligned}
\]

What is the condition of the initial data \(g(x)\) for equation (11) to have time global solution? There are some answers:

- \(N \geq 3\) (Chihara [4]). Smooth, rapidly decay, and sufficiently small.
- \(N \geq 2\) (Hayashi, Miao and Naumkin [8]). \(g \in H^{[n/2]+5}_x\), rapidly decay, and sufficiently small.

**Question:** Can we weaken the smoothness assumption for \(g(x)\)?

**Answer:** Yes if the non-linear term has a "structure"!

Instead of (11), we consider

\[
\begin{aligned}
(i\partial_t - p(D)^2) u(t, x) &= |\sigma(X, D_x)u|^N \\
u(0, x) &= g(x), \quad t \in \mathbb{R}, \ x \in \mathbb{R}^n,
\end{aligned}
\]

where

\[
\begin{aligned}
\sigma(x, \xi) &\geq 0, \quad \sigma(x, \xi) = 0 \quad \text{on} \quad \Gamma_p, \\
\sigma(x, \xi) &\sim |\xi|.
\end{aligned}
\]

- Examples of nonlinear terms which satisfy (13) in the case \(p(D)^2 = -\Delta_x\):

\[
\sigma(x, \xi) = \left| \frac{x}{|x|} \wedge \xi \right|^2 |\xi|^{-1} \quad \text{for large } |x|
\]

**Theorem 9.** Suppose \(n \geq 2\), \(s > (n+3)/2\), and \(N \geq 3\). Assume that \((x)\langle D_x\rangle^2 g \in L^2\) and its \(L^2\)-norm is sufficiently small. Then equation (12) has a time global solution. (In the case \(N = 2\), we need more structure.)

Key point to the proof of Theorem 9. Use Corollary 8 with \(f = |\sigma(X, D_x)u|^N\). The space \(H^s_t(H^\theta_x)\) is an algebra if \(s > 1/2\) and \(\tilde{s} > n/2\). Then we have

\[
\begin{aligned}
\left\langle x \right\rangle^{3/2} |\sigma(X, D_x)u|^N \leq \left\langle x \right\rangle^{1/2} |\sigma(X, D_x)u|^N \leq \left\langle x \right\rangle^{1/2} |\sigma(X, D_x)u|^N
\end{aligned}
\]

if \(N \geq 3\).
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