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Kyoto University
Distributions of exponential growth with support in a proper convex cone

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1 Introduction

In this talk we treated the space $H'(\mathbb{R}^n, K)$ of distributions of exponential growth. The spaces of distributions of exponential growth for the 1-dimensional case, direct product case or global case were investigated by many authors ([3], [5], [6], [10], [11], [12], [13], [15], [17]). In [3] M.Hasumi studied the space $H(\mathbb{R}^n, \mathbb{R}^n)$ and the dual space $H'(\mathbb{R}^n, \mathbb{R}^n)$ (see Definition 3.2 and Definition 3.5). In [10] M.Morimoto studied the space $H(\mathbb{R}^n, K)$ and the dual space $H'(\mathbb{R}^n, K)$ (see Definition 3.2 and Definition 3.5). The purpose of this talk was to treat the space of distributions of exponential growth supported by a proper convex cone $\overline{\Gamma} \subset \mathbb{R}^n$, (denote by $H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$).

In §3 we shall state the base space $H(\mathbb{R}^n, K)$ and its dual space $H'(\mathbb{R}^n, K)$. The main purpose in this section is to introduce the structure theorem for $H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$, the space of distributions of exponential growth supported by a set $\overline{\Gamma} \subset \mathbb{R}^n$ (Theorem 3.7). Therefore as corollary we obtain the structure theorem for $H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$, where $\overline{\Gamma} \subset \mathbb{R}^n$ is a proper convex cone, (Corollary 3.8), and the result which G.Lysik obtained for the case of direct product support of half lines ([6]). Furthermore we have the decomposition theorem for distributions of exponential growth with support in $\overline{\Gamma}_+ \cup \overline{\Gamma}_-$, (Corollary 3.10).

In §4 we shall characterize the space $H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$ by the heat kernel method (Theorem 4.1), which T.Matsuzawa introduced for the spaces of distributions, ultradistributions and hyperfunctions [2], [7], [8], [9].

In §5 we shall introduce the Paley-Wiener theorem for $H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$. Then we showed that the Fourier-Laplace transform of $T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, K)$ is a holomorphic function constructed by a finite sum of functions which are holomorphic on the domains whose imaginary parts are proper convex cones with

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vertex at the elements of $K$ and with some polynomial growth conditions and conversely such a holomorphic function can be represented by the Fourier-Laplace transform of a distribution of exponential growth $T \in H_{\overline{\Gamma}}'({\mathbb R}^n, K)$. Then we can see that $T$ is constructed by a finite sum of distributions of exponential growth supported by a proper convex cone $\overline{\Gamma}$ (Theorem 5.5). As corollary we have the result which M.Morimoto showed for the 1-dimensional case [10].

In §6 we shall state the space of the image by the Fourier-Laplace transform of $T \in H_{\overline{\Gamma}}'({\mathbb R}^n, K)$. Then by using the Paley-Wiener theorem given in §5, we can obtain the Edge-of-the-Wedge theorem for this space (Theorem 6.10). These results are generalizations of the work which M.Morimoto showed for the case of direct product ([11], Theorem 2).

2 Preliminaries

Definition 2.1. We define some notations:

\[ x = (x_1, \cdots, x_n) \in {\mathbb R}^n, \quad \langle x, \xi \rangle = \sum_{j=1}^{n} x_j \xi_j \quad \text{for } x, \xi \in {\mathbb R}^n, \quad x^2 = \langle x, x \rangle. \]

\[ z = (z_1, \cdots, z_n) \in {\mathbb C}^n, \quad z_j = x_j + \iota y_j, \quad j = 1, \cdots, n. \]

\[ \alpha = (\alpha_1, \cdots, \alpha_n) \in {\mathbb N}^n, \quad D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}, \quad \Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}. \]

\[ E(x, t) = (4\pi t)^{-\frac{d}{2}} \exp(-x^2/4t), \quad t > 0. \]

For $\zeta \in {\mathbb C}^n$, $\zeta = (\zeta_1, \cdots, \zeta_n)$, we put $|\zeta| = \sqrt{|\zeta_1|^2 + \cdots + |\zeta_n|^2}$.

Definition 2.2. Let $K$ be a convex compact set in ${\mathbb R}^n$. Then we define supporting function of $K$ by $h_K(x) = \sup_{\xi \in K} \langle x, \xi \rangle$.

Definition 2.3. Let $\Omega$ be an open set in ${\mathbb C}^n$. We denote by $H(\Omega)$ the space of holomorphic functions on $\Omega$ and by $C(\Omega)$ the space of continuous functions on $\Omega$.

Definition 2.4. $S({\mathbb R}^n)$ is the space of rapidly decreasing $C^\infty$ functions and $S'({\mathbb R}^n)$ is the space of tempered distributions.
**Definition 2.5.** Let $A$ be a set in $\mathbb{R}^n$. Then we denote by $A^o$ the interior of $A$, $\overline{A}$ the closure of $A$, for $\varepsilon > 0$, $A_\varepsilon = \{x \in \mathbb{R}^n; \text{dis}(x, A) \leq \varepsilon\}$ and by $\text{ch}(A)$ convex hull of $A$.

**Definition 2.6.** Let $\Gamma$ be a cone with vertex at 0. If $\overline{\text{ch}\Gamma}$ contains no straight line, then we call $\Gamma$ proper cone.

**Definition 2.7 ([4],[16]).** Let $\Gamma$ be a cone. We put

$$\Gamma' := \{\xi \in \mathbb{R}^n; \langle y, \xi \rangle \geq 0 \text{ for all } y \in \Gamma\}.$$

Then we call $\Gamma'$ dual cone of $\Gamma$.

**Definition 2.8.** Let $\Gamma$ be a cone. Then we denote by $\text{pr}\Gamma$ the intersection of $\Gamma$ and the unit sphere. The cone $\Gamma_1$ is said to be a compact cone in the cone $\Gamma_2$ if $\text{pr}\Gamma_1 \subset \text{pr}\Gamma_2$ and we write $\Gamma_1 \subset \subset \Gamma_2$.

**Proposition 2.9 ([16]).** Following conditions are equivalent:

1. $\Gamma$ is proper cone.
2. $(\Gamma')^o \neq \emptyset$.
3. For any $C \subset (\Gamma')^o$, there exists a number $\sigma = \sigma(C) > 0$ such that $\langle \xi, x \rangle \geq \sigma|\xi||x|$, $\xi \in C$, $x \in \text{ch}\Gamma$.

**Proposition 2.10 ([16]).** $(\Gamma')' = \overline{\text{ch}\Gamma}$ and $(\Gamma_1 \cap \Gamma_2)' = \text{ch}(\Gamma_1' \cup \Gamma_2')$. Furthermore for a convex cone $\Gamma$, we have $\Gamma = \Gamma + \Gamma$.

**Definition 2.11.** Let $\Gamma_+$ be a cone with vertex at 0. Then we put $\Gamma_- = -\Gamma_+$.

**Definition 2.12.** Let $A$ be a set in $\mathbb{R}^n$. We put $S_A^\prime := \{T \in S'(\mathbb{R}^n); \text{supp} T \subset A\}$.

### 3 Distributions of exponential growth

In this section, we shall introduce $H'(\mathbb{R}^n, K)$, the space of distributions of exponential growth, and give the structure theorem of $H'(\mathbb{R}^n, K)$. 
**Definition 3.1.** Let $K$ be a convex compact set in $\mathbb{R}^n$ and $\varepsilon > 0$. Then we define $H_b(\mathbb{R}^n, K_\varepsilon)$ as follows:

$$H_b(\mathbb{R}^n, K_\varepsilon) := \{ \varphi \in C^\infty(\mathbb{R}^n); \sup_{x \in \mathbb{R}^n} |D^p \varphi(x)e^{h_K(x)+\varepsilon|x|} < +\infty, \text{ for } \forall p \in \mathbb{N}^n \}. $$

**Definition 3.2.** We define the spaces $H(\mathbb{R}^n, \mathbb{R}^n)$ and $H(\mathbb{R}^n, K)$ as follows:

$$H(\mathbb{R}^n, \mathbb{R}^n) := \lim_{\varepsilon \to 0} H_b(\mathbb{R}^n, K_\varepsilon), \quad H(\mathbb{R}^n, K) := \lim_{\varepsilon \to 0} H_b(\mathbb{R}^n, K_\varepsilon),$$

where $\lim_{\varepsilon \to 0}$ means projective limit and $\lim_{\varepsilon \to 0}$ means inductive limit.

**Remark 3.3.** Now we give the relations of $H(\mathbb{R}^n, K)$ and the other function spaces:

(i) If $\{0\} \subset K$, then $H(\mathbb{R}^n, K) \subset \mathcal{S}$.

(ii) Let $r \geq 0$, $s \geq 0$, $S_r^s(\mathbb{R}^n)$ be Gel'fand-Shilov space and $S_r(\mathbb{R}^n) = \lim_{s \to \infty} S_r^s(\mathbb{R}^n)$. Then it is known that

$$S_1(\mathbb{R}^n) = \{ f \in C^\infty(\mathbb{R}^n); \exists \delta > 0 \sup_{x \in \mathbb{R}^n} |D^2 f(x)e^{\delta|x|} < \infty \},$$

(for details we refer the reader [12]). Therefore

(a) If $K = \{0\}$, then $H(\mathbb{R}^n, K) = S_1(\mathbb{R}^n)$.

(b) If $\{0\} \subset K$, then $H(\mathbb{R}^n, K) \subset S_1(\mathbb{R}^n)$.

(iii) The space $H(\mathbb{R}^n, K)$ is slightly different from $\mathcal{B}_E$ in [1]. In fact

$$\varphi(x) \in H(\mathbb{R}^n, K) \iff \exists \varepsilon > 0 \forall p \in \mathbb{N}^n \text{ s.t. } \sup_{x \in \mathbb{R}^n} |D^p \varphi(x)e^{h_K(x)+\varepsilon|x|} < \infty.$$

$$\varphi(x) \in \mathcal{B}_E \iff \forall p \in \mathbb{N}^n \exists k > 0 \text{ s.t. } \sup_{x \in \mathbb{R}^n} |D^p \varphi(x)e^{h_K(x)+|x|} < \infty.$$

Therefore if $\{0\} \subset K$, then $H(\mathbb{R}^n, K) \subset \mathcal{B}_E$.

**Remark 3.4.** L. Hörmander treated the base space $\mathcal{S}_f$ so that $\mathcal{D} \subset \mathcal{S}_f \subset H(\mathbb{R}^n, K)$ and the Fourier-Laplace transform of $\mathcal{S}_f$. For the details we refer the reader to [5].
**Definition 3.5.** We denote by $H'(\mathbb{R}^n, \mathbb{R}^n)$ the dual space of $H(\mathbb{R}^n, \mathbb{R}^n)$ and by $H'(\mathbb{R}^n, K)$ the dual space of $H(\mathbb{R}^n, K)$. The elements of $H'(\mathbb{R}^n, \mathbb{R}^n)$ and $H'(\mathbb{R}^n, K)$ are called distributions of exponential growth.

**Definition 3.6.** We put $H'_A(\mathbb{R}^n, K):= \{T \in H'(\mathbb{R}^n, K); \text{supp } T \subset \overline{A}\}$.

Now we have the structure theorem for distributions of exponential growth with support $\overline{A} \subset \mathbb{R}^n$:

**Theorem 3.7 ([14]).** Let $A$ be a set in $\mathbb{R}^n$ and $T \in H'_A(\mathbb{R}^n, K)$. Then for every $\epsilon > 0$ there exist $S(x) \in \mathcal{S}'_A$, $n_0 \in \mathbb{N}$ and $t_j \in K$, $j = 1, 2, \cdots, n_0$ such that

$$T = S(x)e^{\epsilon \sqrt{1+x^2}} \sum_{1 \leq j \leq n_0} e^{t_j x}.$$ 

For $H'_\Gamma(\mathbb{R}^n, K)$, we have the following corollaries:

**Corollary 3.8 ([14]).** Let $\Gamma$ be a proper open convex cone in $\mathbb{R}^n$ and let $T \in H'_\Gamma(\mathbb{R}^n, K)$. Then for any $\epsilon > 0$ there exist $m_\epsilon \in \mathbb{N}$ and bounded continuous functions $F_{\epsilon, \alpha}(x)$, $|\alpha| \leq m_\epsilon$, supp$(F_{\epsilon, \alpha}(x)) \subset \overline{\Gamma}$ such that

$$T = \sum_{|\alpha| \leq m_\epsilon} \left( \frac{\partial}{\partial x} \right)^\alpha \{e^{h(x) + \epsilon |x|} F_{\epsilon, \alpha}(x) \}.$$ 

**Corollary 3.9 ([14]).** Let $\Gamma$ be a proper open convex cone in $\mathbb{R}^n$ and let $T \in H'_\Gamma(\mathbb{R}^n, K)$. Then for any $\epsilon > 0$ there exist $n_0$, a partial differential operator with finite order $P_{\epsilon}(D)$ and a polynomially bounded continuous function $G_\epsilon(x)$, supp $(G_\epsilon(x)) \subset \overline{\Gamma}$ such that

$$T = P_{\epsilon}(D)G_\epsilon(x) \times F^*(x), \quad F^*(x) = e^{\epsilon \sqrt{1+x^2}} \sum_{1 \leq n \leq n_0} e^{t_n x},$$

where $t_n \in K$, $(n = 1, \cdots, n_0)$.

**Corollary 3.10 ([14]).** Let $T \in H'_{I_+ \cup I_-}(\mathbb{R}^n, K)$. Then there exist $T_+ \in H'_{I_+}(\mathbb{R}^n, K)$ and $T_- \in H'_{I_-}(\mathbb{R}^n, K)$ such that

$$T = T_+ + T_-.$$ 

**Remark 3.11.** M.Morimoto obtained this result for the 1-dimensional case in [10].
Example 3.12 (Example for Corollary 3.8). Let $n = 2$, $K = \{(x_1, x_2) \in \mathbb{R}^2; |x| \leq 1\}$ and $\Gamma := \{(x_1, x_2) \in \mathbb{R}^2; x_1^2 - x_2^2 > 0, x_1 > 0\}$. We define $T(x)$ by

$$T(x) = \begin{cases} \sqrt{x_1^2 - x_2^2}e^{|x|}, & x_1^2 - x_2^2 > 0, \ x_1 > 0, \\ 0, & \text{otherwise}. \end{cases}$$

Then $h_K(x) = |x|$, $T(x) \in H^r_\Gamma(\mathbb{R}^2, K)$ and for $\varepsilon > 0$,

$$T(x) = \sqrt{x_1^2 - x_2^2}e^{-\varepsilon|x|}e^{|x|} = F_\varepsilon(x)e^{h_K(x)+\varepsilon|x|},$$

where

$$F_\varepsilon(x) = \begin{cases} \sqrt{x_1^2 - x_2^2}e^{-\varepsilon|x|}, & x_1^2 - x_2^2 > 0, \ x_1 > 0, \\ 0, & \text{otherwise}. \end{cases}$$

Then $F_\varepsilon(x)$ is a bounded continuous function and $\text{supp}(F_\varepsilon) \subset \overline{\Gamma}$.

Example 3.13. Let $n = 1$, $K = \{1\}$ and $\Gamma := (0, \infty)$. We define $T(x)$ by

$$T(x) = \begin{cases} e^x, & x \in \Gamma = (0, \infty), \\ 0, & x \in (-\infty, 0]. \end{cases}$$

Then $T \in H^r_\Gamma(\mathbb{R}, K)$ and for $\varepsilon > 0$

$$T = \sum_{k=0}^{1} \left( \frac{\partial}{\partial x} \right)^k \{F_{\varepsilon,k}(x)e^{x+\varepsilon|x|}\},$$

where $F_{\varepsilon,k}(x) = (-1)^{k+1}\chi_+(x)e^{-\varepsilon|x|}$ and

$$\chi_+(x) = \begin{cases} x, & x \in \Gamma = (0, \infty), \\ 0, & x \in (-\infty, 0]. \end{cases}$$

Then $F_{\varepsilon,k}(x)$ is a bounded continuous function and $\text{supp}(F_{\varepsilon,k}) \subset \overline{\Gamma}$.

4 Distributions of exponential growth supported by a proper convex cone

In this section, we shall characterize $H^r_\Gamma(\mathbb{R}^n, K)$ by the heat kernel method.
Theorem 4.1 ([14]). Let $\Gamma \subset \mathbb{R}^n$ be a proper open convex cone, $T \in H^r_\Gamma(\mathbb{R}^n, K)$ and $U(x, t) = \langle T_y, E(x - y, t) \rangle$. Then $U(x, t) \in C^\infty(\mathbb{R}^n \times (0, \infty))$ satisfying the following conditions:

$$\left(\frac{\partial}{\partial t} - \Delta\right) U(x, t) = 0,$$

(1)

$$U(x, t) \to T, \ (t \to 0_+), \ \text{in} \ H'(\mathbb{R}^n, K),$$

(2)

$$\forall \varepsilon > 0 \ \exists N_\varepsilon > 0 \ \exists C_\varepsilon > 0 \ \text{s.t.} \ |U(x, t)| \leq C_\varepsilon t^{-N_\varepsilon} e^{-\frac{\text{dist}(x, \partial \Gamma)^2}{16} t} e^{h_K(x) + \varepsilon |x|}, \ 0 < t < 1, \ x \in \mathbb{R}^n. \ (3)$$

Conversely, for a function $U(x, t) \in C^\infty(\mathbb{R}^n \times (0, \infty))$ satisfying (1) and (3), there exists a unique $T \in H^r_\Gamma(\mathbb{R}^n, K)$ such that $\langle T_y, E(x - y, t) \rangle = U(x, t)$. 

Corollary 4.2 ([14]). Let $T \in H^r_\Gamma(\mathbb{R}^n, K)$ and $U(x, t) = \langle T_y, E(x - y, t) \rangle$. Then $U(x, t) \in C^\infty(\mathbb{R}^n \times (0, \infty))$ satisfies the following conditions:

$$\left(\frac{\partial}{\partial t} - \Delta\right) U(x, t) = 0,$$

(4)

$$U(x, t) \to T, \ (t \to 0_+), \ \text{in} \ H'(\mathbb{R}^n, K),$$

(5)

$$\forall \varepsilon > 0 \ \exists N \ \exists C > 0 \ \text{s.t.} \ |U(x, t)| \leq C t^{-N} e^{h_K(x) + \varepsilon |x|}, \ 0 < t < 1, \ x \in \mathbb{R}^n \ \text{and} \ U(x, t) \to 0, \ (t \to 0_+), \ \text{uniformly for all compact sets in} \ \mathbb{R}^n \setminus \overline{\Gamma}. \ (6)$$

Conversely, for a function $U(x, t) \in C^\infty(\mathbb{R}^n \times (0, \infty))$ satisfying (4) and (6), there exists a unique $T \in H^r_\Gamma(\mathbb{R}^n, K)$ such that $\langle T_y, E(x - y, t) \rangle = U(x, t)$.

5 Paley-Wiener theorem for $H^r_\Gamma(\mathbb{R}^n, K)$

In this section, we shall see the Paley-Wiener theorem for $H^r_\Gamma(\mathbb{R}^n, K)$. For the 1-dimensional case, it is given in [10].

Definition 5.1. Let $\Gamma$ be a proper open convex cone, $K$ be a compact set and $\varepsilon' > 0$. Then we denote $L$ by

$$L = \left\{ \bigcap_{u \in K} (\{u\} + (\overline{\Gamma})^\circ) \right\}^\circ.$$
**Proposition 5.2.** \( L \neq \emptyset \).

**Definition 5.3 ([10], [16]).** For \( T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, K) \), we define the Fourier-Laplace transform \( \mathcal{L}\mathcal{F}(T) \) of \( T \) by

\[
\mathcal{L}\mathcal{F}(T)(\zeta) := \frac{1}{(2\pi)^\frac{n}{2}} \langle T_{x}, e^{\zeta x} \rangle, \quad \zeta \in \mathbb{C}^n.
\]

The right hand side means

\[
\langle T_{x}, e^{\zeta x} \rangle = \langle T_{x}, \chi(x)e^{\zeta x} \rangle,
\]

where \( \chi(x) \in C^\infty(\mathbb{R}^n) \) which satisfies

\[
\chi(x) = \begin{cases} 
1 & x \in \overline{\Gamma}_{\epsilon} \\
0 & x \not\in \overline{\Gamma}_{2\epsilon}, \quad \epsilon > 0.
\end{cases}
\]

**Definition 5.4.** Let \( \Gamma \) be a proper open convex cone and \( K \) be a compact set. For \( \epsilon > 0 \) and \( u_j \in K \), \( j = 1, \cdots, j_0 \), we set the following notations:

\[
[\Gamma]^j = (\{u_j\} + \overline{\Gamma})^c, \quad [\Gamma]_{-\epsilon}^j = \mathbb{R}^n \setminus (\mathbb{R}^n \setminus [\Gamma]^j)_\epsilon.
\]
**Theorem 5.5 ([14]).** Let \( \Gamma \) be a proper open convex cone, \( K \) be a convex compact set, \( T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, K) \) and \( f(\zeta) = \mathcal{L}\mathcal{F}(T)(\xi + i\eta) \). Then for every \( \epsilon > 0 \) there exist \( j_0 \in \mathbb{N} \), \( l_\epsilon \geq 0 \) and the families \( \{u_j\}_{j=1}^{j_0} \subset K \), \( \{f_j(\zeta)\}_{j=1}^{j_0} \) satisfying the conditions (7), (8), (9):

\[
f_j(\zeta) \in \mathcal{H}(\mathbb{R}^n + i[\Gamma'])^j.
\] (7)

\[
\forall \Gamma_C \Subset \Gamma^o \exists M_{\epsilon, \Gamma_C} \geq 0 \text{ such that } |f_j(\zeta)| \leq M(1 + |\zeta|)^{l_\epsilon}, \quad \zeta \in \mathbb{R}^n + i[\Gamma_C]^j_{-2\epsilon}.
\] (8)

\[
f(\zeta) = \sum_{1 \leq j \leq j_0} f_j(\zeta).
\] (9)

In particular, \( f(\zeta) \in \mathcal{H}(\mathbb{R}^n + iL) \).

Conversely if \( f(\zeta) \in \mathcal{H}(\mathbb{R}^n + iL) \) satisfies the conditions (7), (8) and (9), then there exists a unique \( T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, K) \) such that \( f(\zeta) = \frac{1}{(2\pi)^{n/2}} \langle T_x, e^{i\zeta x} \rangle \).

Furthermore \( T \) is given by the following formula:

\[
T = \sum_{1 \leq j \leq j_0} T_j, \quad T_j \in H'_{\overline{\Gamma}}(\mathbb{R}^n, \{u_j\}),
\] (10)

\[
f_j(\zeta) = \frac{1}{(2\pi)^{n/2}} \langle T_j, e^{i\zeta x} \rangle.
\] (11)

**Corollary 5.6 ([14]).** Let \( \Gamma \) be a proper open convex cone, \( T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, \{0\}) \) and \( f(\zeta) = \mathcal{L}\mathcal{F}(T)(\xi + i\eta) \). Then for \( \epsilon > 0 \) there exists \( l_\epsilon \geq 0 \) satisfying the conditions (12), (13):

\[
f(\zeta) \in \mathcal{H}(\mathbb{R}^n + iL).
\] (12)

\[
\forall \Gamma_C \Subset \Gamma^o \exists M_{\epsilon, \Gamma_C} \geq 0 \text{ such that } |f(\zeta)| \leq M(1 + |\zeta|)^{l_\epsilon}, \quad \zeta \in \mathbb{R}^n + i[\Gamma_C]^j_{-2\epsilon}.
\] (13)

Conversely if \( f(\zeta) \in \mathcal{H}(\mathbb{R}^n + iL) \) satisfies the conditions (12) and (13), then there exists a unique \( T \in H'_{\overline{\Gamma}}(\mathbb{R}^n, \{0\}) \) such that \( f(\zeta) = \frac{1}{(2\pi)^{n/2}} \langle T_x, e^{i\zeta x} \rangle \).
Remark 5.7 (Remark for Corollary 5.8). Now we consider more general Fourier-Laplace transforms. That is, if $T \in \mathcal{D}'$ and $e^{-\eta x}T \in \mathcal{S}'$, then we can define the Fourier-Laplace transform $\mathcal{F}(T)(\zeta)$ of $T$. Furthermore it is known that we can obtain the Paley-Wiener theorem for $T \in \mathcal{D}'$ if $\Gamma_T^\circ$ is not empty where $\Gamma_T := \{\eta \in \mathbb{R}^n; e^{-\langle \cdot, \eta \rangle}T \in \mathcal{S}'\}$ (see Theorem 7.4.2 in [4]).

So we can assert that for the Paley-Wiener theorem for $T \in \mathcal{D}'$ (that is, for Theorem 7.4.2 in [4]) we can take the element of the space $H_{\overline{\Gamma}}'(\mathbb{R}^n, \{0\})$ as $T \in \mathcal{D}'$ if and only if the conditions of Corollary 5.6 are satisfied.

Example 5.8 (Example for Theorem 5.5). Let $n = 2$, $K = \{0\} \times [-1, 1]$ and $\Gamma := \{x = (x_1, x_2) \in \mathbb{R}^2; x_1^2 - x_2^2 > 0, x_1 > 0\} = (\overline{\Gamma})^\circ$. We define $T(x)$ by

$$T(x) = \begin{cases} e^{i|x_2|}, & x_1^2 - x_2^2 > 0, x_1 > 0, \\ 0, & \text{otherwise}. \end{cases}$$

We can see $T \in H_{\overline{\Gamma}}'(\mathbb{R}^2, K)$ and if $\eta \in L := \{\eta = (\eta_1, \eta_2); \{(1, 0)\} + (\overline{\Gamma})^\circ\}$, then

$$\langle T_x, e^{i\zeta x} \rangle = \frac{1}{i\zeta_1(i\zeta_1 + i\zeta_2 + 1)} - \frac{1}{i\zeta_1(i\zeta_1 - i\zeta_2 + 1)} = f_1(\zeta) + f_2(\zeta).$$

Then we can see $f_1(\zeta) \in \mathcal{H}(\mathbb{R}^2 + iL_1)$ and $f_2(\zeta) \in \mathcal{H}(\mathbb{R}^2 + iL_2)$, where

$L_1 := \{\eta = (\eta_1, \eta_2); \{(0, 1)\} + (\overline{\Gamma})^\circ\}$, $L_2 := \{\eta = (\eta_1, \eta_2); \{(0, -1)\} + (\overline{\Gamma})^\circ\}$,

and $L = L_1 \cap L_2$. Now we define

$$T_1 = \begin{cases} e^{x_2}, & x_1 > x_2, x_2 > 0, \\ 0, & \text{otherwise}, \end{cases} \quad T_2 = \begin{cases} e^{-x_2}, & x_1 > -x_2, x_2 < 0, \\ 0, & \text{otherwise}. \end{cases}$$

Then we have $T_1 \in H_{\overline{\Gamma}}'(\mathbb{R}^2, \{(0, 1)\})$, $T_2 \in H_{\overline{\Gamma}}'(\mathbb{R}^2, \{(0, -1)\})$ and

$$\langle T_1, e^{i\zeta x} \rangle = f_1(\zeta), \quad \langle T_2, e^{i\zeta x} \rangle = f_2(\zeta), \quad T = T_1 + T_2.$$
6 Edge-of-the-Wedge theorem

In this section we shall see the Edge-of-the-Wedge theorem for the space of the image by the Fourier-Laplace transform of $T \in H'_F(\mathbb{R}^n, K)$. First we introduce some spaces of holomorphic functions. For details we refer the reader to [10], [11].

**Definition 6.1.** For a subset $A$ of $\mathbb{R}^n$, we define a set $\mathcal{T}(A)$ by $\mathcal{T}(A) = \mathbb{R}^n \times iA$. 

**Definition 6.2.** For a convex compact set $K$ of $\mathbb{R}^n$ and $\epsilon > 0$,

$$Q_\alpha(\mathcal{T}(K_\epsilon)) := \left\{ \varphi(\zeta) \in \mathcal{H}(\mathcal{T}(K_\epsilon^o)) \cap \mathcal{C}(\mathcal{T}(K_\epsilon)); \sup_{\zeta \in \mathcal{T}(K_\epsilon)} |\zeta^\alpha \varphi(\zeta)| < \infty \text{ for } \forall \alpha \in \mathbb{N}^n \right\},$$

$$Q(\mathcal{T}(K)) := \lim_{\epsilon \to 0} Q_\alpha(\mathcal{T}(K_\epsilon)).$$
Definition 6.3. The dual space $Q'(\mathcal{T}(K))$ of $Q(\mathcal{T}(K))$ is called tempered ultrahyperfunctions [10], [11].

We have the following theorem for the spaces $H(\mathbb{R}^n, K)$ and $Q(\mathcal{T}(K))$:

**Theorem 6.4 ([10]).** Let $\varphi(x) \in H(\mathbb{R}^n, K)$. The Fourier inverse transform

$$\mathcal{F}^{-1}(\varphi)(\zeta) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \varphi(x)e^{-i\zeta x}dx$$

establishes a topological isomorphism of $H(\mathbb{R}^n, K)$ onto $Q(\mathcal{T}(K))$. The inverse mapping $\mathcal{F}$ is given by

$$\mathcal{F}(\psi)(x) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \psi(\xi + \eta)e^{i(\xi+\eta)\cdot x}d\xi, \quad \eta \in K^\circ, \quad \psi \in Q_0(\mathcal{T}(K_\epsilon)).$$

**Remark 6.5.** In (14), we notice that $\mathcal{F}(\psi)(x)$ is independent of $\eta \in K^\circ$ by Cauchy's integral theorem.

**Definition 6.6 ([10]).** For $T \in H'(\mathbb{R}^n, K)$, we define the dual Fourier transform $\mathcal{F}(T)$ as a continuous linear functional on $Q(\mathcal{T}(K))$ by the formula

$$\langle \mathcal{F}(T), \psi \rangle = \langle T, \mathcal{F}(\varphi) \rangle, \quad \text{for } \psi \in Q(\mathcal{T}(K)).$$

As a consequence of Theorem 6.4, we have the following theorem:

**Theorem 6.7 ([10]).** The dual Fourier transform (15) gives topological isomorphisms

$$\mathcal{F} : H'(\mathbb{R}^n, K) \to Q'(\mathcal{T}(K)).$$

**Definition 6.8.** Let $K = \{u\}, \psi \in Q_0(\mathcal{T}(K_{\epsilon_1}))$ and assume that $f(\zeta) \in H(\mathbb{R}^n + i\mathbb{L})$ satisfies

$$\forall \epsilon > 0 \exists \epsilon' > 0 \forall C \subset (\overline{\Gamma}^\circ) \exists M_{\epsilon,C} \geq 0 \text{ s.t.}$$

$$|f(\zeta)| \leq M(1 + |\zeta|)^l, \quad \zeta \in \mathbb{R}^n + i[\overline{\Gamma} \cap -\epsilon].$$

Then we define $\langle f(\zeta), \psi(\zeta) \rangle$ by

$$\langle f(\zeta), \psi(\zeta) \rangle := \langle f(\xi + i\eta), \psi(\xi + i\eta) \rangle$$

$$= \int_{\mathbb{R}^n} f(\xi + i\eta)\psi(\xi + i\eta)d\xi,$$

where $\eta \in (\{u\} + (\overline{\Gamma}^\circ) \cap (K_{\epsilon_1})^\circ)$. 
Definition 6.9. Let $K = \{u\}$, $T \in H'_F(\mathbb{R}^n, K)$ and $\psi \in Q(T(K))$, $\psi \in Q_0(T(K_{\epsilon}))$. By Theorem 5.5 and Definition 6.8, we define $\langle \mathcal{L}\mathcal{F}(T)(\zeta), \psi(\zeta) \rangle$ by

$$\langle \mathcal{L}\mathcal{F}(T)(\zeta), \psi(\zeta) \rangle := \langle \mathcal{L}\mathcal{F}(T)(\xi + \eta_0), \psi(\xi + \eta_0) \rangle,$$

where $\eta_0 \in (\{u\} + (\overline{\Gamma})^\circ) \cap (K_{\epsilon_0}^*)$.

Now we can show Edge-of-the-Wedge theorem. For the direct product case, it is given in [11].

Theorem 6.10 (Edge-of-the-Wedge Theorem [14]). Let $\Gamma_1$, $\Gamma_2$ be proper open convex cones in $\mathbb{R}^n$,

$$L_m = \{u_m\} + (\overline{\Gamma}_m)^\circ, \quad m = 1, 2.$$ 

Assume that $F_1(\zeta) \in \mathcal{H}(\mathbb{R}^n + iL_1)$ and $F_2(\zeta) \in \mathcal{H}(\mathbb{R}^n + iL_2)$ satisfy

$$\forall \epsilon > 0 \exists \epsilon_m \geq 0 \forall \Gamma_{Cm} \in (\overline{\Gamma}_m)^\circ \exists M_{\epsilon, \Gamma_{Cm}} \geq 0 \text{ s.t.}$$

$$|F_m(\zeta)| \leq M_{\epsilon, \Gamma_{Cm}}(1 + |\zeta|)^{l_{\epsilon}}, \quad \zeta \in \mathbb{R}^n + i[\overline{\Gamma}_{Cm}]_{-\epsilon}, \quad m = 1, 2,$$

where $[\overline{\Gamma}_{Cm}]_{-\epsilon} = \mathbb{R}^n \setminus (\mathbb{R}^n \setminus (\{u_m\} + \overline{\Gamma}_{Cm})^\circ)_{\epsilon}$.

Let $K$ be a convex compact set which contains the segment with $\{u_1\}$ and $\{u_2\}$ as extremal point. Assume that

$$\langle F_1(\zeta), \psi(\zeta) \rangle = \langle F_2(\zeta), \psi(\zeta) \rangle \quad \forall \psi(\zeta) \in Q(T(K)).$$

Then there exists $F(\zeta) \in \mathcal{H}(\mathbb{R}^n + i(L_1' \cup L_2'))$ such that

$$F(\zeta)|_{(\mathbb{R}^n + iL_1')} = F_1(\zeta), \quad F(\zeta)|_{(\mathbb{R}^n + iL_2')} = F_2(\zeta),$$

where $L_1' = \{u_1\} + (\overline{\Gamma}_1' \cup \overline{\Gamma}_2')^\circ$ and $L_2' = \{u_2\} + (\overline{\Gamma}_1' \cup \overline{\Gamma}_2')^\circ$. Furthermore

(i) if $\overline{\Gamma}_1 \cap \overline{\Gamma}_2 = \{0\}$, then $F(\zeta)$ is polynomial,

(ii) if $\{u_1\} = \{u_2\} (=: \{u\})$, then we have

$$F(\zeta) \in \mathcal{H}(\mathbb{R}^n + i(\{u\} + (\overline{\Gamma}_1' \cup \overline{\Gamma}_2')^\circ))$$

and

$$\forall \epsilon > 0 \exists \epsilon \geq 0 \forall \Gamma_{C} \in (\overline{\Gamma}_1' \cup \overline{\Gamma}_2')^\circ \exists M_{\epsilon, \Gamma_{C}} \geq 0$$

$$|F(\zeta)| \leq M(1 + |\zeta|)^l, \quad \zeta \in \mathbb{R}^n + i[\overline{\Gamma}_{C}]_{-\epsilon},$$

where $[\overline{\Gamma}_{C}]_{-\epsilon} = \mathbb{R}^n \setminus (\mathbb{R}^n \setminus (\{u\} + \overline{\Gamma}_{C})^\circ)_{\epsilon}$. 


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References


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