Title
Super Generalized Central Limit Theorem —Limit Distributions for Sums of Non-identical Random Variables with Power Laws—

Author(s)
Shintani, Masaru; Umeno, Ken

Citation
Journal of the Physical Society of Japan (2018), 87(4)

Issue Date
2018-04-15

URL
http://hdl.handle.net/2433/250230

Right
©2018 The Author(s) This article is published by the Physical Society of Japan under the terms of the Creative Commons Attribution 4.0 License. Any further distribution of this work must maintain attribution to the author(s) and the title of the article, journal citation, and DOI.

Type
Journal Article

Textversion
publisher
Kyoto University
Super Generalized Central Limit Theorem
—Limit Distributions for Sums of Non-identical Random Variables with Power Laws—

Masaru Shintani and Ken Umeno

Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Sakyo, Kyoto 606-8501, Japan

(Received November 2, 2017; revised February 16, 2018; accepted March 2, 2018; published online April 2, 2018)

The power law is present ubiquitously in nature and in our societies. Therefore, it is important to investigate the characteristics of power laws in the current era of big data. In this paper we prove that the superposition of non-identical stochastic processes with power laws converges in density to a unique stable distribution. This property can be used to explain the universality of stable laws that the sums of the logarithmic returns of non-identical stock price fluctuations follow stable distributions.

Introduction—In this world, there are several types of data that follow the power laws. Examples of recent studies include, for example, the distribution of people’s assets, the distribution of waiting times between earthquake occurrences, and the dependence of the number of wars on its intensity. It is therefore important to investigate the general characteristics of power laws.

In particular, as for the data in the financial market, Mandelbrot (1) firstly argued that the distribution of the price fluctuations of cotton follows a stable law. Since the 1990s, there has been a controversy as to whether the central limit theorem or the generalized central limit theorem (GCLT), as sums of power-law distributions can be applied to the data of the logarithmic return of stock price fluctuations. In particular, Mantegna, Stanley, (2,3) and Yanagawa (4) argued that the logarithmic return follows a stable distribution. Even recently, some researchers (5–8) have argued whether a distribution of logarithmic returns follows power-laws with \( \alpha > 2 \) or stable laws with \( \alpha < 2 \). On the other hand, it is necessary to prepare very large data sets to elucidate the true tail behavior of distributions. (13) In this respect, recent study (3) showed that the large and high-frequency arrowhead data of the Tokyo stock exchange (TSE) support stable laws with \( 1 < \alpha < 2 \).

In this study, we show that the sums of logarithmic returns of multiple stock price fluctuations that follow stable laws can be described from a theoretical point of view. We will extend the GCLT to the sums of independent non-identical stochastic processes. We call this the Super Generalized Central Limit Theorem (SGCLT).

Summary of stable distributions and the GCLT—A probability density function \( S(x; \alpha, \beta, \gamma, \mu) \) of random variable \( X \), following a stable distribution, is defined with its characteristic function \( \phi(t) \) as:

\[
S(x; \alpha, \beta, \gamma, \mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(t) e^{-\imath tx} \, dt,
\]

where \( \phi(t; \alpha, \beta, \gamma, \mu) \) is expressed as:

\[
\phi(t; \alpha, \beta, \gamma, \mu) = \exp \left\{ \mu t + \frac{\gamma}{2} \left| t \right|^\alpha \left( 1 \pm \beta \sgn(t) w(\alpha, t) \right) \right\},
\]

\[
w(\alpha, t) = \begin{cases} 
\tan(\pi \alpha / 2) & \text{if } \alpha \neq 1 \\
-2 / \alpha \log |t| & \text{if } \alpha = 1.
\end{cases}
\]

The parameters \( \alpha, \beta, \gamma, \) and \( \mu \) are real constants satisfying \( 0 < \alpha \leq 2, -1 \leq \beta \leq 1, \gamma > 0 \), and denote the indices for power-law in stable distributions, the skewness, the scale parameter, and the location, respectively. When \( \alpha = 2 \) and \( \beta = 0 \), the probability density function obeys a normal distribution. Note that explicit forms of stable distributions are not known for general parameters \( \alpha, \beta \) except for a few cases such as the Cauchy distribution \( (\alpha = 1, \beta = 0) \).

A stable random variable satisfies the following property for the scale and location parameters. A random variable \( X \) follows \( S(\alpha, \beta, \gamma, \mu) \), when

\[
X = \begin{cases} 
\gamma X_0 + \mu & \text{if } \alpha \neq 1 \\
\gamma X_0 + \mu + \frac{\gamma}{\pi} \beta \ln \gamma & \text{if } \alpha = 1,
\end{cases}
\]

where \( X_0 = S(\alpha, \beta, 1, 0) \). When the random variables \( X_j \) satisfy \( X_j \sim S(\alpha, \beta_j, \gamma_j, 0) \), the superposition \( Z_n = (X_1 + \cdots + X_n)/n^{\alpha / 25} \) of independent random variables \( \{X_j\}_{j=1}^{n} \) that have different parameters, except for \( \alpha \), is also in the stable distribution family as:

\[
Z_n \sim S(\alpha, \beta, \gamma, \mu),
\]

where the parameters \( \beta, \gamma, \) and \( \mu \) are expressed as:

\[
\beta = \frac{\sum_{j=1}^{n} \beta_j \gamma_j^\alpha}{\sum_{j=1}^{n} \gamma_j^\alpha}, \quad \gamma = \left[ \frac{\sum_{j=1}^{n} \gamma_j^\alpha}{n} \right]^{\frac{\alpha}{\alpha}} \text{ and }
\]

\[
\mu = \begin{cases} 
0 & \text{if } \alpha \neq 1 \\
-2 \ln n - \sum_{j=1}^{n} \beta_j \gamma_j & \text{if } \alpha = 1.
\end{cases}
\]

We can prove this immediately by using the characteristic function for the sums of random variables expressed as the product of their characteristic functions:

\[
\phi(t; \alpha, \beta, \gamma, \mu) = \prod_{j=1}^{n} \phi(t/n^2; \alpha, \beta_j, \gamma_j, 0).
\]

Next, we focus on the GCLT. Let \( f \) be a probability density function of a random variable \( X \) for \( 0 < \alpha < 2 \):

\[
f(x) \approx \begin{cases} 
c_x x^{-(\alpha+1)} & \text{for } x \to -\infty \\
c_-|x|^{-(\alpha+1)} & \text{for } x \to \infty
\end{cases},
\]

\[
043003-1 \quad \text{©2018 The Author(s)}
\]
with real constants \(c_+, c_- > 0\). We define \(D \) (domain of attraction according to \(1)\) as a set whose elements are distribution functions satisfying (3), that is, \(f \in D_{c_+, c_-} \iff (3)\).

Then, according to the GCLT, \(1)\), the superposition of independent, identically distributed random variables \(X_1, \ldots, X_n\) converges in density to a unique stable distribution \(S(x; \alpha, \beta, \gamma, 0)\) for \(n \to \infty\), that is,

\[
Y_n = \sum_{i=1}^{n} X_i - A_n \xrightarrow[n \to \infty]{d} S(\alpha, \beta, \gamma, 0) \quad \text{for} \quad n \to \infty.
\]

(4)

where \(X_i\) is a characteristic function of \(X\) as the expected value of \(\exp(itX)\). \(\mathbb{E}[X]\) is the expectation value of \(X\). \(\mathfrak{S}\) is an imaginary part of the argument, and parameters \(\beta\) and \(\gamma\) are expressed as:

\[
\beta = \frac{c_+ - c_-}{c_+ + c_-}, \quad \gamma = \left\{ \frac{\pi(c_+ + c_-)}{2\alpha \sin(\pi/2)\Gamma(\alpha)} \right\}^{\frac{1}{2}},
\]

with \(\Gamma\) being the Gamma function. When \(\alpha = 2\), we obtain \(\mu = \int xf(x)dx\), \(\sigma^2 = \int x^2f(x)dx\) and the superposition \(Y_n\) of the independent, identically distributed random variables converges in density to a normal distribution:

\[
Y_n = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n\sigma^2}} \xrightarrow[n \to \infty]{d} \mathcal{N}(0, 1), \quad \text{for} \quad n \to \infty.
\]

**Our generalization**—We consider an extension of this existing theorem for sums of non-identical random variables. In what follows, we assume that the random variables \(\{X_i\}_{i=1,...,n}\) satisfy the following two conditions.

**Condition 1:** The random variables \(C_+ > 0\) and \(C_- > 0\) obey respectively the distributions \(P_{c_+}(c)\) and \(P_{c_-}(c)\), and satisfy \(\mathbb{E}[C_+] < \infty\) and \(\mathbb{E}[C_-] < \infty\).

**Condition 2:** The probability distribution function \(f(x)\) of the random variables \(X_i\) is satisfied when \(0 < \alpha < 2\):

\[
f_i(x) \approx \begin{cases} 
    c_{i+}x^{-(\alpha+1)} & \text{for} \quad x \to \infty \\
    c_{i-}x^{-(\alpha+1)} & \text{for} \quad x \to -\infty
\end{cases},
\]

where \(c_{i+}\) and \(c_{i-}\) are samples obtained by \(C_+\) and \(C_-\), respectively. The main claim of this paper is the following generalization of GCLT: The following superposition \(S_N\) of non-identical random variables with power laws converges in density to a unique stable distribution \(S(x; \alpha, \beta^*, \gamma^*, 0)\) for \(n \to \infty\), where

\[
S_N = \sum_{i=1}^{n} X_i - A_{N^2} \xrightarrow[n \to \infty]{d} S(x; \alpha, \beta^*, \gamma^*, 0) \quad \text{for} \quad n \to \infty,
\]

(5)

where \(\mathfrak{S}\) is a characteristic function of \(X_i\) as the expected value of \(\exp(itX_i)\), and parameters \(\beta^*, \gamma^*, \beta_i, \gamma_i\) are expressed as:

\[
\begin{align*}
\beta^* &= \frac{E_{C_+, C_-}[\gamma^3\beta_3]}{E_{C_+, C_-}[\gamma^2\beta_2]} \quad \gamma^* = \left\{ \frac{E_{C_+, C_-}[\gamma^2]}{\frac{\pi}{2} \sin(\pi/2)\Gamma(\alpha)} \right\}^{\frac{1}{2}}, \\
\beta_i &= \frac{c_{i+} - c_{i-}}{c_{i+} + c_{i-}}, \quad \gamma_i = \left\{ \frac{\pi(c_{i+} + c_{i-})}{2\alpha \sin(\pi/2)\Gamma(\alpha)} \right\}^{\frac{1}{2}},
\end{align*}
\]

where \(E_{C_+, C_-}[X]\) denotes the expectation value of \(X\) with respect to the random parameter distributions \(P_{c_+}\) and \(P_{c_-}\).

Note that when we have an explicit value (\(A_0\)) from the given random variables, this result corresponds to the result derived by substituting \((c_+\) and \((c_-)\) into \(c_+\) and \(c_-\) in the GCLT \(\beta\) and \(\gamma\), where \((c_+) = \mathbb{E}[P_{c_+}[c]]\) and \((c_-) = \mathbb{E}[P_{c_-}[c]].\)

**Outline of the proof**—Although the following is not mathematically rigorous, we give the following intuitive proof.

The probability distribution function of the random variables \(\{X_i\}_{i=1,...,N}\) satisfying the Conditions 1–2 are expressed as:

\[
f_j(x) \approx \begin{cases} 
    c_{j+}x^{-(\alpha+1)} & \text{for} \quad x \to \infty \\
    c_{j-}x^{-(\alpha+1)} & \text{for} \quad x \to -\infty
\end{cases},
\]

where \(c_{j+} > 0\) and \(c_{j-} > 0\) satisfy \(\mathbb{E}[C_+] > 0\) and \(\mathbb{E}[C_-] > 0\). The superposition \(S_N\) is then defined as:

\[
S_N = \frac{\sum_{j=1}^{N} X_j - A_N}{N^2}, \quad \text{if} \quad 0 < \alpha < 1
\]

\[
A_N = \begin{cases} 
    0 & \text{if} \quad 0 < \alpha < 1 \\
    \sum_{j=1}^{N} \mathfrak{S}[X_j] & \text{if} \quad 1 < \alpha < 2
\end{cases}
\]

with \(\mathfrak{S}\) being a characteristic function \(X_j\) of the expected value of \(\exp(itX_j)\), and parameters \(\beta^*, \gamma^*, \beta_i, \gamma_i\) are expressed as:

\[
\begin{align*}
\beta^* &= \frac{E_{C_+, C_-}[\gamma^3\beta_3]}{E_{C_+, C_-}[\gamma^2\beta_2]} \quad \gamma^* = \left\{ \frac{E_{C_+, C_-}[\gamma^2]}{\frac{\pi}{2} \sin(\pi/2)\Gamma(\alpha)} \right\}^{\frac{1}{2}}, \\
\beta_i &= \frac{c_{i+} - c_{i-}}{c_{i+} + c_{i-}}, \quad \gamma_i = \left\{ \frac{\pi(c_{i+} + c_{i-})}{2\alpha \sin(\pi/2)\Gamma(\alpha)} \right\}^{\frac{1}{2}},
\end{align*}
\]

Here, we do not consider the convergence of \(S_N\) in density for \(N \to \infty\), but consider the superposition \(S_N\) for \(N' \to \infty\), since the superposition \(S_N\) converges to the same limiting distribution of \(S_{N'}\) if \(S_N\) converges in density.

We focus on the convergence in density of \(S_N\) for \(M \to \infty\) and \(N' \to \infty\) as follows. With regard to the previous \(A_N\) in \(S_N\), we express it as \(A_N = \sum_{j=1}^{N} A_{M_j}\) with the following \(A_{M_j}\) \((j = 1, \ldots, N),\)
Here, the superposition $S_N$ is described as:

$$S_N = \sum_{i=1}^{M} \sum_{j=1}^{N} X_{ij} - A_{N}$$

where $A_N = \int_0^1 M^2 N^3 \ln(\varphi_j(1/MN)) \, d\varphi_j$ and $M = \mathbb{E}[X]$. For $0 < \alpha < 1$,

$$A_M = \begin{cases} M^2 N^3 \ln(\varphi_j(1/MN)) & \text{if } \alpha = 1 \\ M^2 N^3 & \text{if } 1 < \alpha < 2 \end{cases}$$

When $\alpha \neq 1$, let $Y_M$ be the superposition ($\sum_{i=1}^{M} X_{ij} - A_{M}$) $M^2$. Then, $Y_M$ converges in density to $S(\alpha, \beta, \gamma, 0)$ for $M \to \infty$ according to the GCLT (4), that is,

$$Y_M = \frac{\sum_{i=1}^{M} X_{ij} - A_{M}}{M^2} \overset{d}{\to} S(\alpha, \beta, \gamma, 0) \quad \text{for } M \to \infty.$$

where $\beta_j$ and $\gamma_j$ are

$$\beta_j = \frac{c_{ij} - c_{-j}}{c_{ij} + c_{-j}}, \quad \gamma_j = \frac{\pi (c_{ij} + c_{-j})}{2 \alpha \sin(\pi \alpha/2 \Gamma(\alpha))}.$$

Thus, with the stable property (2), we obtain the convergence of the superposition $S_N$ as follows:

$$S_N = \sum_{i=1}^{M} \sum_{j=1}^{N} X_{ij} - A_{N} \overset{d}{\to} S(\alpha, \beta, \gamma, 0) \quad \text{for } M \to \infty,$$

thus proving that the superposition $S_N$ converges in density to $S(\alpha, \beta^*, \gamma^*, 0)$. Figure 1 illustrates the concept of this proof.

To verify the main claim numerically, we use two kinds of tests: two-samples Kolmogorov–Smirnov (KS) test and two-samples Anderson–Darling (AD) test with $5\%$ significance level. We generate two sets of data by using different methods, and find out the $P$-values of both the tests. Then, unless the null hypothesis is rejected, we judge that both the data follow the same distribution. For the first data, we generate non-identical stochastic processes satisfying Conditions 1–2, and prepare the superposition obtained in the same way as (5). For the second data, we generate random numbers that follow the stable distribution, where the first data converges to the stable distribution according to (5).

For the first data, let us consider the chaotic dynamical system $x_{n+1} = g(x_n)$, where $g(x)$ is defined to be\(^{17}\) as follows:

$$g(x) = \begin{cases} \frac{1}{\delta_1} \left( 1 - \frac{1}{2} \delta_1 x^{1/\alpha} \right)^{1/\alpha} & \text{for } x > \frac{1}{\delta_1} \\ \frac{1}{\delta_2} \left( 1 - \frac{1}{2} \delta_2 x^{1/\alpha} \right)^{1/\alpha} & \text{for } 0 < x < \frac{1}{\delta_1} \\ \frac{1}{\delta_2} \left( 1 - \frac{1}{2} \delta_2 x^{1/\alpha} \right)^{1/\alpha} & \text{for } -\frac{1}{\delta_2} < x < 0 \\ -1 & \text{for } x < -\frac{1}{\delta_2} \end{cases}$$

This mapping has a mixing property and an ergodic invariant density for almost all initial points $x_0$. One of the authors (KU) obtained the following explicit asymmetric power-law distribution as an invariant density:\(^{17}\)

$$\rho_{\alpha, \delta_1, \delta_2}(x) = \begin{cases} \frac{\alpha}{\pi \delta_1^\alpha} x^{(1/\alpha) - 1} & \text{if } x \geq 0 \\ \frac{\alpha}{\pi \delta_2^\alpha} x^{(1/\alpha) - 1} & \text{if } x < 0 \end{cases}$$

This asymmetric distribution behaves as follows for $x \to \pm \infty$:

$$\rho_{\alpha, \delta_1, \delta_2}(x) \propto \begin{cases} \alpha x^{(\alpha - 1)} & \text{for } x \to +\infty \\ \alpha x^{(\alpha - 1)} & \text{for } x \to -\infty \end{cases}$$
This is exactly the same expression with the condition of the GCLT (3) for random variables in X. Then, by marking the variables $\delta_1$ and $\delta_2$ distributed, we can obtain various different distributions with the same power laws. We regard the parameters $\delta_1$ and $\delta_2$ as random samples obtained from $\Delta_1 \sim P_{\alpha}(\delta)$ and $\Delta_2 \sim P_{\beta}(\delta)$, respectively, where $E[\Delta_1] < \infty$ and $E[\Delta_2] < \infty$ are satisfied. Then, the parameters $c_{+\iota}$ and $c_{-\iota}$ are given as $c_{+\iota} = \frac{a}{\sin(\Theta_{+\iota})}$ and $c_{-\iota} = \frac{a}{\sin(\Theta_{-\iota})}$, and $E[C_{\iota}] < \infty$ and $E[C_{\iota}^{-}] < \infty$ are also satisfied. As shown above, we can get some stochastic processes satisfying Conditions 1–2.

For the second data, the random numbers generated with the following procedure follow a stable distribution. Let $\Theta$ and $\Omega$ be independent random numbers: $\Theta$ is uniformly distributed in $(\frac{\pi}{2}, \frac{3\pi}{2})$, while $\Omega$ is exponentially distributed with mean 1. In addition, let $R$ be as follows:

$$R = \begin{cases} \frac{\sin((\alpha - 1)\Theta)}{(\cos(\alpha \Theta) \cos(\Theta))^{1/\alpha}} & (\alpha \neq 1) \\ \frac{\pi}{2} \left( \frac{\pi}{2} + \Theta \tan(\Theta) - \beta \log \left( \frac{\pi}{2} \cos(\Theta) \right) \right) & (\alpha = 1) \end{cases}$$

for $0 < \alpha \leq 2$, where $\theta_0 = \arctan(\beta \tan(\pi \alpha/2))$. Then, it follows that $R \sim S(x; \alpha, \beta, 1, 0)$. We get arbitrary stable distributions by using the property (1) about the scale parameter and location.

With the two sets of data obtained accordingly, we see whether the superposition $S_N = (\sum_{i=1}^{N} X_i - N\theta_0)/N^{1/\alpha}$ numerically converges to a dense stable distribution $S(x; \alpha, \beta^*, \gamma^*, 0)$ or not. Tables I and II show the $P$-values of the KS test and AD test for each $\alpha$, $\Delta_1$, and $\Delta_2$. The constant $L$ is the length of the sequence, and $N$ is the number of sequences used for the superposition. $U(a, b)$ indicates the uniform distribution in $(a, b)$. Figure 2 illustrates an example of correspondence when $\alpha = 1$. “Crand(0, 1)” represents the random numbers that follow the standard Cauchy distribution.

As can be seen from Tables I and II, we cannot reject the null hypothesis in any case for $\alpha$. In other words, the distribution of superposition $S_N$ and the stable distribution $S(x; \alpha, \beta^*, \gamma^*, 0)$ are close enough in density according to our SGCLT. In Fig. 3, we can see that the superposition of non-identical distributed random variables converges.

**Remarks**– As mentioned at the end of the section *Our generalization*, when we have an explicit value ($A_\alpha$) from random variables, the result of SGCLT corresponds to the result derived by substituting $(c_{+\iota})$ and $(c_{-\iota})$ for $c_{+\iota}$ and $c_{-\iota}$ in the GCLT parameters $\beta$ and $\gamma$. We have the following interpretation of our outline of proof.

First, let three random variables $(c_{+, \iota}, c_{-, \iota}, x)$ have a joint probability distribution $P(c_{+, \iota}, c_{-, \iota}, x) = P_{C_{+\iota}}(c_{+, \iota})P_{C_{-\iota}}(c_{-, \iota}) \times f(c_{+, \iota}, c_{-, \iota}, x)$, where $P_{C_{+\iota}}(c_{+, \iota})$ and $P_{C_{-\iota}}(c_{-, \iota})$ satisfy Condition 1, and $f(c_{+, \iota}, c_{-, \iota}, x)$, which satisfies Condition 2 of our SGCLT, is a conditional probability distribution. A conditional probability distribution function satisfies $f(c_{+, \iota}^*, c_{-, \iota}^*, x) = D_{c_{+, \iota}^*}$. Here, we assume that a unique probability distribution $P(c_{+, \iota}^*, c_{-, \iota}^*, x)$ exists, and consider random variable sequences $\{c_{+, \iota}^*, c_{-, \iota}^*, x \}_i$ that obey $P(c_{+, \iota}^*, c_{-, \iota}^*, x)$. Then, the number sequence $\{x_i\}_i$ derived from $\{c_{+, \iota}^*, c_{-, \iota}^*, x\}_i$ and the relation

---

**Table I.** $P$-values of two tests.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$P_{h}(\delta)$</th>
<th>$P_{h}(\delta)$</th>
<th>$N$</th>
<th>$L$</th>
<th>$P$-value (KS test)</th>
<th>$P$-value (AD test)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>1(const)</td>
<td>1(const)</td>
<td>10000</td>
<td>50000</td>
<td>0.122</td>
<td>0.074</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>U(1, 2)</td>
<td>10000</td>
<td>100000</td>
<td>0.561</td>
<td>0.413</td>
</tr>
<tr>
<td>0.5</td>
<td>U(0.5, 1)</td>
<td>U(1, 2)</td>
<td>10000</td>
<td>100000</td>
<td>0.865</td>
<td>0.546</td>
</tr>
</tbody>
</table>

**Table II.** $P$-values of two tests.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$P_{h}(\delta)$</th>
<th>$P_{h}(\delta)$</th>
<th>Random variables</th>
<th>$N$</th>
<th>$L$</th>
<th>$P$-value (KS test)</th>
<th>$P$-value (AD test)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>3</td>
<td>1</td>
<td>$X_i - i/N$</td>
<td>2000</td>
<td>10000</td>
<td>0.136</td>
<td>0.110</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>1</td>
<td>$X_i - \text{Crand}(0, 1)$</td>
<td>10000</td>
<td>100000</td>
<td>0.289</td>
<td>0.190</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>1</td>
<td>$X_i - \text{Crand}(0, 1)$</td>
<td>10000</td>
<td>100000</td>
<td>0.145</td>
<td>0.093</td>
</tr>
<tr>
<td>1.5</td>
<td>3</td>
<td>1</td>
<td>$X_i - \text{Crand}(0, 1)$</td>
<td>10000</td>
<td>100000</td>
<td>0.371</td>
<td>0.286</td>
</tr>
</tbody>
</table>

---
that $P(c'_+, c'_-, x') = P_{C_+}(c'_%)P_{C_-}(c'_-)$ is satisfied. Since the distribution of the probability vector $F(c_+, c_-, x)$ is calculated as

$$P(F) = \int \int \delta(F(c'_+, c'_-, x) - F)P(c'_+, c'_-, x) dc'_+ dc'_- dx,'$$

for the case $F(c_+, c_-, x) = x$, we have $P(x) = \int f(c'_+, c'_-, x)P_{C_+}(c'_+)P_{C_-}(c'_-) dc'_+ dc'_-$. If we establish $P(x) \in D_{(c_+),(c_-)}$, then we can get the result of SGCLT by applying GCLT to $P(x)$. This is another interpretation of our outline of proof. The relation $P(x) \in D_{(c_+),(c_-)}$ is derived as follows:

The following limits do not depend on $c'_+ (c'_-)$ for $x \rightarrow \infty (x \rightarrow -\infty)$ as:

$$\lim_{x \rightarrow \infty} x^{1+\alpha} \int f(c'_+, c'_-, x')P_{C_+}(c'_+)P_{C_-}(c'_-) dc'_+ dc'_-$$

$$= \int c'_+ P_{C_+}(c'_+) dc'_+ \int P_{C_-}(c'_-) dc'_-$$

$$= \int c'_+ P_{C_+}(c'_+) dc'_+ = (c_+),$$

$$\lim_{x \rightarrow -\infty} |x|^{1+\alpha} \int f(c'_+, c'_-, x')P_{C_+}(c'_+)P_{C_-}(c'_-) dc'_+ dc'_-$$

$$= \int c'_- P_{C_-}(c'_-) dc'_- \int P_{C_+}(c'_+) dc'_+$$

$$= \int c'_- P_{C_-}(c'_-) dc'_- = (c_-).$$

These equations imply

$$P(x) = \int \int f(c'_+, c'_-, x')P_{C_+}(c'_+)P_{C_-}(c'_-) dc'_+ dc'_-$$

$$\approx \begin{cases} 
(c_+)x^{-(\alpha+1)} & \text{for } x' \rightarrow \infty \\
(c_-)|x|^{-(\alpha+1)} & \text{for } x' \rightarrow -\infty.
\end{cases}$$

Thus, we get the relation $P(x) \in D_{(c_+),(c_-)}$.

Note that we do not assume that $f(c'_+, c'_-, x)$ is uniquely dependent on $c'_+$ and $c'_-$. In other words, the imperative condition that SGCLT can be applied is an asymptotic tail behavior of some $f_i(c_{\iota+i}, c_{\iota-}; x)$ in the limit $x \rightarrow \pm \infty$ such as that $f_i \in D_{c_{\iota+i}, c_{\iota-}}$. Note that each characteristic of $f_i(c_{\iota+i}, c_{\iota-}; x)$ can be different (non-identical) as illustrated in Fig. 3, where $f_i(c_{\iota+i}, c_{\iota-}; x)$ have different (non-identical) $(\mathbb{F}_i)$, which corresponds to non-identical distributions of random variables for the superposition in SGCLT.

Conclusions—We further generalize the GCLT for the sums of independent non-identical stochastic processes with the same power-law index $\alpha$. Our main claim of SGCLT can have more general applications, since various type of different power laws exist in nature. Thus, our SGCLT can support the argument on the ubiquitous nature of stable laws that the logarithmic return of multiple stock price fluctuations would follow a stable distribution with $1 < \alpha < 2$ by regarding them as the sums of non-identical random variables with power laws.

Acknowledgment The authors would like to thank Dr. Shin-itiro Goto (Kyoto University) for stimulating discussions.