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Super Generalized Central Limit Theorem —Limit Distributions for Sums of Non-identical Random Variables with Power Laws—

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The power law is present ubiquitously in nature and in our societies. Therefore, it is important to investigate the characteristics of power laws in the current era of big data. In this paper we prove that the superposition of non-identical stochastic processes with power laws converges in density to a unique stable distribution. This property can be used to explain the universality of stable laws that the sums of the logarithmic returns of non-identical stock price fluctuations follow stable distributions.

Introduction— In this world, there are several types of data that follow the power laws. Examples of recent studies include, but are not limited to, the financial market,[1–8] the distribution of people's assets,[9] the distribution of waiting times between earthquake occurrences,[10] and the dependence of the number of wars on its intensity.[11] It is therefore important to investigate the general characteristics of power laws.

In particular, as for the data in the financial market, Mandelbrot[1] firstly argued that the distribution of the price fluctuations of cotton follows a stable law. Since the 1990s, there has been a controversy as to whether the central limit theorem or the generalized central limit theorem (GCLT),[12] as sums of power-law distributions can be applied to the data of the logarithmic return of stock price fluctuations. In particular, Mantegna, Stanley,[2,3] and Yanagawa[4] argued that the logarithmic return follows a stable distribution with the power-law index $\alpha < 2$,[2–4] and later Gopikrishnana et al. denied their own argument by introducing the cubic laws ($\alpha = 3$).[5] Even recently, some researchers[6–8] have argued whether a distribution of logarithmic returns follows power-laws with $\alpha > 2$ or stable laws with $\alpha < 2$. On the other hand, it is necessary to prepare very large data sets to elucidate the true tail behavior of distributions.[13] In this respect, recent study[8] showed that the large and high-frequency arrowhead data of the Tokyo stock exchange (TSE) support stable laws with $1 < \alpha < 2$.

In this study, we show that the sums of logarithmic returns of multiple stock price fluctuations that follow stable laws can be described from a theoretical background. We will extend the GCLT to the sums of independent non-identical stochastic processes. We call this the Super Generalized Central Limit Theorem (SGCLT).

Summary of stable distributions and the GCLT— A probability density function $S(x;\alpha,\beta,\gamma,\mu)$ of random variable X, following a stable distribution,[14] is defined with its characteristic function $\phi(t)$ as:

$$S(x;\alpha,\beta,\gamma,\mu) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(t)e^{-ixt}\,\mathrm{d}x,$$

where $\phi(t;\alpha,\beta,\gamma,\mu)$ is expressed as:

$$\phi(t) = \exp\{i\mu t - \gamma^{\alpha}|t|^{\alpha}(1 - i\beta\,\mathrm{sgn}(t)w(\alpha,t))\},$$

$$w(\alpha,t) = \begin{cases} \tan(\pi\alpha/2) & \text{if } \alpha \neq 1 \\ -2/\pi \log|t| & \text{if } \alpha = 1 \end{cases}.$$

The parameters α, β, γ, and μ are real constants satisfying $0 < \alpha \leq 2$, $-1 \leq \beta \leq 1$, $\gamma > 0$, and denote the indices for power-law in stable distributions, the skewness, the scale parameter, and the location, respectively. When $\alpha = 2$ and $\beta = 0$, the probability density function obeys a normal distribution. Note that explicit forms of stable distributions are not known for general parameters α and β, except for a few cases such as the Cauchy distribution ($\alpha = 1$, $\beta = 0$).

A stable random variable satisfies the following property for the scale and location parameters. A random variable X follows $S(\alpha,\beta,\gamma,\mu)$, when

$$X \stackrel{d}{=} \begin{cases} \gamma X_0 + \mu & \text{if } \alpha \neq 1 \\ \gamma X_0 + \mu + \dfrac{2}{\pi}\beta\gamma\ln\gamma & \text{if } \alpha = 1 \end{cases}, \quad (1)$$

where $X_0 = S(\alpha,\beta,1,0)$. When the random variables X_j satisfy $X_j \sim S(x;\alpha,\beta_j,\gamma_j,0)$, the superposition $Z_n = (X_1 + \cdots + X_n)/n^{\frac{1}{\alpha}}$ of independent random variables $\{X_j\}_{j=1,\ldots,n}$ that have *different* parameters, except for α, is also in the stable distribution family as:

$$Z_n \sim S(\alpha,\hat{\beta},\hat{\gamma},\hat{\mu}), \quad (2)$$

where the parameters $\hat{\beta}$, $\hat{\gamma}$, and $\hat{\mu}$ are expressed as:

$$\hat{\beta} = \frac{\sum_{j=1}^{n}\beta_j\gamma_j^{\alpha}}{\sum_{j=1}^{n}\gamma_j^{\alpha}}, \quad \hat{\gamma} = \left\{\frac{\sum_{j=1}^{n}\gamma_j^{\alpha}}{n}\right\}^{\frac{1}{\alpha}} \quad \text{and}$$

$$\hat{\mu} = \begin{cases} 0 & \text{if } \alpha \neq 1 \\ -\dfrac{2\ln n}{n\pi}\sum_{j=1}^{n}\beta_j\gamma_j & \text{if } \alpha = 1 \end{cases}.$$

We can prove this immediately by using the characteristic function for the sums of random variables expressed as the product of their characteristic functions:

$$\phi(t;\alpha,\hat{\beta},\hat{\gamma},\hat{\mu}) = \prod_{j=1}^{n}\phi(t/n^{\frac{1}{\alpha}};\alpha,\beta_j,\gamma_j,0).$$

Next, we focus on the GCLT. Let f of x be a probability density function of a random variable X for $0 < \alpha < 2$:

$$f(x) \simeq \begin{cases} c_{+}x^{-(\alpha+1)} & \text{for } x \to \infty \\ c_{-}|x|^{-(\alpha+1)} & \text{for } x \to -\infty \end{cases}, \quad (3)$$

with real constants $c_+, c_- > 0$. We define D (*domain of attraction* according to[12]) as a set whose elements are distribution functions satisfying (3), that is, $f \in D_{c_+,c_-} \Leftrightarrow$ (3). Then, according to the GCLT,[12] the superposition of independent, identically distributed random variables $X_1, \ldots, X_n$ converges in density to a unique stable distribution $S(x;\alpha,\beta,\gamma,0)$ for $n \to \infty$, that is,

$$Y_n = \frac{\sum_{i=1}^{n} X_i - A_n}{n^{\frac{1}{\alpha}}} \xrightarrow{d} S(\alpha,\beta,\gamma,0) \text{ for } n \to \infty, \qquad (4)$$

$$A_n = \begin{cases} 0 & \text{if } 0 < \alpha < 1 \\ n^2 \Im \ln(\varphi_X(1/n)) & \text{if } \alpha = 1 \\ n\mathbb{E}[X] & \text{if } 1 < \alpha < 2 \end{cases},$$

where φ_X is a characteristic function of X as the expected value of $\exp(itX)$, $\mathbb{E}[X]$ is the expectation value of X, $\Im$ is an imaginary part of the argument, and parameters β and γ are expressed as:

$$\beta = \frac{c_+ - c_-}{c_+ + c_-}, \quad \gamma = \left\{\frac{\pi(c_+ + c_-)}{2\alpha\sin(\pi\alpha/2)\Gamma(\alpha)}\right\}^{\frac{1}{\alpha}},$$

with Γ being the Gamma function. When $\alpha = 2$, we obtain $\mu = \int x f(x)\,\mathrm{d}x$, $\sigma^2 = \int x^2 f(x)\,\mathrm{d}x$ and the superposition Y_n of the independent, identically distributed random variables converges in density to a normal distribution:

$$Y_n = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\sigma} \xrightarrow{d} \mathcal{N}(0,1), \text{ for } n \to \infty.$$

Our generalization—We consider an extension of this existing theorem for sums of non-identical random variables. In what follows, we assume that the random variables $\{X_i\}_{i=1,\ldots,n}$ satisfy the following two conditions.

Condition 1: The random variables $C_+ > 0$ and $C_- > 0$ obey respectively the distributions $P_{c_+}(c)$ and $P_{c_-}(c)$, and satisfy $\mathbb{E}[C_+] < \infty$ and $\mathbb{E}[C_-] < \infty$.

Condition 2: The probability distribution function $f_i(x)$ of the random variables X_i is satisfied when $0 < \alpha < 2$:

$$f_i(x) \simeq \begin{cases} c_{+i} x^{-(\alpha+1)} & \text{for } x \to \infty \\ c_{-i}|x|^{-(\alpha+1)} & \text{for } x \to -\infty \end{cases},$$

where c_{+i} and c_{-i} are samples obtained by C_+ and C_-, respectively. The main claim of this paper is the following generalization of GCLT: The following superposition S_n of *non-identical* random variables with power laws converges in density to a *unique stable* distribution $S(x;\alpha,\beta^*,\gamma^*,0)$ for $n \to \infty$, where

$$S_n = \frac{\sum_{i=1}^{n} X_i - A_n}{n^{\frac{1}{\alpha}}} \xrightarrow{d} S(x;\alpha,\beta^*,\gamma^*,0) \text{ for } n \to \infty, \qquad (5)$$

$$A_n = \begin{cases} 0 & \text{if } 0 < \alpha < 1 \\ n\sum_{i=1}^{n} \Im \ln(\varphi_i(1/n)) & \text{if } \alpha = 1 \\ \sum_{i=1}^{n} \mathbb{E}[X_i] & \text{if } 1 < \alpha < 2 \end{cases},$$

with φ_i being a characteristic function of X_i as the expected value of $\exp(itX_i)$, and parameters β^*, γ^*, β_i, γ_i are expressed as:

$$\beta^* = \frac{\mathrm{E}_{C_+,C_-}[\beta_i\gamma_i^\alpha]}{\mathrm{E}_{C_+,C_-}[\gamma_i^\alpha]}, \quad \gamma^* = \{\mathrm{E}_{C_+,C_-}[\gamma_i^\alpha]\}^{\frac{1}{\alpha}},$$

$$\beta_i = \frac{c_{+i} - c_{-i}}{c_{+i} + c_{-i}}, \quad \gamma_i = \left\{\frac{\pi(c_{+i} + c_{-i})}{2\alpha\sin(\pi\alpha/2)\Gamma(\alpha)}\right\}^{\frac{1}{\alpha}},$$

where $\mathrm{E}_{C_+,C_-}[X]$ denotes the expectation value of X with respect to the random parameter distributions P_{c_+} and P_{c_-}.

Note that when we have an explicit value $\langle A_n \rangle$ from the given random variables, this result corresponds to the result derived by substituting $\langle c_+ \rangle$ and $\langle c_- \rangle$ into c_+ and c_- in the GCLT β and γ, where $\langle c_+ \rangle = \mathbb{E}_{\mathbb{P}_{\mathbb{C}_+}}[c]$ and $\langle c_- \rangle = \mathbb{E}_{\mathbb{P}_{\mathbb{C}_-}}[c]$.

Outline of the proof—Although the following is not mathematically rigorous, we give the following intuitive proof.

The probability distribution function of the random variables $\{X_j\}_{j=1,\ldots,N}$ satisfying the Conditions 1–2 are expressed as:

$$f_j(x) \simeq \begin{cases} c_{+j} x^{-(\alpha+1)} & \text{for } x \to +\infty \\ c_{-j}|x|^{-(\alpha+1)} & \text{for } x \to -\infty \end{cases},$$

where $c_{+j} > 0$ and $c_{-j} > 0$ satisfy $\mathbb{E}[C_+] > 0$ and $\mathbb{E}[C_-] > 0$. The superposition S_N is then defined as:

$$S_N = \frac{\sum_{j=1}^{N} X_j - A_N}{N^{\frac{1}{\alpha}}},$$

$$A_N = \begin{cases} 0 & \text{if } 0 < \alpha < 1 \\ N\sum_{j=1}^{N} \Im \ln(\varphi_j(1/N)) & \text{if } \alpha = 1 \\ \sum_{j=1}^{N} \mathbb{E}[X_j] & \text{if } 1 < \alpha < 2 \end{cases},$$

where φ_j is a characteristic function of X_j. Let N' be $M \times N$ with a positive integer M and $\{X_{ij}\}_{i=1,\ldots,M,j=1,\ldots,N}$ be *copies* of the stochastic variables X_j in the sense that the characteristic parameters α, c_+, and c_- of the probability density functions of X_{ij} are the *same* as that of X_j. Then, we define the superposition $S_{N'}$ as follows:

$$S_{N'} = \frac{\sum_{i=1}^{M}\sum_{j=1}^{N} X_{ij} - A_{N'}}{N'^{\frac{1}{\alpha}}},$$

$$A_{N'} = \begin{cases} 0 & \text{if } 0 < \alpha < 1 \\ M^2 N\sum_{j=1}^{N} (\Im \ln(\varphi_j(1/(MN)))) & \text{if } \alpha = 1 \\ M\sum_{j=1}^{N} \mathbb{E}[X_j] & \text{if } 1 < \alpha < 2 \end{cases}.$$

Here, we do not consider the convergence of S_N in density for $N \to \infty$, but consider the superposition $S_{N'}$ for $N' \to \infty$, since the superposition S_N converges to the same limiting distribution of $S_{N'}$ if S_N converges in density.

We focus on the convergence in density of $S_{N'}$ for $M \to \infty$ and $N \to \infty$ as follows. With regard to the previous $A_{N'}$ in $S_{N'}$, we express it as $A_{N'} = \sum_{j=1}^{N} A_{M_j}$ with the following A_{M_j} $(j = 1, \ldots, N)$,

$$A_{M_j} = \begin{cases} 0 & \text{if } 0 < \alpha < 1 \\ M^2 N \Im \ln(\varphi_j(1/MN)) & \text{if } \alpha = 1 \\ M\mathbb{E}[X_j] & \text{if } 1 < \alpha < 2 \end{cases}.$$

Here, the superposition $S_{N'}$ is described as:

$$S_{N'} = \frac{\sum_{i=1}^{M}\sum_{j=1}^{N} X_{ij} - A_{N'}}{N'^{\frac{1}{\alpha}}} = \frac{\frac{\sum_{i=1}^{M} X_{i1} - A_{M_1}}{M^{\frac{1}{\alpha}}} + \cdots + \frac{\sum_{i=1}^{M} X_{iN} - A_{M_N}}{M^{\frac{1}{\alpha}}}}{N^{\frac{1}{\alpha}}}.$$

When $\alpha \neq 1$, let Y_{M_j} be the superposition $(\sum_{i=1}^{M} X_{ij} - A_{M_j})/M^{\frac{1}{\alpha}}$. Then, Y_{M_j} converges in density to $S(\alpha, \beta_j, \gamma_j, 0)$ for $M \to \infty$ according to the GCLT (4), that is,

$$Y_{M_j} = \frac{\sum_{i=1}^{M} X_{ij} - A_{M_j}}{M^{\frac{1}{\alpha}}} \xrightarrow{d} S(\alpha, \beta_j, \gamma_j, 0) \text{ for } M \to \infty,$$

where β_j and γ_j are

$$\beta_j = \frac{c_{+j} - c_{-j}}{c_{+j} + c_{-j}}, \quad \gamma_j = \left\{ \frac{\pi(c_{+j} + c_{-j})}{2\alpha \sin(\pi\alpha/2)\Gamma(\alpha)} \right\}^{\frac{1}{\alpha}}.$$

Thus, with the stable property (2), we obtain the convergence of the superposition $S_{N'}$ as follows:

$$\begin{aligned} S_{N'} &= \frac{\sum_{j=1}^{N} Y_{M_j}}{N^{\frac{1}{\alpha}}} \\ &\xrightarrow{d} \frac{\sum_{j=1}^{N} Y_j}{N^{\frac{1}{\alpha}}} \text{ for } M \to \infty, \ (Y_j \sim S(\alpha, \beta_j, \gamma_j, 0)) \\ &\xrightarrow{d} S(x; \alpha, \beta^*, \gamma^*, 0) \text{ for } N \to \infty, \end{aligned}$$

where β^* and γ^* are:

$$\beta^* = \lim_{N\to\infty} \frac{\sum_{j=1}^{N} \beta_j \gamma_j^{\alpha}}{\sum_{j=1}^{N} \gamma_j^{\alpha}} = \frac{\mathrm{E}_{C_+, C_-}[\beta_j \gamma_j^{\alpha}]}{\mathrm{E}_{C_+, C_-}[\gamma_j^{\alpha}]},$$

$$\gamma^* = \lim_{N\to\infty} \left\{ \frac{\sum_{j=1}^{N} \gamma_j^{\alpha}}{N} \right\}^{\frac{1}{\alpha}} = \{\mathrm{E}_{C_+, C_-}[\gamma_j^{\alpha}]\}^{\frac{1}{\alpha}}.$$

This proves that the superposition $S_{N'}$ converges in density to $S(\alpha, \beta^*, \gamma^*, 0)$. Figure 1 illustrates the concept of this proof.

As shown above, the superposition $S_{N'}$ of non-identical stochastic processes converges in density to a unique stable distribution. Since the limiting distribution of $S_{N'}$ is the same as that of S_N, S_N also converges to $S(x, \alpha, \beta^*, \gamma^*, 0)$. When $\alpha = 1$, this statement does not hold because of dependence between M and N in A_{M_j}; however, we find that the limit distribution of the superposition S_N generally converges in density to $S(x; \alpha, \beta^*, \gamma^*, 0)$ in the following numerical examples.

Numerical confirmation— As below, we confirm the claim of SGCLT (5) by some numerical experiments.

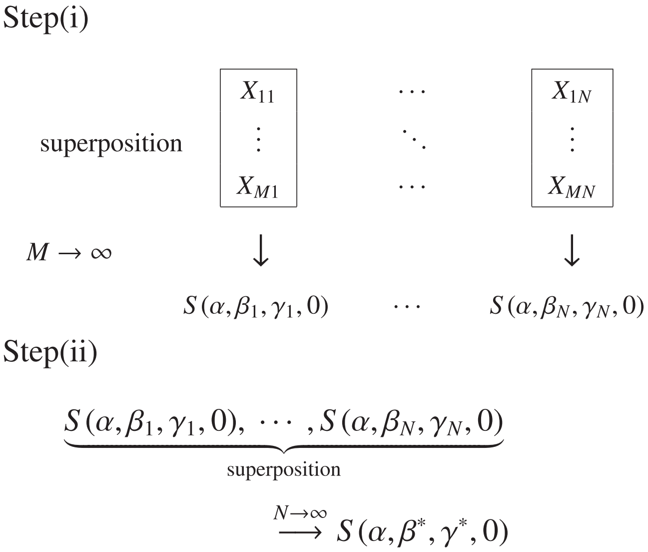

Fig. 1. Concept of the convergence (when $\alpha \neq 1$).

To verify the main claim numerically, we use two kinds of tests: two-samples Kolmogorov–Smirnov (KS) test[15] and two-samples Anderson–Darling (AD) test[16] with 5% significance level. We generate two sets of data by using different methods, and find out the *P-values* of both the tests. Then, unless the null hypothesis is rejected, we judge that both the data follow the same distribution. For the first data, we generate non-identical stochastic processes satisfying Conditions 1–2, and prepare the superposition obtained in the same way as (5). For the second data, we generate random numbers that follow the stable distribution, where the first data converges to the stable distribution according to (5).

For the first data, let us consider the chaotic dynamical system $x_{n+1} = g(x_n)$, where $g(x)$ is defined to be[17] as follows for $0 < \alpha < 2$:

$$g(x) = \begin{cases} \dfrac{1}{\delta_1^2 |x|}\left(\dfrac{|\delta_1 x|^{2\alpha} - 1}{2}\right)^{1/\alpha} & \text{for } x > \dfrac{1}{\delta_1} \\ -\dfrac{1}{\delta_1 \delta_2 |x|}\left(\dfrac{1 - |\delta_1 x|^{2\alpha}}{2}\right)^{1/\alpha} & \text{for } 0 < x < \dfrac{1}{\delta_1} \\ \dfrac{1}{\delta_1 \delta_2 |x|}\left(\dfrac{1 - |\delta_2 x|^{2\alpha}}{2}\right)^{1/\alpha} & \text{for } -\dfrac{1}{\delta_2} < x < 0 \\ -\dfrac{1}{\delta_2^2 |x|}\left(\dfrac{|\delta_2 x|^{2\alpha} - 1}{2}\right)^{1/\alpha} & \text{for } x < -\dfrac{1}{\delta_2} \end{cases}.$$

This mapping has a mixing property and an ergodic invariant density for almost all initial points x_0. One of the authors (KU) obtained the following explicit asymmetric power-law distribution as an invariant density:[17]

$$\rho_{\alpha,\delta_1,\delta_2}(x) = \begin{cases} \dfrac{\alpha \delta_1^{\alpha} x^{\alpha-1}}{\pi(1 + \delta_1^{2\alpha} x^{2\alpha})} & \text{if } x \geq 0 \\ \dfrac{\alpha \delta_2^{\alpha} |x|^{\alpha-1}}{\pi(1 + \delta_2^{2\alpha} |x|^{2\alpha})} & \text{if } x < 0 \end{cases}.$$

This asymmetric distribution behaves as follows for $x \to \pm\infty$:

$$\rho_{\alpha,\delta_1,\delta_2}(x) \simeq \begin{cases} \dfrac{\alpha}{\pi \delta_1^{\alpha}} x^{-(\alpha+1)} & \text{for } x \to +\infty \\ \dfrac{\alpha}{\pi \delta_2^{\alpha}} |x|^{-(\alpha+1)} & \text{for } x \to -\infty \end{cases}.$$

Table I. *P-values* of two tests.

α	$\mathrm{P}_{\delta_1}(\delta)$	$\mathrm{P}_{\delta_2}(\delta)$	N	L	*P-value* (KS test)	*P-value* (AD test)
0.5	1(const)	1(const)	10000	50000	0.122	0.074
	U(1, 2)	U(1, 2)	1000	100000	0.561	0.413
	U(0.5, 1)	U(1, 2)	1000	100000	0.865	0.546
1	1	1	1000	100000	0.226	0.308
	U(1, 2)	U(1, 2)	1000	100000	0.741	0.497
	U(0.5, 1)	U(1, 2)	1000	100000	0.659	0.301
1.5	1	1	1000	100000	0.916	0.529
	U(1, 1.2)	U(1, 1.2)	10000	20000	0.768	0.548
	U(0.5, 1)	U(1.5, 2)	10000	30000	0.108	0.099

Table II. *P-values* of two tests.

α	$\mathrm{P}_{\delta_1}(\delta)$	$\mathrm{P}_{\delta_2}(\delta)$	Random variables	N	L	*P-value* (KS test)	*P-value* (AD test)
0.5	3	1	$X_i - i/N$	2000	10000	0.136	0.110
	3	1	$X_i - \mathrm{Crand}(0,1)$	1000	10000	0.289	0.190
1	3	1	$X_i - i/N$	1000	10000	0.305	0.081
	3	1	$X_i - \mathrm{Crand}(0,1)$	2000	10000	0.145	0.093
1.5	3	1	$X_i - \mathrm{Crand}(0,1)$	1000	10000	0.371	0.286

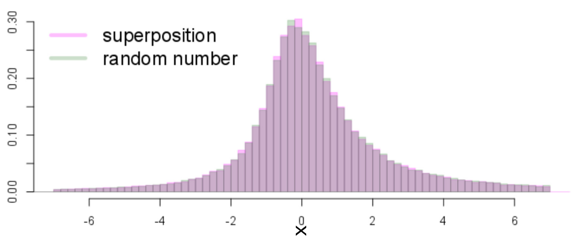

Fig. 2. (Color online) Comparison of two probability densities: the superposition [$N = 10^3$ and $L = 10^5$ for $\alpha = 1$, $\Delta_1 \sim \mathrm{U}(0.5, 1)$, $\Delta_2 \sim \mathrm{U}(1, 2)$] and a stable distribution ($L = 10^5$ for $\alpha = 1$, $\beta^* = 1/3$, $\gamma^* = 1$).

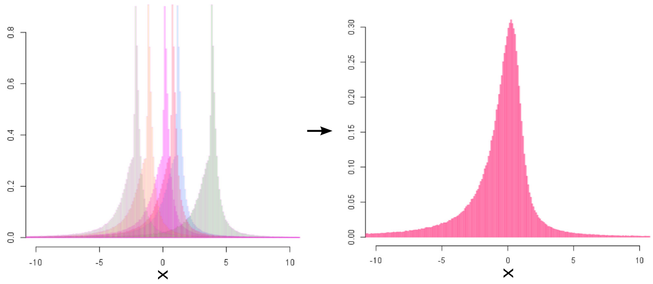

Fig. 3. (Color online) Image of the convergence process: The figure on the left shows some samples of *non-identical* random variables $X_i - \mathrm{Crand}(0, 1)$, where $\alpha = 1$, $\delta_1 = 3$, and $\delta_2 = 1$. Their integration does not have an explicit expression because of the indefinite mean of the Cauchy distribution. However, the sum (the right figure) converges to $S(1, -0.5, 2/3, 0)$.

This is exactly the same expression with the condition of GCLT (3) for random variables in X. Then, by marking the variables δ_1 and δ_2 distributed, we can obtain various *different* distributions with the same power laws. We regard the parameters δ_{1i} and δ_{2i} as random samples obtained from $\Delta_1 \sim \mathrm{P}_{\delta_1}(\delta)$ and $\Delta_2 \sim \mathrm{P}_{\delta_2}(\delta)$, respectively, where $\mathbb{E}[\Delta_1] < \infty$ and $\mathbb{E}[\Delta_2] < \infty$ are satisfied. Then, the parameters c_{+i} and c_{-i} are given as $c_{+i} = \frac{\alpha}{\pi\delta_{1i}^\alpha}$ and $c_{-i} = \frac{\alpha}{\pi\delta_{2i}^\alpha}$, and $\mathbb{E}[C_+] < \infty$ and $\mathbb{E}[C_-] < \infty$ are also satisfied. As shown above, we can get some stochastic processes satisfying Conditions 1–2.

For the second data, the random numbers generated with the following procedure follow a stable distribution.[18] Let Θ and Ω be independent random numbers: Θ is uniformly distributed in $(-\frac{\pi}{2}, \frac{\pi}{2})$, while Ω is exponentially distributed with mean 1. In addition, let R be as follows:

$$R = \begin{cases} \dfrac{\sin(\alpha(\theta_0 + \Theta))}{(\cos(\alpha\theta_0)\cos\Theta)^{1/\alpha}} \left[\dfrac{\cos((\alpha-1)\Theta)}{\Omega}\right]^{(1-\alpha)/\alpha} & (\alpha \neq 1) \\ \dfrac{2}{\pi}\left[\left(\dfrac{\pi}{2} + \beta\Theta\right)\tan\Theta - \beta\log\left(\dfrac{\frac{\pi}{2}\Omega\cos\Theta}{\frac{\pi}{2} + \beta\Theta}\right)\right] & (\alpha = 1) \end{cases},$$

for $0 < \alpha \leq 2$, where $\theta_0 = \arctan(\beta\tan(\pi\alpha/2))$. Then, it follows that $R \sim S(x; \alpha, \beta, 1, 0)$. We get arbitrary stable distributions by using the property (1) about the scale parameter and location.

With the two sets of data obtained accordingly, we see whether the superposition $S_N = (\sum_{i=1}^N X_i - A_N)/N^{1/\alpha}$ numerically converges in density to a stable distribution $S(x; \alpha, \beta^*, \gamma^*, 0)$ or not. Tables I and II show the *P-values* of the KS test and AD test for each α, Δ_1, and Δ_2. The constant L is the length of the sequence, and N is the number of sequences used for the superposition. $\mathrm{U}(a, b)$ indicates the uniform distribution in (a, b). Figure 2 illustrates an example of correspondence when $\alpha = 1$. "Crand(0, 1)" represents the random numbers that follow the standard Cauchy distribution.

As can be seen from Tables I and II, we cannot reject the null hypothesis in any case for α. In other words, the distribution of superposition S_N and the stable distribution $S(x; \alpha, \beta^*, \gamma^*, 0)$ are close enough in density according to our SGCLT. In Fig. 3, we can see that the superposition of non-identical distributed random variables converges.

Remarks– As mentioned at the end of the section *Our generalization*, when we have an explicit value $\langle A_n \rangle$ from given random variables, the result of SGCLT corresponds to the result derived by substituting $\langle c_+ \rangle$ and $\langle c_- \rangle$ for c_+ and c_- in the GCLT parameters β and γ. We have the following interpretation of our outline of proof.

First, let three random variables (c_+, c_-, x) have a joint probability distribution $\mathrm{P}(c'_+, c'_-, x') = \mathrm{P}_{C_+}(c'_+)\mathrm{P}_{C_-}(c'_-) \times f(c'_+, c'_-; x')$, where $\mathrm{P}_{C_+}(c'_+)$ and $\mathrm{P}_{C_-}(c'_-)$ satisfy Condition 1, and $f(c'_+, c'_-; x')$, which satisfies Condition 2 of our SGCLT, is a conditional probability distribution. A conditional probability distribution function satisfies $f(c'_+, c'_-; x') \in D_{c'_+, c'_-}$.

Here, we assume that a unique probability distribution $\mathrm{P}(c'_+, c'_-, x')$ exists, and consider random variable sequences $\{(c_{+i}, c_{-i}, x)\}_i$ that obey $\mathrm{P}(c'_+, c'_-, x')$. Then, the number sequence $\{x_i\}_i$ derived from $\{(c_{+i}, c_{-i}, x)\}_i$ and the relation that $P(c'_+, c'_-, x') = P_{C_+}(c'_+)P_{C_-}(c'_-)f(c'_+, c'_-; x')$ is satisfied. Since the distribution of the probability vector $F(c_+, c_-, x)$ is calculated as

$$P(F') = \iiint \delta(F(c'_+, c'_-, x') - F')P(c'_+, c'_-, x')\, dc'_+\, dc'_-\, dx',$$

for the case $F(c_+, c_-, x) = x$, we have $P(x') = \iint f(c'_+, c'_-; x')P_{C_+}(c'_+)P_{C_-}(c'_-)\, dc'_+\, dc'_-$. If we establish $P(x') \in D_{\langle c_+\rangle,\langle c_-\rangle}$, then we can get the result of SGCLT by applying GCLT to $P(x')$. This is another interpretation of our outline of proof. The relation $P(x') \in D_{\langle c_+\rangle,\langle c_-\rangle}$ is derived as follows:

The following limits do not depend on $c'_-(c'_+)$ for $x \to \infty (x \to -\infty)$ as:

$$\begin{aligned}
&\lim_{x\to\infty} x'^{1+\alpha} \iint f(c'_+, c'_-; x')P_{C_+}(c'_+)P_{C_-}(c'_-)\, dc'_+\, dc'_- \\
&\quad = \int c'_+ P'_{C_+}(c'_+)\, dc'_+ \int P'_{C_-}(c'_-)\, dc'_- \\
&\quad = \int c'_+ P'_{C_+}(c'_+)\, dc'_+ = \langle c_+\rangle, \\
&\lim_{x\to-\infty} |x'|^{1+\alpha} \iint f(c'_+, c'_-; x')P_{C_+}(c'_+)P_{C_-}(c'_-)\, dc'_+\, dc'_- \\
&\quad = \int c'_- P'_{C_-}(c'_-)\, dc'_- \int P'_{C_+}(c'_+)\, dc'_+ \\
&\quad = \int c'_- P'_{C_-}(c'_-)\, dc'_- = \langle c_-\rangle.
\end{aligned}$$

These equations imply

$$\begin{aligned}
P(x') &= \iint f(c'_+, c'_-; x')P_{C_+}(c'_+)P_{C_-}(c'_-)\, dc'_+\, dc'_- \\
&\simeq \begin{cases} \langle c_+\rangle x'^{-(\alpha+1)} & \text{for } x' \to \infty \\ \langle c_-\rangle |x'|^{-(\alpha+1)} & \text{for } x' \to -\infty. \end{cases}
\end{aligned}$$

Thus, we get the relation $P(x') \in D_{\langle c_+\rangle,\langle c_-\rangle}$.

Note that we do not assume that $f(c'_+, c'_-; x')$ is uniquely dependent on c'_+ and c'_-. In other words, the imperative condition that SGCLT can be applied is an asymptotic *tail* behavior of some $f_i(c_{+i}, c_{-i}; x)$ in the limit $x \to \pm\infty$ such as that $f_i \in D_{c_{+i},c_{-i}}$. Note that each characteristic of $f_i(c_{+i}, c_{-i}; x)$ can be *different (non-identical)* as illustrated in Fig. 3, where $f_i(c_{+i}, c_{-i}; x)$ have *different (non-identical)* $\langle A_n\rangle$, which corresponds to non-identical distributions of random variables for the superposition in SGCLT.

Conclusions— We further generalize the GCLT for the sums of independent *non-identical* stochastic processes with the same power-law index α. Our main claim of SGCLT can have more general applications, since various type of different power laws exist in nature. Thus, our SGCLT can support the argument on the ubiquitous nature of stable laws that the logarithmic return of multiple stock price fluctuations would follow a stable distribution with $1 < \alpha < 2$ by regarding them as the sums of *non-identical* random variables with power laws.

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