

# Are two nucleons bound in lattice QCD for heavy quark masses? Consistency check with Lüscher's finite volume formula

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On the basis of Lüscher's finite volume formula, a simple test (consistency check or sanity check) is introduced and applied to inspect the recent claims of the existence of the nucleon-nucleon ( $NN$ ) bound state(s) for heavy quark masses in lattice QCD. We show that the consistency between the scattering phase shifts at  $k^2 > 0$  and/or  $k^2 < 0$  obtained from the lattice data and the behavior of phase shifts from the effective range expansion (ERE) around  $k^2 = 0$  exposes the validity of the original lattice data; otherwise, such information is hidden in the energy shift  $\Delta E$  of the two nucleons on the lattice. We carry out this consistency check for all the lattice results in the literature claiming the existence of the  $NN$  bound state(s) for heavy quark masses and find that (i) some of the  $NN$  data show a clear inconsistency between the behavior of ERE at  $k^2 > 0$  and that at  $k^2 < 0$ , (ii) some of the  $NN$  data exhibit a singular behavior of the low-energy parameter (such as the divergent effective range) at  $k^2 < 0$ , (iii) some of the  $NN$  data have the unphysical residue for the bound-state pole in the  $S$  matrix, and (iv) the rest of the  $NN$  data are inconsistent among themselves. Furthermore, we raise a caution of using the ERE in the case of the multiple bound states. Our finding, together with the fake plateau problem previously pointed out by the present authors, brings a serious doubt on the existence of the  $NN$  bound states for pion masses heavier than 300 MeV in the previous studies.

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## I. INTRODUCTION

In recent years, hadron-hadron interactions in lattice QCD have been investigated by two approaches. The first approach is the direct method, where the ground state energy is extracted from the temporal correlation function on a finite lattice volume. If the interaction is attractive at low energies, the energy shift  $\Delta E$  in the center of mass system defined by the ground state energy of two hadrons relative to the sum of hadron masses is always negative in the finite volume: For bound states (scattering states),  $\Delta E$  remains negative (approaches to zero) in the infinite volume limit. If the interaction is repulsive,  $\Delta E$  is positive in the finite volume, and the scattering phase shift at the corresponding energy can be determined via Lüscher's finite volume formula [1]. The second approach is the HAL QCD method [2–4], where the energy-independent nonlocal potential between hadrons is defined and extracted from the spacetime dependence of the Nambu-Bethe-Salpeter (NBS) wave function: Observables such as the binding energies and the scattering phase shifts are obtained by solving the Schrödinger-type

equation with the potential. The HAL QCD method has been extensively applied to various two-hadron systems [5–19] as well as three-hadron systems [20] using the derivative expansion with respect to the nonlocality of potentials.

For the volume larger than the range of the interactions, the asymptotic behavior of the NBS wave function encodes the phase shift of the  $S$  matrix. This phase shift can be extracted from the two-particle energy via Lüscher's finite volume formula [1] or from the potential through the Schrödinger equation [4,6]. As the two methods utilize the property of the same NBS wave function [4,6,21], they in principle give the same results, and they indeed agree quantitatively well with each other in the case of the  $I = 2$   $\pi\pi$  scattering [22], while results for the two nucleons ( $NN$ ) for heavy quark masses show disagreement (for example, see Fig. 8 in Ref. [23]): All studies with the direct method [24–31] indicate that bound states appear in both  $^1S_0$  (dineutron) and  $^3S_1$  (deuteron) channels. On the other hand, the HAL QCD method shows no bound states in both channels for heavy quarks [2–7,9].

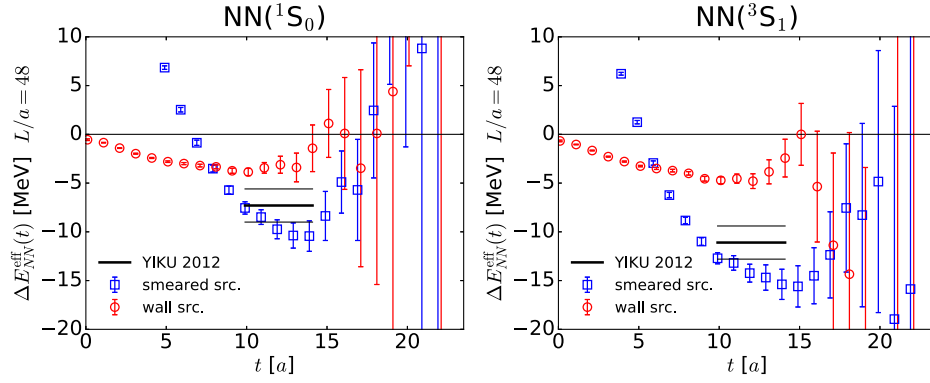


FIG. 1. Effective energy shift  $\Delta E_{NN}^{\text{eff}}(t) = E_{NN}^{\text{eff}}(t) - 2m_N^{\text{eff}}(t)$  in the  $NN(^1S_0)$  channel (left) and the  $NN(^3S_1)$  channel (right) at  $m_\pi = 0.51$  GeV,  $L = 4.3$  fm, and  $a \approx 0.09$  fm, from the smeared source (blue squares) and the wall source (red circles) with the nonrelativistic operator. Here  $E_{NN}^{\text{eff}}(t)$  and  $m_N^{\text{eff}}(t)$  are the effective energy of  $NN$  and the effective mass of  $N$ , respectively. The black solid line represents the fit to the plateau of data in Ref. [25], in which  $\Delta E_{NN}^{\text{eff}}(t)$  was calculated from the same smeared source on the same gauge configurations but with smaller statistics. These figures are adapted from Ref. [33].

In our previous papers [32–34], we have studied the origin of this discrepancy. The direct method is based on the plateau fitting of the effective energy shift  $\Delta E_{\text{eff}}(t)$  as a function of the imaginary time  $t$ . In principle, one can make a reliable calculation by taking sufficiently large  $t$  compared to the inverse of the excitation energy, while relatively small time regions  $t \approx 1$ – $2$  fm were used in all previous studies. We pointed out that the plateau identification in the direct method for such small imaginary time regions suffers a serious systematic bias from the excited-state contaminations. Such a bias is inevitable, since the multibaryon on the lattice has elastic scattering states whose excitation energies approach zero as the lattice volume increases. We have demonstrated this situation, by using mock data, that even the 10% contamination of the excited state can easily produce fake plateaus (which we called “mirage” in [33]) at small  $t$ . Moreover, we have shown that such fake plateaus are indeed observed in lattice data [33]. An example with real data is recapitulated in Fig. 1, where plateaus for  $\Delta E_{\text{eff}}(t)$  are found to be inconsistent between smeared and wall quark sources for  $NN$  source operators.<sup>1</sup>

Inconsistent plateaus in the direct method are also observed in other studies claiming the existence of the  $NN$  bound states. In CalLat2017 [31] and NPL2013 [28,29] papers,  $NN(^1S_0)$  and  $NN(^3S_1)$  were studied with the three-flavor degenerate quark masses at  $m_\pi = 0.81$  GeV and  $a \approx 0.145$  fm. The same gauge configurations with the spatial extension  $L/a = 24$  and 32 are used among these studies. They exclusively employ the smeared quark source<sup>2</sup>

to construct the single-nucleon operator. It is then used to construct several types of two-nucleon source operators: CalLat2017 studied both zero and nonzero displacements between two nucleons. NPL2013 used only a zero displacement between two nucleons, while the center of mass is boosted with the momentum,  $\vec{P} = (2\pi/L) \cdot \vec{n}$ .

For the energy shift  $\Delta E$  at each  $L$ , the results of CalLat2017 and NPL2013 must agree with each other within errors no matter what kind of displacement is taken or what kind of boost is given as long as the boost is not too large. The latter is due to the fact that the data at  $n \equiv |\vec{n}| = 0$  and  $n = 2$  are almost identical on these volumes according to the finite volume formula [1,35].<sup>3</sup> The actual lattice results, however, exhibit a significant inconsistency as shown in Fig. 2.<sup>4</sup> This is another manifestation of the fake plateau (mirage) problem described in Ref. [33].

Note here that CalLat2017 interpreted two values of  $\Delta E$  in their data as the indication of the existence of two states with  $\Delta E < 0$  by speculating that the source with a zero (nonzero) displacement couples dominantly to the deeper (shallower) bound state. However, such an interpretation can be justified only after a sophisticated variational analysis [36] is performed.<sup>5</sup>

The above observations cast strong doubt on the existence of the  $NN$  bound states claimed by using the direct method. Note that the method has been abused in the previous literature without a careful analysis of a large systematic bias due to the excited-state contamination as

<sup>1</sup>A strong sink operator dependence is also observed with the smeared quark source. See Appendix A in Ref. [33].

<sup>2</sup>While both CalLat2017 and NPL2013 employed the Gaussian smearing, the detailed implementations are slightly different from each other: CalLat2017 employed the Coulomb gauge fixing, while NPL2013 employed the gauge covariant smearing. Parameters for the Gaussian are also different.

<sup>3</sup>Their difference is less than 1.0% (0.2%) at  $\Delta E \leq -15$  MeV for  $L/a = 24$  (32).

<sup>4</sup>In the  $NN(^1S_0)$  channel on  $L/a = 24$ , datum corresponding to the nonzero displacement was not given in CalLat2017.

<sup>5</sup>Also, the ERE used by CalLat2017 for two states with  $\Delta E < 0$  cannot be theoretically justified as will be discussed in the next section and Appendix B.

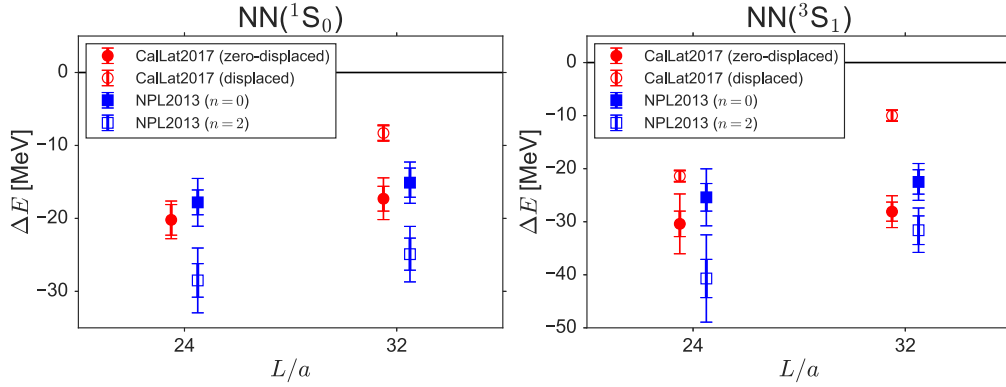


FIG. 2. The energy shifts  $\Delta E$  on the  $L/a = 24$  and  $32$  lattices, in the  $NN(^1S_0)$  channel (left) and the  $NN(^3S_1)$  channel (right), from CalLat2017 (red circles) and from NPL2013 (blue squares). Inner and outer error bars represent the statistical errors and statistical and systematic errors added in quadrature, respectively. Red solid (open) circles for CalLat2017 are obtained from a zero (nonzero) displaced two-nucleon source operator in the center of mass system. Blue solid (open) squares for NPL2013 are obtained from a zero displaced one in the center of mass system (the  $n = 2$  boosted system).

discussed in Ref. [33]. For a further inspection of the results obtained by the direct method, we introduce an alternative and simpler test (consistency check or sanity check) in this paper on the basis of Lüscher's finite volume formula. The basic idea is to investigate the behaviors of the scattering phase shifts in the region of negative squared momentum  $k^2 < 0$ : Consistency between the lattice data as a function of  $k^2$  and the effective range expansion (ERE) around  $k^2 = 0$  exposes the reliability or unreliability of the lattice data, the information otherwise hidden in the energy shift  $\Delta E$ .

In Sec. II, we discuss the theoretical basis behind our consistency check. In Sec. III, we summarize all the  $NN$  data sets to be analyzed in this paper, together with tables of numerical data in Appendix D. They are taken from the previous literature claiming the  $NN$  bound states for heavy quarks. In Sec. IV, consistency checks of these  $NN$  data are presented in detail. Section V is devoted to the conclusion and discussions. In Appendix A, we demonstrate typical behaviors of the phase shift using analytic solutions for the square-well potential. The phase shifts of NPL2013 and CalLat2017 will not be considered in the main text but given in Appendix B, as the mirage problems are already observed. Typical examples of the phase shifts with hyperons are presented in Appendix C. Data used in the paper are collected in Appendix D. We note that a preliminary account of this study was given in Ref. [37].

## II. FINITE VOLUME FORMULA

Lüscher's finite volume formula [1] (and the extensions thereof, e.g., for boosted systems [35] and arbitrary spin or partial waves [38,39]) provides a relation between the scattering phase shifts and the energies on a finite box. If we focus on the elastic  $S$ -wave scattering of two baryons with identical mass  $m$  in the center of mass system, the scattering phase shift  $\delta_0(k)$  is given by

$$k \cot \delta_0(k) = \frac{1}{\pi L} \sum_{\vec{n} \in \mathbb{Z}^3} \frac{1}{\vec{n}^2 - q^2}, \quad q = \frac{kL}{2\pi}, \quad (1)$$

where  $k$  is defined through  $\Delta E = E_{BB} - 2m_B \equiv 2\sqrt{k^2 + m_B^2} - 2m_B$  with  $E_{BB}$  being the energy of the two-baryon state measured in lattice QCD on a finite box with the spatial extension  $L$ . Only the discrete sets of points  $(k^2, k \cot \delta_0(k))$  which satisfy Lüscher's finite volume formula are realized on a given volume. Vice versa, by measuring the energy of the two-particle system on a box, the scattering phase shift at the corresponding energy can be extracted from lattice QCD. If the interaction between two hadrons is attractive, we have  $\Delta E < 0$  ( $k^2 < 0$ ), so that Eq. (1) provides a way to make the analytic continuation of  $k \cot \delta_0(k)$  to the negative  $k^2$  region.

The relation between  $k^2$  and  $k \cot \delta_0(k)$  characterizes the underlying baryon-baryon interaction at low energies, which can be best seen through the ERE around  $k^2 = 0$ :

$$k \cot \delta_0(k) = \frac{1}{a_0} + \frac{r_0}{2} k^2 + \sum_{n=2}^{\infty} P_0^{(n)} k^{2n}, \quad (2)$$

where  $a_0$ ,  $r_0$ , and  $P_0^{(n)}$  are the scattering parameters representing the scattering length, the effective range, and shape parameters, respectively.

In the upper panels in Fig. 3, we illustrate the ERE up to next-to-leading order (NLO) by the red lines in which the empirical  $NN$  scattering lengths ( $a_0$ ) and effective ranges ( $r_0$ ) are used. Figure 3 (upper left) corresponds to the  $NN(^1S_0)$  case with  $a_0 m_\pi = 16.8$  and  $r_0 m_\pi = 1.9$  with no bound state in the infinite volume ( $L = \infty$ ). In Fig. 3 (upper right), we show the ERE line corresponding to the  $NN(^3S_1)$  case with  $a_0 m_\pi = -3.8$  and  $r_0 m_\pi = 1.3$ . The

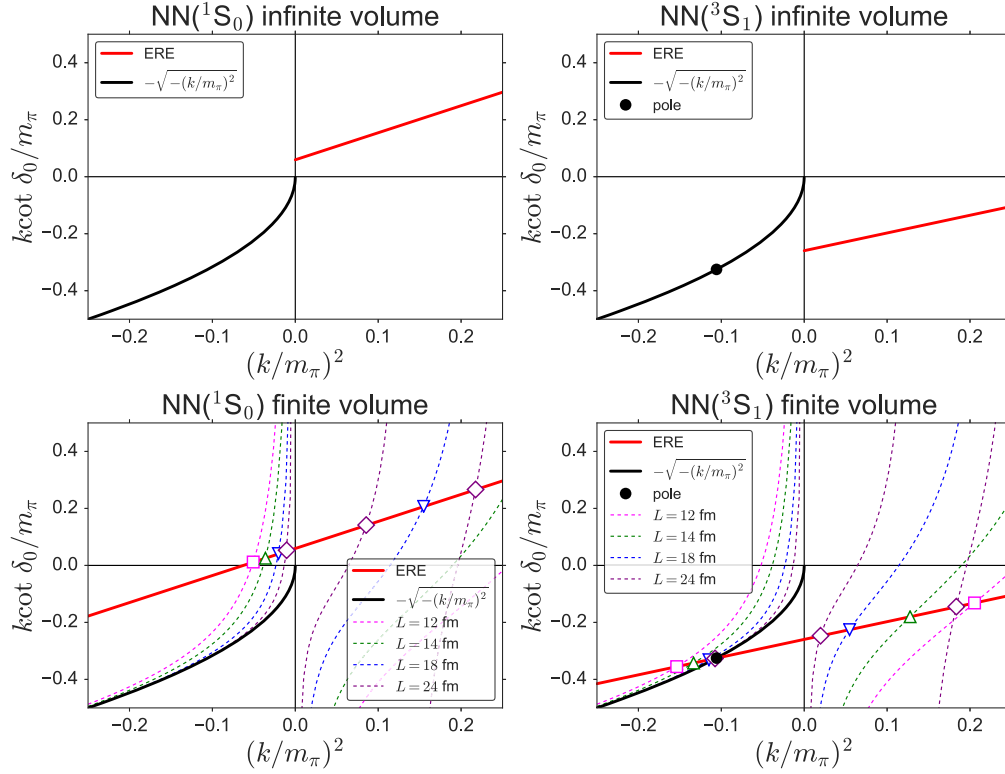


FIG. 3. The relation between  $k \cot \delta_0(k)/m_\pi$  and  $(k/m_\pi)^2$  in the infinite volume for  $NN(^1S_0)$  (upper left) and  $NN(^3S_1)$  (upper right) with  $m_\pi = 0.14$  GeV. The red solid lines denote empirical ERE relations, and the black solid lines are the condition for the bound-state pole. In the upper right figure, the bound state is identified as the solid black point. The lower panels show the relation between  $k \cot \delta_0(k)/m_\pi$  and  $(k/m_\pi)^2$  on finite volumes. The colored dashed lines represent Lüscher's formula for each finite volume  $L$ . Realized on each volume are the discrete points which satisfy both Lüscher's formula and the ERE relation, as shown by open squares, up and down triangles, and diamonds for  $L = 12, 14, 18$ , and  $24$  fm, respectively.

bound-state pole (deuteron) can be identified as the point where  $k \cot \delta_0(k)/m_\pi = -\sqrt{-(k/m_\pi)^2}$  is satisfied (the solid black circle).

For finite volumes ( $L < \infty$ ), two-particle spectra are quantized, so that only the discrete values satisfying Lüscher's formula Eq. (1) are realized on the ERE line. They are indicated by the open square, up and down triangle, and diamond symbols in Fig. 3 (lower left and lower right), where Eq. (1) is drawn by the dashed lines for different values of the lattice volume  $L = 12, 14, 18$ , and  $24$  fm. As the volume becomes larger, the state density increases for  $k^2 \geq 0$  to form the continuous ERE line. On the other hand, for  $k^2 < 0$ , the discrete points constitute a sequence which leads to an accumulation point corresponding to either the  $k^2 = 0$  scattering state at the threshold energy (lower left) or the bound-state pole (lower right).

It is in order here to discuss general properties of  $(k^2, k \cot \delta_0(k))$  obtained from the analytic properties of the  $S$  matrix for systems with a bound state(s). Suppose we have a bound state at momentum  $k = i\kappa_b$  with  $\kappa_b > 0$ . Then the  $S$  matrix  $S(k) = e^{2i\delta_0(k)}$  has the corresponding simple pole and simple zero at  $k = i\kappa_b$  and  $k = -i\kappa_b$ , respectively. By using the identity

$$k \cot \delta_0(k) = ik \cdot \frac{S(k) + 1}{S(k) - 1}, \quad (3)$$

one obtains the bound-state condition  $k \cot \delta_0(k) = -\sqrt{-k^2}$  at  $k^2 = -\kappa_b^2$ , as mentioned above. In addition, the  $S$  matrix near the pole corresponding to the bound state is known to obey the formula [40]

$$S(k \sim i\kappa_b) \simeq \frac{-i\beta_b^2}{k - i\kappa_b}, \quad (4)$$

where  $\beta_b^2$  is real and positive for physical poles. Consequently, the  $S$  matrix with a pure imaginary momentum near the bound-state pole diverges as

$$\lim_{\epsilon \rightarrow 0} S(k)|_{k=i(\kappa_b \pm \epsilon)} = \mp \lim_{\epsilon \rightarrow 0} \frac{\beta_b^2}{\epsilon} \rightarrow \mp \infty. \quad (5)$$

Also, we have

$$\frac{d}{dk^2} \left[ k \cot \delta_0(k) - \left( -\sqrt{-k^2} \right) \right] \Big|_{k^2 = -\kappa_b^2} = -\frac{1}{\beta_b^2} < 0, \quad (6)$$

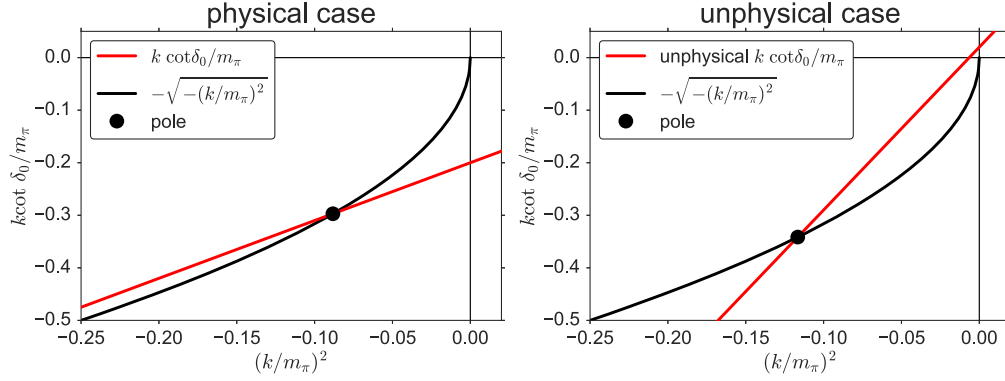


FIG. 4. Schematic illustration for the system with the pole satisfying the physical condition (left) or the pole having the unphysical residue (right). The red solid lines denote  $k \cot \delta_0(k)$  obtained by fitting the lattice QCD data at finite volumes. The bound state is identified as the crossing point of  $k \cot \delta_0(k)$  and the bound-state condition (black solid line), as indicated by the black solid circle. In the left (right) figure, the behavior of  $k \cot \delta_0(k)$  near the bound-state pole is consistent (inconsistent) with the condition Eq. (6), and thus the bound-state pole has a physical (unphysical) residue in the  $S$  matrix.

which implies that the slope of  $k \cot \delta_0(k)$  as a function of  $k^2$  must be smaller than that of  $-\sqrt{-k^2}$  at the bound-state pole. We note here that the conditions (4)–(6) hold as long as  $\kappa_b^2$  is smaller than the possible lowest-lying left-hand singularity,<sup>6</sup> while the ERE [Eq. (2)] is valid only for small  $k^2$ .

In Fig. 4 (left), we show an example for a system with one bound state which satisfies the condition (6). (Here, for simplicity, we assume that the binding energy is sufficiently small, so that the NLO ERE is valid.) This corresponds to the situation of the deuteron pole shown in Fig. 3 (right panels) except for the small S/D mixing. In Fig. 4 (right), we show an unphysical case which does not satisfy the condition (6). If the fit of the lattice data indicates such behavior, it is clear evidence that the data are not reliable.

Let us now consider the case where there exist multiple bound states. Then the conditions (5) and (6) must be satisfied for each bound state. This poses a further constraint on the behavior of  $(k^2, k \cot \delta_0(k))$ . To illustrate this, consider the system with two bound states at  $k = i\kappa_{b_1}$  and  $i\kappa_{b_2}$  with  $\kappa_{b_1} > \kappa_{b_2} > 0$ . Then we have  $S(k)|_{k=i(\kappa_{b_1}-\epsilon)} = +\infty$  and  $S(k)|_{k=i(\kappa_{b_2}+\epsilon)} = -\infty$ . Since  $S(k)$  is real for pure imaginary  $k$  (for it is defined by the ratio of the Jost functions [40,42]), there exists at least one  $\kappa_c$  between  $\kappa_{b_1}$  and  $\kappa_{b_2}$  which satisfies  $S(k)|_{k=i(\kappa_c \pm \epsilon)} = 1 \pm \epsilon$ . Combining this with the identity (3), we obtain

$$k \cot \delta_0(k)|_{k^2=-(\kappa_c \pm \epsilon)^2} = \mp \infty; \quad (7)$$

i.e. the  $k \cot \delta_0(k)$  must diverge at least once between two bound-state poles. The generalization of this to the case with more than two bound states is straightforward.

<sup>6</sup>These conditions may not be valid beyond the left-hand singularity [40–43].

Shown in Fig. 5 (left) is a case with two bound states, taken from the exactly solvable three-dimensional square-well potential with the radius  $b$ .<sup>7</sup> The deeply bound state at  $(kb)^2 \simeq -16.4$  and the shallow bound state at  $(kb)^2 = -0.1$  are denoted by the black solid circles, while the  $k \cot \delta_0(k)$  is plotted by the red solid line. One finds the condition (6) is satisfied for both bound states, so that they are indeed physical. Note here that  $k \cot \delta_0(k)$  diverges between two bound states at  $(kb)^2 \simeq -5.4$ , so that ERE of  $k \cot \delta_0(k)$  around  $k^2 = 0$  has clearly finite convergence radius.

Figure 5 (right) illustrates a case where ERE is erroneously applied beyond the convergence radius. The unphysical crossing violating the condition (6) at the deeper pole indicates that the use of ERE is incorrect. In the real lattice data, we do not know the black solid circles from the beginning. They are rather obtained as a result of the fitting of the lattice QCD data which are all located above the black solid line for  $k^2 < 0$ . If one finds that the naive ERE fitting of the lattice data shows the situation such as Fig. 5 (right), i.e. the unphysical crossing of the red line and the black line, one needs to try the proper fitting of  $k \cot \delta_0(k)$  without using ERE or to doubt the original lattice data.

Having now established the general properties of  $k \cot \delta_0(k)$  at  $k^2 < 0$ , we present its novel applications in lattice QCD assuming that there is at most one bound state whose binding energy is small enough within the convergence radius of ERE around  $k^2 = 0$ , Eq. (2). Then one may extract the scattering parameters at  $k^2 = 0$  such as the scattering length  $a_0$  and the effective range  $r_0$  through the ERE fitting of the lattice data either at  $k^2 > 0$  or at  $k^2 < 0$  (or both). Such an analysis for the data at  $k^2 < 0$  with the exact Lüscher's formula has never been conducted in

<sup>7</sup>See Appendix A for notations and analysis.

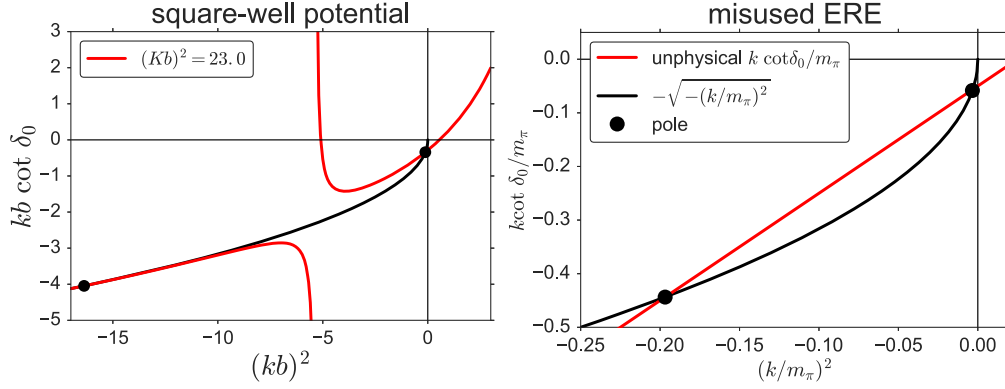


FIG. 5. (Left) The  $k \cot \delta_0(k)$  (the red line) for a three-dimensional square-well potential. The radius of the potential is denoted by  $b$ , and the potential depth is chosen so that there exist two bound states. The black solid line is the condition for the bound-state poles, which are denoted by the black solid circles. Note that  $k \cot \delta_0(k)$  diverges between two bound states. (Right) Illustration of a misuse of ERE beyond its convergence radius, where the left crossing point between the red line and black line violates the condition (6).

previous lattice studies for two-baryon systems in the direct method [24–30], except for the one in Ref. [31]. [See also [44–46] for reviews with meson(s).]

Furthermore, the method can be used to test the reliability of lattice data, which we call a “consistency check”: Self-inconsistent and/or singular behaviors of ERE lines around  $k^2 = 0$  and/or the unphysical behaviors as shown in Figs. 4 (right) and 5 (right) indicate that the systematic errors of the original  $\Delta E$  on the lattice are substantially underestimated. A main source of the systematic errors is likely to be the excited-state contaminations, which generate fake plateaus in the direct method, as pointed out in Ref. [33] and recapitulated in Sec. I. A potential danger of these fake plateaus applies to  $NN$  data in Refs. [24–31]. In addition, the general properties of  $k \cot \delta_0(k)$  at the  $k^2 < 0$  region tells us the proper use of the ERE for claiming more than one bound state as we discussed above. This applies to the data of Ref. [31].

In the next sections, we apply this consistency check to existing lattice data which claim the existence of bound states for two-baryon systems at heavy pion masses.

### III. DATA SETS

Lattice data to be checked are summarized in Table I. Numerical results of  $(k/m_\pi)^2$  and  $k \cot \delta_0(k)/m_\pi$  together with  $\Delta E$  are recapitulated in the tables in Appendix D: Table V for YKU2011 [24], Table VI for YIKU2012 [25] and YIKU2015 [26], Table VII for NPL2012 [27], Tables VIII and IX for NPL2013 [28,29], Table X for NPL2015 [30], and Table XI for CalLat2017 [31]. For YKU2011, NPL2013, NPL2015, and CalLat2017, data for excited states are also given. [We tabulated only the data below the possible lowest-lying left-hand singularity,  $|(k/m_\pi)^2| < 0.25$ .] Two nucleon source operators with zero displacement under quark-source smearing are employed in all these literature. CalLat2017 used nonzero displacement additionally as mentioned in Sec. I.

Strictly speaking, the  $^3S_1$  channel mixes with the  $l = 2$  partial wave ( $^3D_1$  channel) due to the presence of the tensor interaction. In addition, each of the  $^1S_0$  and  $^3S_1$  channels mixes with  $l = 4, 6, \dots$  partial waves due to the breaking of the rotational symmetry on a cubic box. In the above references, however, binding energies of  $NN(^1S_0)$  and  $NN(^3S_1)$  are extracted without explicitly taking into

TABLE I. Summary of references for lattice data used in this paper. NPL2013 and CalLat2017 employed the same set of lattice configurations. NPL2012 employed the anisotropic lattice with  $a_s/a_t \approx 3.5$ , where  $a_s(\equiv a)$  and  $a_t$  are spatial and temporal lattice spacings, respectively. (\*) We use the lattice spacing  $a = 0.1453(16)$  fm given in NPL2013 for  $L$  in the present table.

Name	Ref.	$N_f$	$a$ [fm]	$L$ [fm]	$m_\pi$ [GeV]	$m_N$ [GeV]	$m_\Lambda$ [GeV]	$m_\Xi$ [GeV]
YKU2011	[24]	0	0.128	3.1, 4.1, 6.1, 12.3	0.80	1.62	...	...
YIKU2012	[25]	2 + 1	0.090	2.9, 3.6, 4.3, 5.8	0.51	1.32	...	...
YIKU2015	[26]	2 + 1	0.090	4.3, 5.8	0.30	1.05	...	...
NPL2012	[27]	2 + 1	0.123 (aniso.)	2.9, 3.9	0.39	1.17	1.23	1.34
NPL2013	[28,29]	3	0.145	3.5 <sup>(*)</sup> , 4.6 <sup>(*)</sup> , 7.0 <sup>(*)</sup>	0.81	1.64	1.64	1.64
NPL2015	[30]	2 + 1	0.117	2.8, 3.7, 5.6	0.45	1.23	1.31	1.42
CalLat2017	[31]	3	0.145	3.5, 4.6	0.81	1.64	1.64	1.64

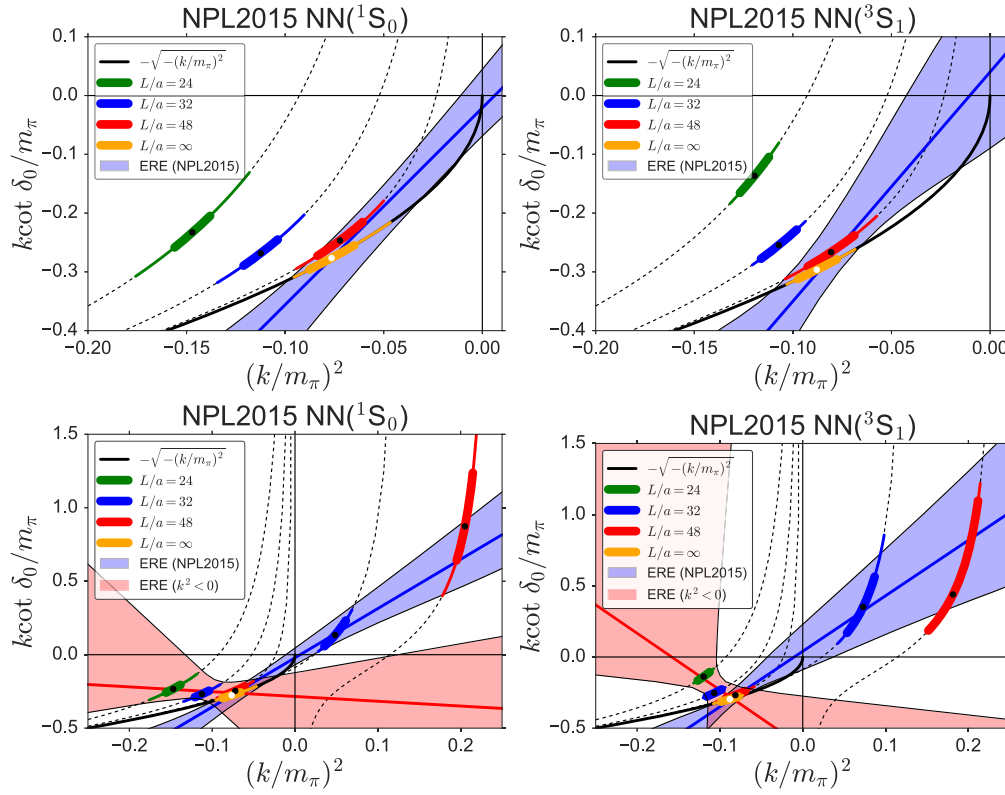


FIG. 6.  $k \cot \delta_0(k)/m_\pi$  as a function of  $(k/m_\pi)^2$  for  $NN(^1S_0)$  (left) and  $NN(^3S_1)$  (right) of NPL2015. Black dashed lines correspond to Lüscher's formula for each finite volume, while the black solid line represents the bound-state condition that  $-\sqrt{-(k/m_\pi)^2}$ . Upper panels show the data at  $(k/m_\pi)^2 < 0$ , while lower ones include the data at  $(k/m_\pi)^2 > 0$ . Light blue bands correspond to ERE with statistical and systematic errors added in quadrature given in NPL2015, obtained from data  $k^2 > 0$  and the binding energy in the infinite volume; this is called  $\text{ERE}_{k^2 > 0, BE}$  in the text. Light red bands in lower panels correspond to the ERE obtained by using data at  $k^2 < 0$  on finite volumes; this is called  $\text{ERE}_{k^2 < 0}$  in the text. The red (blue) lines in the middle of the red (blue) bands correspond to the best fits.

account these higher partial waves. Correspondingly, if the numerical values of  $\Delta E$ ,  $(k/m_\pi)^2$ , and  $k \cot \delta_0(k)/m_\pi$  are not explicitly given in the references in Table I, we calculate them by using Lüscher's formula for the  $S$  wave, Eq. (1).<sup>8</sup> Both statistical and systematic errors evaluated in the original references are taken into account in the tables in Appendix D. The systematic errors originating from the scale setting given in NPL2012, NPL2013, and NPL2015 are not considered, since we analyze only the dimensionless quantities normalized by  $m_\pi$  in this paper.

Although we focus on the  $NN$  states in this paper, we also tabulate  $\Lambda\Lambda(^1S_0)$  and  $\Xi\Xi(^1S_0)$  states (NPL2012) and two octet-baryon states in **1**, **8<sub>A</sub>**, and **10** irreducible representations of flavor  $SU(3)$  (NPL2013) in Appendix D.

<sup>8</sup>For this conversion, the statistical and systematic errors for pion and baryon masses are neglected, since they are much smaller compared to other errors.

## IV. CONSISTENCY CHECK FOR EACH LATTICE DATA

### A. NPL2015

We first consider the data from NPL2015, in which  $NN(^1S_0)$  and  $NN(^3S_1)$  were studied in  $(2+1)$ -flavor QCD at  $m_\pi = 0.45$  GeV. The data contain not only the ground states ( $k^2 < 0$ ) but also excited states ( $k^2 > 0$ ) and, thus, are a particularly useful data set for the full consistency check.

Figure 6 shows  $k \cot \delta_0(k)/m_\pi$  as a function of  $(k/m_\pi)^2$  for  $NN(^1S_0)$  (left) and  $NN(^3S_1)$  (right). Upper panels focus on the data at  $k^2 < 0$ , while lower panels include data at  $k^2 > 0$ . Black dashed lines in the figures represent the behavior of Eq. (1) for each volume, and the black solid lines represent  $-\sqrt{-(k/m_\pi)^2}$ . The lattice data  $k \cot \delta_0(k)/m_\pi$  on finite volumes are shown by the solid circles together with statistical (systematic) errors denoted by the thick (thin) line segments.

NPL2015 claims the existence of bound states in both channels indicated by the open circles, where the binding energies were obtained by the infinite volume extrapolation

TABLE II. Summary of the scattering parameters obtained from NPL2015 data [30].  $\text{ERE}_{k^2>0, BE}$  is the ERE fit using data at  $k^2 > 0$  and the binding energy in the infinite volume.  $\text{ERE}_{k^2<0}$  is the ERE fit using data at  $k^2 < 0$  on finite volumes.

Name	Ref.	$NN(^1S_0)$		$NN(^3S_1)$	
		$(a_0 m_\pi)^{-1}$	$r_0 m_\pi$	$(a_0 m_\pi)^{-1}$	$r_0 m_\pi$
$\text{ERE}_{k^2>0, BE}$	[30]	$-0.021(^{+0.036}_{-0.028})(^{+0.063}_{-0.032})$	$6.7(^{+1.0}_{-0.8})(^{+2.0}_{-1.3})$	$0.04(^{+0.10}_{-0.07})(^{+0.17}_{-0.08})$	$7.8(^{+2.2}_{-1.5})(^{+3.5}_{-1.7})$
$\text{ERE}_{k^2<0}$	This work	$-0.28(^{+0.06}_{-0.07})(^{+0.10}_{-0.23})$	$-0.65(^{+1.05}_{-1.18})(^{+1.82}_{-4.71})$	$-0.63(^{+0.18}_{-0.49})(^{+0.19}_{-2.02})$	$-8.0(^{+3.4}_{-9.1})(^{+3.7}_{-37.5})$

using the data at  $k^2 < 0$  with the asymptotic expansion [47,48] of Lüscher's formula. In NPL2015, ERE parameters up to NLO were also determined using the data on the finite volume at  $k^2 > 0$  below the lowest-lying left-hand singularity together with the binding energy in the infinite volume (open circles). We call this fit  $\text{ERE}_{k^2>0, BE}$ . Corresponding EREs with statistical and systematic errors added in quadrature are shown by the light blue bands in the figures. As clearly seen in upper panels of Fig. 6, for both channels, the  $\text{ERE}_{k^2>0, BE}$  determined in NPL2015 has the wrong intersection with the bound-state condition in the same way as Fig. 4 (right).

To further check the reliability of the data, we perform the ERE fit using the data only at  $k^2 < 0$  on the finite volumes ( $L/a = 24, 32, 48$ ), which we refer to as  $\text{ERE}_{k^2<0}$ . The results are shown by the light red bands in lower panels in Fig. 6. The two ERE bands (light red and light blue) in the figures are clearly inconsistent with each other for both channels. Indeed, the scattering parameters obtained by  $\text{ERE}_{k^2>0, BE}$  and  $\text{ERE}_{k^2<0}$  do not agree with each other in magnitude and/or sign as summarized in Table II, despite that  $\text{ERE}_{k^2>0, BE}$  and  $\text{ERE}_{k^2<0}$  should be consistent with each other as shown in Fig. 3 (lower right). This observation casts serious doubt on the reliability of the lattice data of NPL2015.

What causes these inconsistencies? The first possibility is that the volume is too small for the finite volume formula (1) applicable. This is, however, unlikely by the

fact that  $m_\pi L \geq 6.4$  in NPL2015. The second possibility is that the ERE up to NLO has large truncation errors. However, this is also unlikely, since the data under consideration are well below the lowest-lying left-hand singularity at  $|(k/m_\pi)^2| = 0.25$ . The third and most plausible possibility is that the energy shifts  $\Delta E$  in NPL2015 are incorrect due to contaminations from excited states nearby. Indeed,  $\Delta E$  in NPL2015 are extracted from the data at  $t \approx 0.6\text{--}1.5$  fm, while fake plateaus due to contamination from the excited states can easily appear at  $t \approx 1\text{--}2$  fm as demonstrated in Ref. [33] and recapitulated in Sec. I.

In summary, the unphysical behavior of  $\text{ERE}_{k^2>0, BE}$  as well as the inconsistency between  $\text{ERE}_{k^2>0, BE}$  and  $\text{ERE}_{k^2<0}$  exposed by our consistency check indicate that  $\Delta E$  in NPL2015 is not reliable enough to claim the existence of  $NN$  bound states at  $m_\pi = 0.45$  GeV.

## B. YKU2011

Next we consider YKU2011, in which  $NN(^1S_0)$  and  $NN(^3S_1)$  were studied in quenched QCD at  $m_\pi = 0.80$  GeV. As in the case of NPL2015, the data in YKU2011 contain both the ground states ( $k^2 < 0$ ) and excited states ( $k^2 > 0$ ) and serve as a useful data set for the consistency check.

Figure 7 shows  $k \cot \delta_0(k)/m_\pi$  as a function of  $(k/m_\pi)^2$  for  $NN(^1S_0)$  (left) and  $NN(^3S_1)$  (right). The existence of the bound state was claimed for both  $NN(^1S_0)$  and

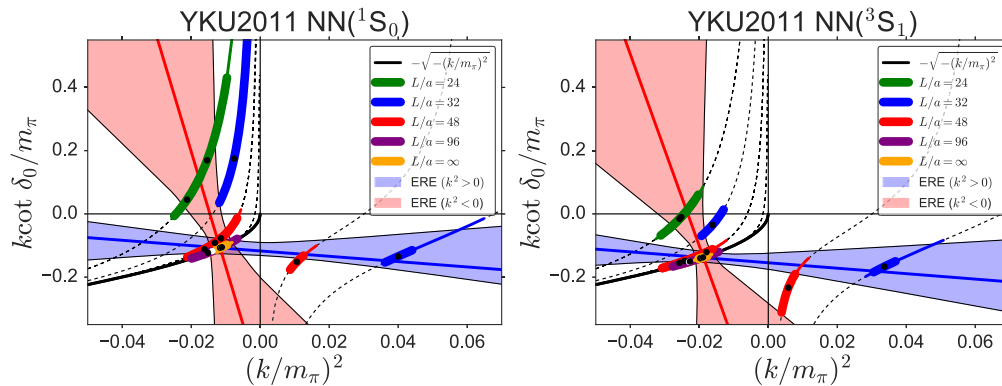


FIG. 7.  $k \cot \delta_0(k)/m_\pi$  as a function of  $(k/m_\pi)^2$  for  $NN(^1S_0)$  (left) and  $NN(^3S_1)$  (right) of YKU2011. Black dashed lines correspond to Lüscher's formula for each volume, while the black solid line represents  $-\sqrt{-(k/m_\pi)^2}$ . EREs corresponding to  $\text{ERE}_{k^2>0, BE}$  and  $\text{ERE}_{k^2<0}$  are shown by the light blue band and light red band, respectively, with statistical and systematic errors added in quadrature.

TABLE III. The same as Table II but from YKU2011 data [24]. YKU2011 [24] evaluated scattering lengths assuming  $r_0 m_\pi = 0$ .

Name	Ref.	$NN(^1S_0)$		$NN(^3S_1)$	
		$(a_0 m_\pi)^{-1}$	$r_0 m_\pi$	$(a_0 m_\pi)^{-1}$	$r_0 m_\pi$
$a_0^{k^2>0, L=32}$	[24]	$-0.137(^{+0.020}_{-0.027})(^{+0.118}_{-0.045})$	0	$-0.164(^{+0.019}_{-0.025})(^{+0.077}_{-0.029})$	0
$a_0^{k^2>0, L=48}$	[24]	$-0.152(^{+0.020}_{-0.026})(^{+0.046}_{-0.001})$	0	$-0.235(^{+0.044}_{-0.069})(^{+0.082}_{-0.017})$	0
$\text{ERE}_{k^2>0, BE}$	This work	$-0.12(^{+0.01}_{-0.01})(^{+0.02}_{-0.01})$	$-1.69(^{+0.81}_{-0.97})(^{+2.20}_{-0})$	$-0.15(^{+0.01}_{-0.01})(^{+0.02}_{-0.01})$	$-1.72(^{+0.67}_{-0.89})(^{+2.00}_{-0.67})$
$\text{ERE}_{k^2<0}$	This work	$-0.53(^{+0.25}_{-1.09})(^{+0.12}_{-0.40})$	$-72.7(^{+39.4}_{-166.7})(^{+15.8}_{-52.6})$	$-0.71(^{+0.32}_{-1.66})(^{+0.15}_{-1.47})$	$-60.6(^{+32.8}_{-169.2})(^{+13.1}_{-144.1})$

$NN(^3S_1)$  by the infinite volume extrapolation from a subset of the data at  $k^2 < 0$  fitted with the asymptotic form [47,48] of Lüscher's formula.

The consistency check on YKU2011 immediately exposes a similar symptom as one observed in NPL2015: The ERE behaviors are inconsistent between those at  $k^2 > 0$  and  $k^2 < 0$  in both  $NN(^1S_0)$  and  $NN(^3S_1)$  channels. In fact,  $\Delta E$  for the ground states is found to be almost independent of the volume, and thus data at  $k^2 < 0$  align on a nearly vertical line. On the other hand, data at  $k^2 > 0$  align on a nearly horizontal line in the figure.

In order to quantify the inconsistency of YKU2011 data, we perform two different ERE analyses in the same manner as those performed for NPL2015 data.<sup>9</sup> In Fig. 7, the ERE lines for  $\text{ERE}_{k^2>0, BE}$  and  $\text{ERE}_{k^2<0}$  are shown with a light blue band and a light red band, respectively. Also in Table III, the scattering parameters are summarized together with scattering lengths evaluated in the YKU2011 paper.

Inconsistency between  $\text{ERE}_{k^2>0, BE}$  and  $\text{ERE}_{k^2<0}$  is apparent in both channels. Quantitatively, one notices that the parameters in  $\text{ERE}_{k^2<0}$  are very singular:  $r_0 m_\pi$  are 1–2 orders of magnitude larger (with negative signs) than their natural value  $r_0 m_\pi \sim \mathcal{O}(1)$ . The singular behavior is caused by the fact that  $\Delta E$  are almost independent of the volume, while claimed binding energies are shallow compared to the size of lattice volumes. To the best of our knowledge, such singular ERE parameters together with the existence of one shallow bound state are very difficult to be realized by any reasonable interactions.

As in the case of NPL2015, the finite volume effect is unlikely to be the origin of the above inconsistency, since  $m_\pi L \geq 12$  and also  $\Delta E$  for  $k^2 < 0$  is almost independent on  $L$ . The breakdown of the ERE is also unlikely, since  $(k/m_\pi)^2$  for YKU2011 data are much smaller than  $|(k/m_\pi)^2| = 0.25$ . Again, the most plausible explanation

is that  $\Delta E$  in YKU2011 suffer serious excited-state contaminations.

In summary, the inconsistency between  $\text{ERE}_{k^2>0, BE}$  and  $\text{ERE}_{k^2<0}$  exposed by our consistency check indicates that  $\Delta E$  in YKU2011 is not reliable enough to claim the existence of  $NN$  bound states at  $m_\pi = 0.80$  GeV.

### C. YIKU2012 and YIKU2015

The ground states for  $NN(^1S_0)$  and  $NN(^3S_1)$  were studied in  $(2+1)$ -flavor QCD at  $m_\pi = 0.51$  GeV (YIKU2012) and  $m_\pi = 0.30$  GeV (YIKU2015). Since the excited states were not studied in these works, we consider only the behavior of  $k \cot \delta_0(k)$  for  $k^2 < 0$ . Figures 8 and 9 show  $k \cot \delta_0(k)/m_\pi$  as a function of  $(k/m_\pi)^2$  for  $NN(^1S_0)$  (left) and  $NN(^3S_1)$  (right) from YIKU2012 and YIKU2015, respectively. The existence of the bound states in both channels was claimed by the infinite volume extrapolation with the asymptotic expansion of Lüscher's formula (YIKU2012) or with the constant fit (YIKU2015).

As can be seen from these figures, the data show singular behaviors in  $^1S_0$  and  $^3S_1$  channels for both YIKU2012 and YIKU2015: Since  $\Delta E$  is almost independent of the volume, the data align almost vertically. Such behavior leads to very singular ERE parameters, i.e. divergent values of  $r_0 m_\pi$  and sometimes of  $(a_0 m_\pi)^{-1}$ .

We perform the NLO ERE fit to quantify the singular behavior in terms of the scattering parameters. In the case of YIKU2012, the results are plotted in Fig. 8 by the red lines with the light red bands where statistical and systematic errors are added in quadrature. Although total errors of the ERE fits are rather large, the central values show the singular behaviors:  $((a_0 m_\pi)^{-1}, r_0 m_\pi) = (5.27, 303.6)$  in the  $NN(^1S_0)$  channel and  $((a_0 m_\pi)^{-1}, r_0 m_\pi) = (-3.84, -129.3)$  in the  $NN(^3S_1)$  channel. In addition, the red line in the  $^1S_0$  channel violates Eq. (6), which must be satisfied for the physical bound state. The fake plateau problem of  $\Delta E$  found in Ref. [33] certainly lead to these singular  $k \cot \delta_0(k)$ .

In the case of YIKU2015, there are only two finite volume data, and thus the degrees of freedom in the NLO ERE fit is zero. We therefore obtain only the central values for ERE parameters:  $((a_0 m_\pi)^{-1}, r_0 m_\pi) = (1.0, 23.3)$  in the

<sup>9</sup>Correlations among data points are neglected in these fits. For  $\text{ERE}_{k^2<0}$ , we perform an additional fit to a part of data ( $L/a = 32$  and 48 from the two-state analysis in Ref. [24]), which are manifestly uncorrelated. We confirm that obtained EREs are consistent with those given in Fig. 7 in both  $NN(^1S_0)$  and  $NN(^3S_1)$  channels.

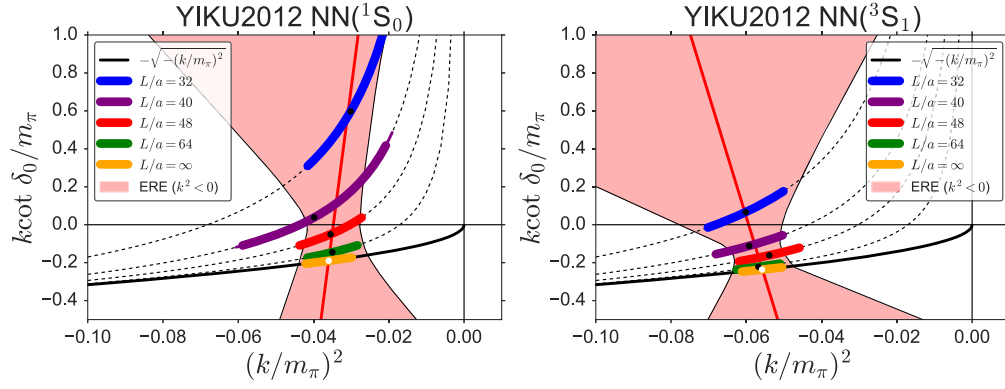


FIG. 8.  $k \cot \delta_0(k)/m_\pi$  as a function of  $(k/m_\pi)^2$  for  $NN(^1S_0)$  (left) and  $NN(^3S_1)$  (right) for data on each volume from YIKU2012, together with YIKU2012's infinite volume extrapolation. Black dashed lines correspond to Lüscher's formula for each finite volume, while the black solid line represents  $-\sqrt{-(k/m_\pi)^2}$ . NLO ERE fits to finite volume data are shown by red lines, together with light red bands corresponding to statistical and systematic errors added in quadrature.

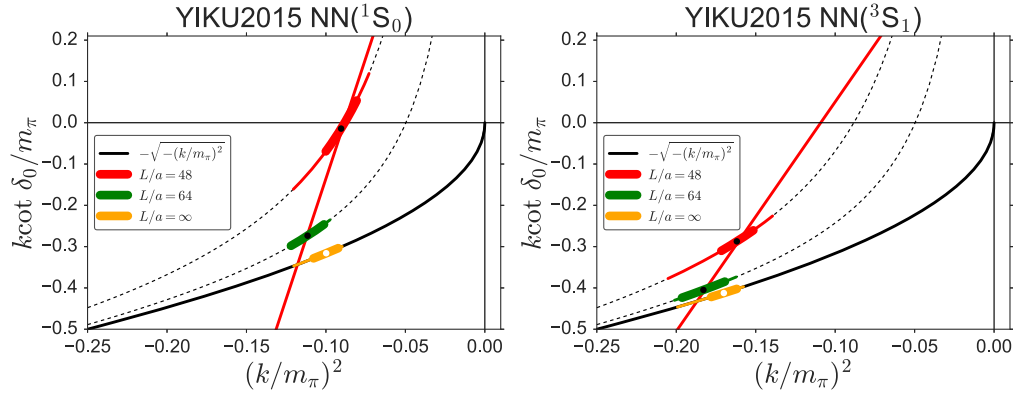


FIG. 9. The same as Fig. 8, but from YIKU2015. Red lines correspond to NLO ERE fits.

$NN(^1S_0)$  channel and  $((a_0 m_\pi)^{-1}, r_0 m_\pi) = (0.61, 11.1)$  in the  $NN(^3S_1)$  channel, where the corresponding ERE lines are plotted in Fig. 9 by red lines. In both channels, the violations of the physical condition Eq. (6) for the intersections and/or the singular ERE behaviors are observed.

Since  $|(k/m_\pi)^2|$  for these data are smaller than 0.25, singular ERE behaviors are very difficult to be realized by any reasonable interactions. We therefore conclude that the values of  $\Delta E$  in YIKU2012 and YIKU2015 are unreliable, most probably due to the excited-state contaminations.

#### D. NPL2012

We perform the consistency check on NPL2012 data in  $(2+1)$ -flavor QCD at  $m_\pi = 0.39$  GeV. Similar to YIKU2012 and YIKU2015, only data for the ground state are available in NPL2012. Figure 10 shows  $k \cot \delta_0(k)/m_\pi$  as a function of  $(k/m_\pi)^2$  for  $NN(^1S_0)$  (left) and  $NN(^3S_1)$  (right). In NPL2012, the binding energies were determined

by the infinite volume extrapolation with the asymptotic expansion of Lüscher's formula.

In the  $NN(^1S_0)$  channel, we observe a singular ERE behavior similar to (but somewhat milder than) those observed in YIKU2011, YIKU2012, and YIKU2015. As shown in Fig. 10 (left),  $k \cot \delta_0(k)/m_\pi$  at  $(k/m_\pi)^2 < 0$  decreases vertically as the volume increases. The NLO ERE fit for data at  $L/a = 24, 32$  gives  $((a_0 m_\pi)^{-1}, r_0 m_\pi) = (-1.06, -32.3)$ , and the corresponding ERE is plotted in Fig. 10 (left) by the red line.

In the  $NN(^3S_1)$  channel, values for ERE parameters are rather reasonable:  $((a_0 m_\pi)^{-1}, r_0 m_\pi) = (-0.24, 0.0)$ , as shown by the red line in Fig. 10 (right). Even if a reasonable behavior is observed, however, it does not guarantee that the data are reliable. Indeed, as seen in Appendix B, NPL2013 and CalLat2017 give nonsingular but manifestly source-dependent  $k \cot \delta_0(k)$  behaviors.

The NPLQCD Collaboration reported [49] the small positive values for  $(k/m_\pi)^2$  with the same lattice setup but on a smaller volume ( $L/a = 20$ ):

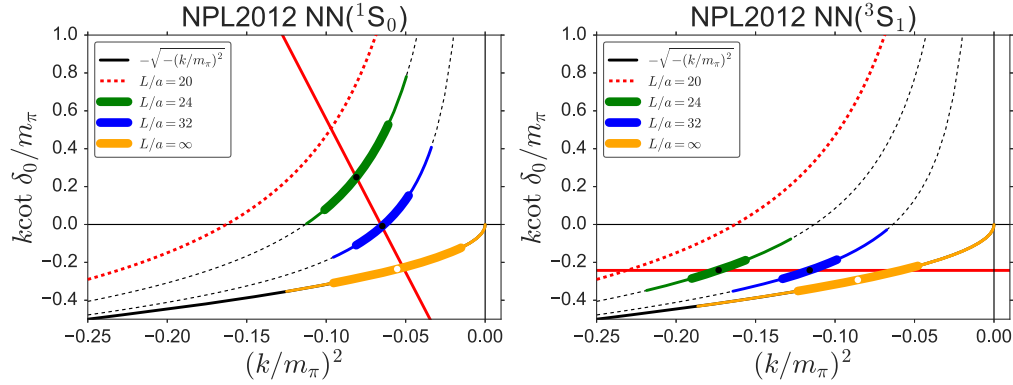


FIG. 10. The same as Fig. 8, but from NPL2012. Red lines correspond to the NLO ERE fits. The red dashed line represents Lüscher's formula for  $L/a = 20$ . Lattice data around  $(k/m_\pi)^2 = 0$  at  $L/a = 20$  [49] are located way out of the plot region of the figures.

$$\begin{aligned} (k/m_\pi)^2 &= 0.030(13)(20) \quad \text{for } NN(^1S_0), \\ (k/m_\pi)^2 &= 0.012(20)(33) \quad \text{for } NN(^3S_1). \end{aligned} \quad (8)$$

Such results clearly conflict with the ERE behaviors obtained from  $L/a = 24, 32$ <sup>10</sup>. In Fig. 10, we show only the lines corresponding to Lüscher's formula for  $L/a = 20$ , as the lattice data around  $(k/m_\pi)^2 = 0$  are located way beyond the plot range of the figure.

Again, the consistency check reveals that at least some of the data in NPL2012 (and their earlier result [49]) are unreliable. Provided that all  $\Delta E$  in NPL2012 were obtained by the plateau identification at early time slices, further investigations which do not rely on the plateau identification are necessary before claiming the existence of  $NN$  bound states.

## V. CONCLUSION AND DISCUSSION

In this paper, we have introduced a simple test (consistency check) to inspect the reliability of the energy shift of two-hadron systems in lattice QCD on the basis of Lüscher's finite volume formula. We have argued that useful information on the hadron-hadron interactions can be extracted from the lattice data in the region of not only positive squared momentum  $k^2 > 0$  but also negative squared momentum  $k^2 < 0$ . Consistency with the ERE around  $k^2 = 0$  exposes the problem of the lattice data which is otherwise hidden in the energy shift  $\Delta E$ .

We have applied the consistency check to lattice results from which the existence of the  $NN$  bound state(s) for heavy quark masses are concluded in the literature. All of them employ the direct method, in which  $\Delta E$  is obtained

<sup>10</sup>Lüscher's formula for  $L/a = 20$  intersects with the NLO ERE at  $(k/m_\pi)^2 = -0.097$  for  $NN(^1S_0)$  and  $(k/m_\pi)^2 = -0.231$  for  $NN(^3S_1)$ , respectively.

by the plateau identification at early time slices. In Table IV, we summarize our consistency checks, together with source independences of the plateaus (the mirage problem) discussed in Ref. [33] and reviewed in Sec. I. In the table, "source independence" means whether  $\Delta E$  is physical in the sense that it is independent of the nucleon source operators, "consistency check (i)" means whether  $\text{ERE}_{k^2 > 0, BE}$  and  $\text{ERE}_{k^2 < 0}$  are consistent with each other, "consistency check (ii)" means whether the scattering parameters obtained by ERE is nonsingular, and "consistency check (iii)" means whether the bound-state pole has a physical residue in Eq. (6). As can be seen from the table, none of these results is free from either the plateau problem or the ERE problem, or both.

Results in this paper, together with those in our previous paper [33], strongly indicate that  $\Delta E$  in the direct method, determined by plateaus at earlier time slices, suffer uncontrolled systematic errors from excited-state contaminations. This conclusion brings a serious doubt on the existence of the  $NN$  bound states for pion masses heavier than 300 MeV, contrary to the claims of YKU2011, YIKU2012, YIKU2015, NPL2012, NPL2013, NPL2015, and CalLat2017. In order to determine correct spectra of two nucleon systems at heavier pion masses by the direct method, a much more sophisticated method than the plateau fitting, such as the variational method [36], must be employed.

An alternative method to determine spectra of multi-hadrons is the HAL QCD method, which does not suffer from the problem of excited-state contaminations in multi-hadron systems by the use of the space-time correlations instead of the temporal correlations [7]. In forthcoming papers [50], we will investigate the source dependence of the potential in the HAL QCD method, which will be also used to analyze the fundamental origin of the mirage problem in the direct method.

TABLE IV. A summary of consistency checks (i) consistency between  $\text{ERE}_{k^2>0, BE}$  and  $\text{ERE}_{k^2<0}$ , (ii) nonsingular ERE parameters, and (iii) physical residue for the bound-state pole, together with the source independence of  $\Delta E$ . Here “no” means that the source independency or consistency check has failed, while the symbol † implies there is none or only insufficient study on the corresponding item. The symbol \* means that an obvious contradiction is not found within the error bars, while it does not necessarily guarantee that the data are reliable. See Appendix B for the meaning of the symbol ? on the consistency check for NPL2013 and CalLat2017.

Data	$NN(^1S_0)$				$NN(^3S_1)$			
	Source independence	Consistency check			Source independence	Consistency check		
		(i)	(ii)	(iii)		(i)	(ii)	(iii)
YKU2011 [24]	†	No	No	*	†	No	No	*
YKU2012 [25]	No	†	No	*	No	†	No	*
YKU2015 [26]	†	†	No	*	†	†	No	No
NPL2012 [27]	†	†	No	*	†	†	*	*
NPL2013 [28,29]	No	*	*	No	No	*	*	?
NPL2015 [30]	†	No	*	No	†	No	*	No
CalLat2017 [31]	No	?	*	No	No	?	*	No

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### APPENDIX A: THE SQUARE-WELL POTENTIAL AND $k \cot \delta_0(k)$

In this Appendix, we consider two nonrelativistic particles with each mass  $M$  interacting through the three-dimensional square-well potential,  $V(\vec{r}) = -v \cdot \theta(b - |\vec{r}|)$ , which leads to

$$k \cot \delta(k) = \frac{k^2 + \sqrt{K^2 + k^2} \cot(\sqrt{K^2 + k^2}b) k \cot(kb)}{k \cot(kb) - \sqrt{K^2 + k^2} \cot(\sqrt{K^2 + k^2}b)}, \quad (\text{A1})$$

where  $k^2 = ME$  and  $K^2 = Mv$ . From the effective range expansion, the scattering length  $a_0$  and the effective range  $r_0$  are obtained as

$$\begin{aligned} a_0/b &= \frac{\tan(Kb)}{Kb} - 1, \\ r_0/b &= 1 - \frac{(Kb)^2}{3(\tan(Kb) - Kb)^2} + \frac{1}{Kb(\tan(Kb) - Kb)}, \end{aligned} \quad (\text{A2})$$

which are plotted as a function of  $(Kb)^2$  in Fig. 11. A number of bound states increases as  $(Kb)^2$  does, and the scattering length diverges at  $(Kb)^2 = (\pi/2)^2, (3\pi/2)^2, \dots$

The  $k \cot \delta_0(k)$  for several interaction strengths are given in Fig. 12: (a) weak repulsion with  $-2.0 \leq (Kb)^2 \leq -0.4$ , (b),(b') weak attraction with  $1.0 \leq (Kb)^2 \leq 6.0$ , (c) moderate attraction with  $15.0 \leq (Kb)^2 \leq 20.0$ , and (d) strong attraction with  $21.0 \leq (Kb)^2 \leq 23.0$ . Solid circles correspond to the bound-state poles. The thin dashed lines in

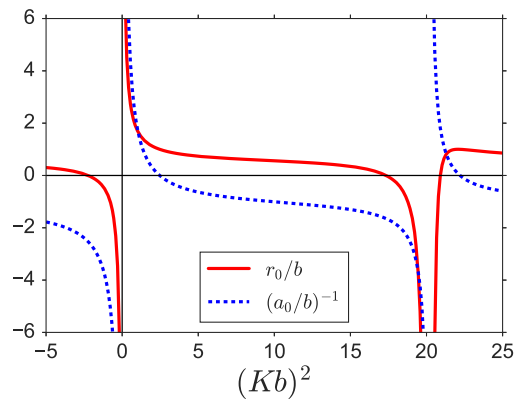


FIG. 11. The inverse scattering length  $a_0^{-1}$  (blue dashed line) and the effective range  $r_0$  (red solid line) as a function of  $(Kb)^2$ . The first bound state appears at  $(Kb)^2 = (\pi/2)^2$  and the second one at  $(Kb)^2 = (3\pi/2)^2$ .

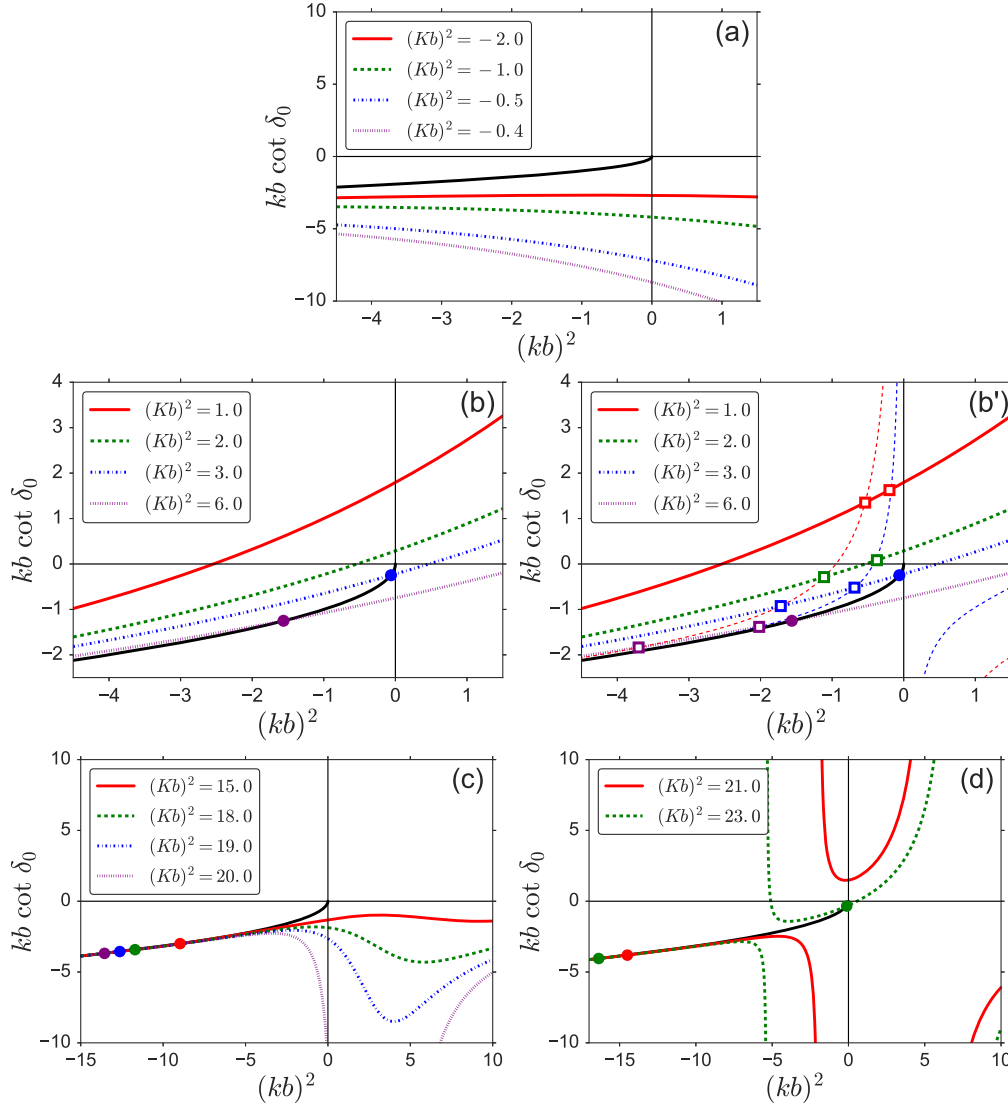


FIG. 12. The  $kb \cot \delta_0(k)$  as a function of  $(kb)^2$  are shown by colored lines. The black solid lines denote the condition for the bound states, and the solid circles correspond to the poles. (a) Weak repulsion. (b) Weak attraction. (b') Weak attraction together with Lüscher's formula at  $L/b = 2$  (3) by the red (blue) thin dashed line, where open squares are finite volume spectra. (c) Moderate attraction. (d) Strong attraction having the 2nd pole at  $(Kb)^2 = 23$ .

Fig. 12(b') represent Lüscher's formula, together with finite volume spectra denoted by open squares.

## APPENDIX B: CONSISTENCY CHECK FOR NPL2013 AND CALLAT2017

In NPL2013 and CalLat2017, although the same gauge configurations are employed for  $L/a = 24$  and 32, mutual and/or self-inconsistencies are found for  $\Delta E$  at  $k^2 < 0$ , as discussed in Sec. I (see Fig. 2). As shown in Fig. 13, a similar mutual inconsistency is also observed for  $\Delta E$  at  $k^2 > 0$ , which are obtained in the center of mass system with a nonzero relative momentum injected between two nucleons at the sink. Here NPL2013 employed the zero displaced two-nucleon source, while the CalLat2017

used the nonzero displaced one. The inconsistency at  $L/a = 32$ , in particular, in the  $NN(^3S_1)$  channel, indicates that the scattering state also fails to satisfy the source independence.<sup>11</sup>

In the rest of this Appendix, we analyze these data in terms of  $k \cot \delta_0(k)$ .

Upper panels in Fig. 14 show  $k \cot \delta_0(k)/m_\pi$  at  $(k/m_\pi)^2 < 0$  for  $^1S_0$  (left) and  $^3S_1$  (right) in the case of NPL2013. Given  $L$ , an apparent inconsistency between  $n = 0$  (open symbols) and  $n = 2$  (black solid symbols)

<sup>11</sup>The details of sink operators may also differ between the two.

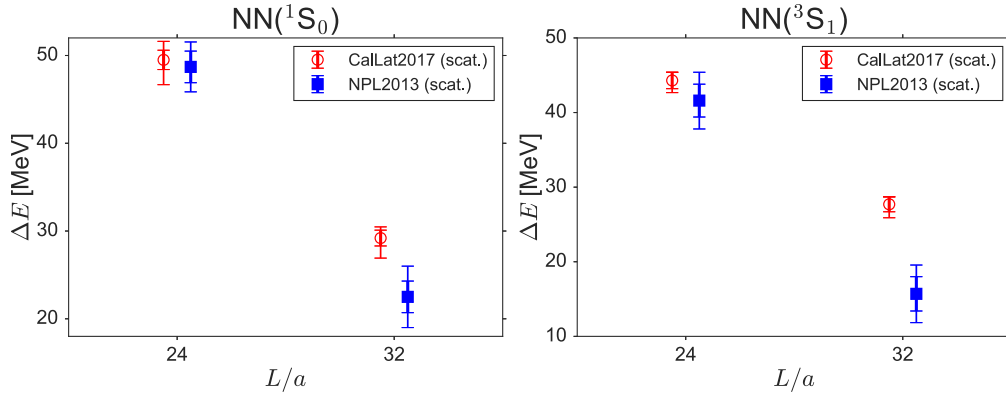


FIG. 13. The same as Fig. 2 but for the excited state ( $\Delta E > 0$ ) from CalLat2017 (red circles) and NPL2013 (blue triangles) in the center of mass system.

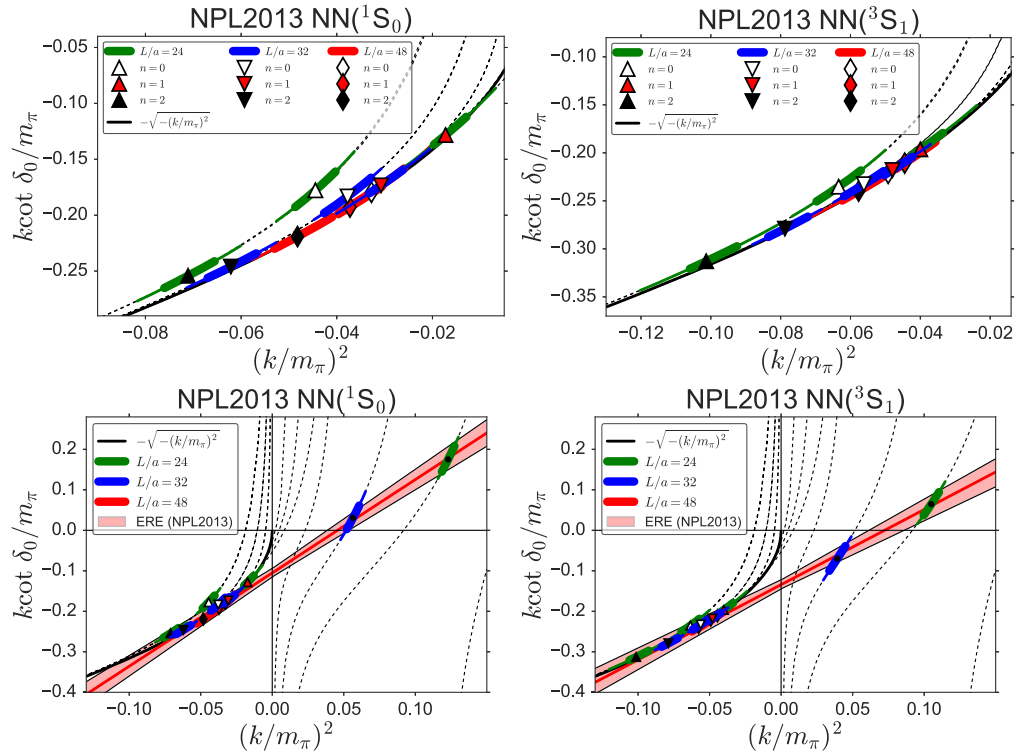


FIG. 14. (Upper) The same as Fig. 8, but from NPL2013. (Lower) The same as upper figures but with data from excited states. Red bands correspond to  $\text{ERE}_{k^2>0, BE}$  given in NPL2013 with statistical and systematic errors added in quadrature.

data<sup>12</sup> in both channels can be seen clearly, which confirms the discussion in Sec. I. Lower panels in Fig. 14 include data at  $(k/m_\pi)^2 > 0$  together with  $\text{ERE}_{k^2>0, BE}$  from NPL2013. While  $\text{ERE}_{k^2>0, BE}$  and data at  $(k/m_\pi)^2 < 0$  with  $n = 0$  show no apparent inconsistency,  $\text{ERE}_{k^2>0, BE}$  themselves violate the physical condition for the residue of the bound-state pole in both channels. Considering the

<sup>12</sup> $n \equiv |\vec{n}|$  corresponds to the boost momentum as  $\vec{P} = (2\pi/L) \cdot \vec{n}$ .

uncertainties, we put “no” and “?” for  $^1S_0$  and  $^3S_1$ , respectively, about the consistency check (iii) in Table IV.

Upper panels in Fig. 15 represent  $k \cot \delta_0(k)/m_\pi$  at  $(k/m_\pi)^2 < 0$  for CalLat2017, while lower panels in Fig. 15 include data at  $(k/m_\pi)^2 > 0$ . As already discussed in Sec. II, the “naive” ERE fits by CalLat2017 contradict the physical pole condition (see the right panel in Fig. 5). If the two bound-state poles are physical,  $k \cot \delta_0(k)$  should diverge at a very narrow interval of  $(k/m_\pi)^2$ , between  $-0.043$  (left blue point) and  $-0.021$  (right blue point) for  $^1S_0$  and between  $-0.070$  (left blue point) and  $-0.053$  (right

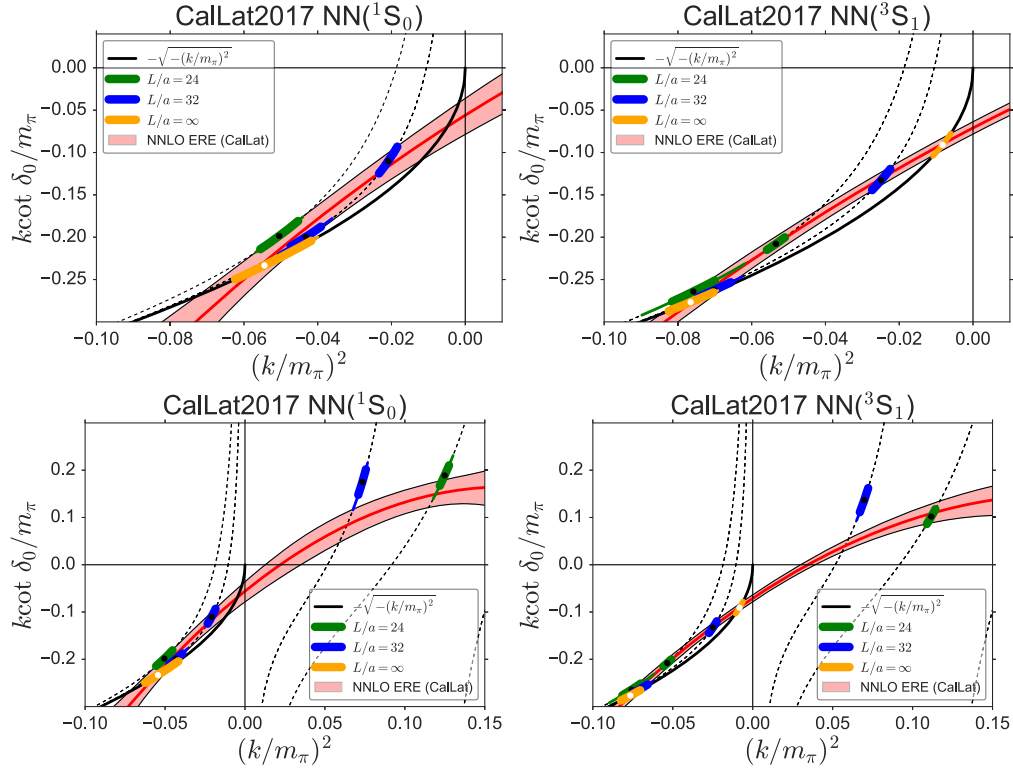


FIG. 15. (Upper) The same as Fig. 8, but from CalLat2017. Red bands correspond to NNLO ERE given in CalLat2017. (Lower) The same as upper figures but with data from excited states.

green point) for  ${}^3S_1$ . This is unlikely if not impossible, which supports our interpretation that two data at  $k^2 < 0$  on each volume are the artifact due to the source operator dependence.

Data at  $(k/m_\pi)^2 > 0$  behave rather differently from those at  $(k/m_\pi)^2 < 0$  (lower panels). As a consequence, their NNLO ERE fit misses the point at  $(k/m_\pi)^2 > 0$  on  $L/a = 32$  in both channels. We thus put “?” on the consistency check (i) in Table IV.

### APPENDIX C: CONSISTENCY CHECK FOR LATTICE DATA WITH HYPERON(S)

Here we present two examples of the consistency check using the data given in Appendix D.

Figure 16 shows  $k \cot \delta_0(k)/m_\pi$  as a function of  $(k/m_\pi)^2$  for  $\Xi\Xi({}^1S_0)$  (left) and  $\Lambda\Lambda({}^1S_0)$  (right) in the case of NPL2012. The best NLO fit for data at  $L/a = 24, 32$  for  $\Xi\Xi({}^1S_0)$ ,  $((am_\pi)^{-1}, rm_\pi) = (1.87, 35.6)$ , violates the

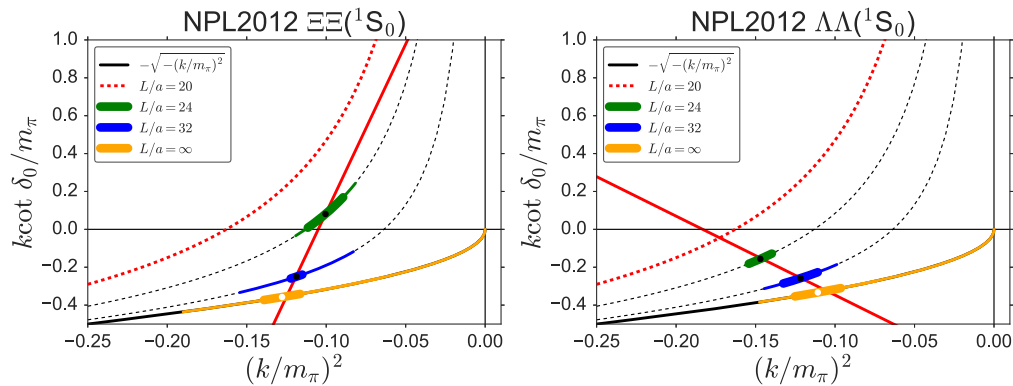


FIG. 16.  $k \cot \delta_0(k)/m_\pi$  as a function of  $(k/m_\pi)^2$  in NPL2012 for  $\Xi\Xi({}^1S_0)$  (left) and  $\Lambda\Lambda({}^1S_0)$  (right). Red lines correspond to the NLO ERE fit using two volumes. The red dashed line represents Lüscher's formula for  $L/a = 20$ , while the corresponding lattice data around  $(k/m_\pi)^2 = 0$  [49] are located way out of the plot region of the figures.

physical pole condition Eq. (6), while that for  $\Lambda\Lambda(^1S_0)$  does not violate the condition and gives  $((am_\pi)^{-1}, rm_\pi) = (-0.76, -8.33)$ .

We also note that the earlier paper by the NPLQCD Collaboration [49] reported the results with the same lattice setup but on a smaller volume ( $L/a = 20$ ):

$$(k/m_\pi)^2 = 0.0247(94)(77) \quad \text{for } \Xi\Xi(^1S_0), \quad (k/m_\pi)^2 = -0.033(09)(11) \quad \text{for } \Lambda\Lambda(^1S_0). \quad (\text{C1})$$

Such results clearly conflict with the ERE behaviors obtained from  $L/a = 24, 32$ , which intersects with Lüscher's formula for  $L/a = 20$  (red dashed line) at  $(k/m_\pi)^2 = -0.173$  for  $\Lambda\Lambda(^1S_0)$ , while it has no intersection for  $\Xi\Xi(^1S_0)$  at  $(k/m_\pi)^2 < 0$ .<sup>13</sup>

#### APPENDIX D: DATA OF $\Delta E$ , $(k/m_\pi)^2$ , AND $k \cot \delta_0(k)/m_\pi$

TABLE V. Summary of the data from YKU2011 [24]. Corresponding  $(k/m_\pi)^2$  and  $k \cot \delta_0(k)/m_\pi$  are calculated by using Eq. (1).

Label	State	$L/a$	$\Delta E$ [MeV]	$(k/m_\pi)^2$	$k \cot \delta_0(k)/m_\pi$
YKU2011	$NN(^1S_0)$ two-states	32	$-3.0(1.7)(^{+0.3}_{-0.7})$	$-0.008(4)(^{+1}_{-2})$	$0.17(^{+45}_{-14})(^{+21}_{-5})$
		48	$-4.5(0.9)(^{+2.1}_{-0.1})$	$-0.011(2)(^{+5}_{-1})$	$-0.08(^{+2}_{-2})(^{+9}_{-1})$
		$\infty$	$-4.4(0.6)(1.0)$	$-0.011(1)(^{+3}_{-3})$	$-0.11(^{+1}_{-1})(^{+1}_{-1})$
		32	$15.8(1.6)(^{+9.6}_{-0.3})$	$0.040(4)(^{+25}_{-2})$	$-0.13(^{+2}_{-2})(^{+12}_{-1})$
		48	$4.2(0.8)(^{+2.1}_{-0.0})$	$0.011(2)(^{+5}_{-1})$	$-0.15(^{+2}_{-2})(^{+5}_{-1})$
	$\mathcal{O}_1$	24	$-6.1(2.3)(2.2)$	$-0.02(1)(^{+1}_{-1})$	$0.17(^{+26}_{-12})(^{+39}_{-10})$
		48	$-5.2(2.6)(0.8)$	$-0.01(1)(^{+0}_{-0})$	$-0.09(^{+8}_{-4})(^{+4}_{-1})$
		96	$-4.6(2.0)(1.1)$	$-0.012(5)(^{+3}_{-3})$	$-0.11(^{+3}_{-2})(^{+2}_{-1})$
	$\mathcal{O}_2$	24	$-8.4(1.5)(0.5)$	$-0.021(4)(^{+1}_{-1})$	$0.05(^{+7}_{-5})(^{+3}_{-2})$
		48	$-6.4(2.0)(0.8)$	$-0.016(5)(^{+2}_{-2})$	$-0.11(^{+4}_{-2})(^{+2}_{-1})$
		96	$-6.0(1.9)(0.5)$	$-0.015(5)(^{+1}_{-1})$	$-0.12(2)(^{+1}_{-1})$
	$NN(^3S_1)$ two-states	32	$-6.4(1.3)(^{+0.1}_{-0.7})$	$-0.016(3)(^{+1}_{-2})$	$-0.03(^{+5}_{-3})(^{+2}_{-2})$
		48	$-7.1(0.7)(^{+2.2}_{-0.1})$	$-0.018(2)(^{+6}_{-1})$	$-0.12(^{+1}_{-1})(^{+4}_{-0})$
		$\infty$	$-7.5(0.5)(0.9)$	$-0.019(1)(^{+2}_{-2})$	$-0.14(^{+1}_{-0})(^{+1}_{-1})$
		32	$13.3(1.3)(^{+6.6}_{-1.7})$	$0.034(3)(^{+17}_{-4})$	$-0.17(^{+2}_{-2})(^{+8}_{-3})$
		48	$2.3(0.8)(^{+2.2}_{-0.1})$	$0.006(2)(^{+6}_{-1})$	$-0.23(^{+4}_{-7})(^{+8}_{-4})$
	$\mathcal{O}_1$	24	$-10.2(2.2)(1.6)$	$-0.03(1)(^{+0}_{-0})$	$-0.02(^{+8}_{-5})(^{+7}_{-4})$
		48	$-9.6(2.6)(0.9)$	$-0.02(1)(^{+0}_{-0})$	$-0.15(^{+3}_{-2})(^{+1}_{-1})$
		96	$-7.8(2.1)(0.4)$	$-0.02(1)(^{+0}_{-0})$	$-0.14(2)(^{+0}_{-0})$
	$\mathcal{O}_2$	24	$-10.0(1.5)(0.5)$	$-0.025(4)(^{+1}_{-1})$	$-0.01(^{+5}_{-4})(^{+2}_{-1})$
		48	$-10.2(2.0)(0.8)$	$-0.026(5)(^{+2}_{-2})$	$-0.15(2)(^{+1}_{-1})$
		96	$-9.0(2.0)(0.5)$	$-0.023(5)(^{+1}_{-1})$	$-0.15(2)(^{+1}_{-0})$

<sup>13</sup> $(k/m_\pi)^2 = 0.0247(94)(77)$  for  $\Xi\Xi(^1S_0)$  corresponds to  $k \cot \delta_0/m_\pi = -5.11(^{+1.26}_{-2.79})(^{+0.83}_{-3.40})$ , which is also incompatible with the ERE from  $L/a = 24, 32$ .

TABLE VI. Summary of the data from YIKU2012 [25] and YIKU2015 [26]. Corresponding  $(k/m_\pi)^2$  and  $k \cot \delta_0(k)/m_\pi$  are calculated by using Eq. (1).

Label	State	$L/a$	$\Delta E$ [MeV]	$(k/m_\pi)^2$	$k \cot \delta_0(k)/m_\pi$
YIKU2012	$NN(^1S_0)$	32	-6.2(2.4)(0.5)	-0.03(1)( $\frac{+0}{-0}$ )	0.60( $\frac{+63}{-29}$ )( $\frac{+24}{-7}$ )
		40	-8.2(4.0)(1.5)	-0.04(2)( $\frac{+1}{-1}$ )	0.04( $\frac{+38}{-15}$ )( $\frac{+24}{-5}$ )
		48	-7.3(1.7)(0.5)	-0.04(1)( $\frac{+0}{-0}$ )	-0.05( $\frac{+9}{-6}$ )( $\frac{+3}{-2}$ )
		64	-7.2(1.4)(0.3)	-0.03(1)( $\frac{+0}{-0}$ )	-0.15( $\frac{+4}{-3}$ )( $\frac{+1}{-1}$ )
		$\infty$	-7.4(1.3)(0.6)	-0.04(1)( $\frac{+0}{-0}$ )	-0.19( $\frac{+2}{-2}$ )( $\frac{+1}{-1}$ )
	$NN(^3S_1)$	32	-12.4(2.1)(0.5)	-0.06(1)( $\frac{+0}{-0}$ )	0.07( $\frac{+11}{-8}$ )( $\frac{+4}{-2}$ )
		40	-12.2(1.9)(0.6)	-0.06(1)( $\frac{+0}{-0}$ )	-0.11( $\frac{+6}{-4}$ )( $\frac{+3}{-2}$ )
		48	-11.1(1.7)(0.3)	-0.05(1)( $\frac{+0}{-0}$ )	-0.16( $\frac{+4}{-3}$ )( $\frac{+1}{-1}$ )
		64	-11.7(1.2)(0.5)	-0.06(1)( $\frac{+0}{-0}$ )	-0.22( $\frac{+2}{-1}$ )( $\frac{+1}{-1}$ )
		$\infty$	-11.5(1.1)(0.6)	-0.06(1)( $\frac{+0}{-0}$ )	-0.24( $\frac{+1}{-1}$ )( $\frac{+1}{-1}$ )
YIKU2015	$NN(^1S_0)$	48	-7.7(0.9)( $\frac{+1.2}{-2.4}$ )	-0.09(1)( $\frac{+1}{-3}$ )	-0.01( $\frac{+7}{-6}$ )( $\frac{+11}{-14}$ )
		64	-9.5(0.9)( $\frac{+0.8}{-0.5}$ )	-0.11(1)( $\frac{+1}{-1}$ )	-0.27( $\frac{+3}{-2}$ )( $\frac{+3}{-1}$ )
		$\infty$	-8.5(0.7)( $\frac{+0.5}{-1.6}$ )	-0.10(1)( $\frac{+1}{-2}$ )	-0.32( $\frac{+1}{-1}$ )( $\frac{+1}{-3}$ )
	$NN(^3S_1)$	48	-13.8(0.9)( $\frac{+1.7}{-3.6}$ )	-0.16(1)( $\frac{+2}{-4}$ )	-0.29( $\frac{+3}{-2}$ )( $\frac{+6}{-9}$ )
		64	-15.6(1.2)( $\frac{+1.3}{-1.0}$ )	-0.18(1)( $\frac{+2}{-1}$ )	-0.40( $\frac{+2}{-1}$ )( $\frac{+2}{-2}$ )
		$\infty$	-14.5(0.7)( $\frac{+0.8}{-2.4}$ )	-0.17(1)( $\frac{+1}{-3}$ )	-0.41( $\frac{+1}{-1}$ )( $\frac{+1}{-3}$ )

TABLE VII. The same as Table V, but from NPL2012 [27]. To evaluate the systematic errors for  $k \cot \delta_0(k)/m_\pi$  with the \* symbol, we impose a constraint that the corresponding  $(k/m_\pi)^2$  is negative, since the pole condition  $k \cot \delta_0(k) = -\sqrt{-k^2}$  is meaningful only for negative  $k^2$ .

Label	State	$L/a$	$\Delta E$ [MeV]	$(k/m_\pi)^2$	$k \cot \delta_0(k)/m_\pi$
NPL2012	$NN(^1S_0)$	24	-10.4(2.6)(3.1)	-0.08(2)( $\frac{+2}{-2}$ )	0.25( $\frac{+28}{-17}$ )( $\frac{+45}{-18}$ )
		32	-8.3(2.2)(3.3)	-0.06(2)( $\frac{+3}{-3}$ )	-0.01( $\frac{+17}{-10}$ )( $\frac{+38}{-13}$ )
		$\infty$	-7.1(5.2)(7.3)	-0.06(4)( $\frac{+6}{-6}$ )	-0.24( $\frac{+11}{-7}$ )( $\frac{+21}{-9}$ )*
	$NN(^3S_1)$	24	-22.3(2.3)(5.4)	-0.17(2)( $\frac{+4}{-4}$ )	-0.24( $\frac{+5}{-5}$ )( $\frac{+15}{-10}$ )
		32	-14.9(2.3)(5.8)	-0.12(2)( $\frac{+5}{-5}$ )	-0.24( $\frac{+5}{-4}$ )( $\frac{+21}{-10}$ )
		$\infty$	-11.0(5.0)(12.0)	-0.09(4)( $\frac{+9}{-9}$ )	-0.29( $\frac{+7}{-6}$ )( $\frac{+28}{-13}$ )*
	$\Lambda\Lambda(^1S_0)$	24	-17.5(0.9)(0.7)	-0.15(1)( $\frac{+1}{-1}$ )	-0.16( $\frac{+3}{-3}$ )( $\frac{+2}{-2}$ )
		32	-14.5(1.3)(2.4)	-0.12(1)( $\frac{+2}{-2}$ )	-0.26( $\frac{+3}{-3}$ )( $\frac{+6}{-5}$ )
		$\infty$	-13.2(1.8)(4.0)	-0.11(1)( $\frac{+3}{-3}$ )	-0.33( $\frac{+2}{-2}$ )( $\frac{+6}{-5}$ )
	$\Xi\Xi(^1S_0)$	24	-11.0(1.3)(1.6)	-0.10(1)( $\frac{+2}{-2}$ )	0.08( $\frac{+9}{-7}$ )( $\frac{+14}{-9}$ )
		32	-13.0(0.5)(3.9)	-0.119(4)( $\frac{+36}{-36}$ )	-0.25( $\frac{+1}{-1}$ )( $\frac{+13}{-8}$ )
		$\infty$	-14.0(1.4)(6.7)	-0.13(1)( $\frac{+6}{-6}$ )	-0.36( $\frac{+2}{-2}$ )( $\frac{+10}{-8}$ )

TABLE VIII. The same as Table V, but from NPL2013 [28,29].  $27(^1S_0)$  and  $\overline{10}(^3S_1)$  irreducible representations of flavor SU(3) correspond to  $NN(^1S_0)$  and  $NN(^3S_1)$ , respectively.  $n \equiv |\vec{n}|$  in the table is related to the boost momentum as  $\vec{P} = (2\pi/L)\vec{n}$ .

Label	State	$n$	$L/a$	$\Delta E$ [MeV]	$(k/m_\pi)^2$	$k \cot \delta_0(k)/m_\pi$
NPL2013	$27(^1S_0)$	0	24	-17.8(1.7)(2.8)	-0.044(4)( $\begin{smallmatrix} +7 \\ -7 \end{smallmatrix}$ )	-0.18( $\begin{smallmatrix} +2 \\ -1 \end{smallmatrix}$ )( $\begin{smallmatrix} +3 \\ -2 \end{smallmatrix}$ )
			32	-15.1(2.0)(2.0)	-0.038(5)( $\begin{smallmatrix} +5 \\ -5 \end{smallmatrix}$ )	-0.18( $\begin{smallmatrix} +2 \\ -1 \end{smallmatrix}$ )( $\begin{smallmatrix} +2 \\ -2 \end{smallmatrix}$ )
			48	-13.1(2.8)(4.3)	-0.03(1)( $\begin{smallmatrix} +1 \\ -1 \end{smallmatrix}$ )	-0.18( $\begin{smallmatrix} +2 \\ -2 \end{smallmatrix}$ )( $\begin{smallmatrix} +4 \\ -3 \end{smallmatrix}$ )
			24	48.7(1.8)(2.2)	0.123(4)( $\begin{smallmatrix} +6 \\ -6 \end{smallmatrix}$ )	0.18( $\begin{smallmatrix} +3 \\ -3 \end{smallmatrix}$ )( $\begin{smallmatrix} +5 \\ -4 \end{smallmatrix}$ )
			32	22.5(1.8)(3.0)	0.056(4)( $\begin{smallmatrix} +8 \\ -8 \end{smallmatrix}$ )	0.03( $\begin{smallmatrix} +3 \\ -3 \end{smallmatrix}$ )( $\begin{smallmatrix} +6 \\ -5 \end{smallmatrix}$ )
		1	24	-6.9(1.8)(3.8)	-0.017(4)( $\begin{smallmatrix} +10 \\ -10 \end{smallmatrix}$ )	-0.13( $\begin{smallmatrix} +2 \\ -1 \end{smallmatrix}$ )( $\begin{smallmatrix} +4 \\ -3 \end{smallmatrix}$ )
			32	-12.3(1.9)(3.6)	-0.031(5)( $\begin{smallmatrix} +9 \\ -9 \end{smallmatrix}$ )	-0.17( $\begin{smallmatrix} +1 \\ -1 \end{smallmatrix}$ )( $\begin{smallmatrix} +3 \\ -2 \end{smallmatrix}$ )
			48	-14.9(2.7)(2.7)	-0.04(1)( $\begin{smallmatrix} +1 \\ -1 \end{smallmatrix}$ )	-0.19( $\begin{smallmatrix} +2 \\ -1 \end{smallmatrix}$ )( $\begin{smallmatrix} +2 \\ -2 \end{smallmatrix}$ )
		2	24	-28.5(2.3)(3.8)	-0.07(1)( $\begin{smallmatrix} +1 \\ -1 \end{smallmatrix}$ )	-0.25( $\begin{smallmatrix} +1 \\ -1 \end{smallmatrix}$ )( $\begin{smallmatrix} +2 \\ -2 \end{smallmatrix}$ )
			32	-24.9(2.2)(3.1)	-0.06(1)( $\begin{smallmatrix} +1 \\ -1 \end{smallmatrix}$ )	-0.25( $\begin{smallmatrix} +1 \\ -1 \end{smallmatrix}$ )( $\begin{smallmatrix} +2 \\ -2 \end{smallmatrix}$ )
			48	-19.3(2.9)(3.3)	-0.05(1)( $\begin{smallmatrix} +0 \\ -1 \end{smallmatrix}$ )	-0.22( $\begin{smallmatrix} +2 \\ -1 \end{smallmatrix}$ )( $\begin{smallmatrix} +2 \\ -2 \end{smallmatrix}$ )
	$\overline{10}(^3S_1)$	0	24	-25.4(2.6)(4.7)	-0.06(1)( $\begin{smallmatrix} +1 \\ -1 \end{smallmatrix}$ )	-0.24( $\begin{smallmatrix} +2 \\ -2 \end{smallmatrix}$ )( $\begin{smallmatrix} +3 \\ -3 \end{smallmatrix}$ )
			32	-22.5(2.3)(2.6)	-0.06(1)( $\begin{smallmatrix} +1 \\ -1 \end{smallmatrix}$ )	-0.23( $\begin{smallmatrix} +1 \\ -1 \end{smallmatrix}$ )( $\begin{smallmatrix} +2 \\ -1 \end{smallmatrix}$ )
			48	-19.7(3.1)(4.1)	-0.05(1)( $\begin{smallmatrix} +1 \\ -1 \end{smallmatrix}$ )	-0.22( $\begin{smallmatrix} +2 \\ -2 \end{smallmatrix}$ )( $\begin{smallmatrix} +3 \\ -2 \end{smallmatrix}$ )
			24	41.6(2.2)(3.1)	0.10(1)( $\begin{smallmatrix} +1 \\ -1 \end{smallmatrix}$ )	0.06( $\begin{smallmatrix} +3 \\ -3 \end{smallmatrix}$ )( $\begin{smallmatrix} +5 \\ -4 \end{smallmatrix}$ )
			32	15.7(2.3)(3.1)	0.04(1)( $\begin{smallmatrix} +1 \\ -1 \end{smallmatrix}$ )	-0.07( $\begin{smallmatrix} +3 \\ -3 \end{smallmatrix}$ )( $\begin{smallmatrix} +4 \\ -4 \end{smallmatrix}$ )
		1	24	-16.0(2.7)(5.9)	-0.04(1)( $\begin{smallmatrix} +1 \\ -1 \end{smallmatrix}$ )	-0.20( $\begin{smallmatrix} +2 \\ -1 \end{smallmatrix}$ )( $\begin{smallmatrix} +4 \\ -3 \end{smallmatrix}$ )
			32	-19.2(2.3)(3.7)	-0.05(1)( $\begin{smallmatrix} +1 \\ -1 \end{smallmatrix}$ )	-0.22( $\begin{smallmatrix} +1 \\ -1 \end{smallmatrix}$ )( $\begin{smallmatrix} +2 \\ -2 \end{smallmatrix}$ )
			48	-17.8(3.6)(3.1)	-0.04(1)( $\begin{smallmatrix} +1 \\ -1 \end{smallmatrix}$ )	-0.21( $\begin{smallmatrix} +2 \\ -1 \end{smallmatrix}$ )( $\begin{smallmatrix} +2 \\ -2 \end{smallmatrix}$ )
		2	24	-40.7(3.6)(7.4)	-0.10(1)( $\begin{smallmatrix} +2 \\ -2 \end{smallmatrix}$ )	-0.31( $\begin{smallmatrix} +1 \\ -1 \end{smallmatrix}$ )( $\begin{smallmatrix} +3 \\ -3 \end{smallmatrix}$ )
			32	-31.6(2.7)(3.2)	-0.08(1)( $\begin{smallmatrix} +1 \\ -1 \end{smallmatrix}$ )	-0.28( $\begin{smallmatrix} +1 \\ -1 \end{smallmatrix}$ )( $\begin{smallmatrix} +2 \\ -1 \end{smallmatrix}$ )
			48	-23.1(3.9)(5.5)	-0.06(1)( $\begin{smallmatrix} +1 \\ -1 \end{smallmatrix}$ )	-0.24( $\begin{smallmatrix} +2 \\ -1 \end{smallmatrix}$ )( $\begin{smallmatrix} +3 \\ -3 \end{smallmatrix}$ )

TABLE IX. The same as Table V, but from NPL2013 [28,29] (continued).

Label	State	$n$	$L/a$	$\Delta E$ [MeV]	$(k/m_\pi)^2$	$k \cot \delta_0(k)/m_\pi$
NPL2013 (continued)	<b>1</b>	0	24	$-77.7(1.8)(3.2)$	$-0.192(4)(\begin{smallmatrix} +8 \\ -8 \end{smallmatrix})$	$-0.438(\begin{smallmatrix} +5 \\ -5 \end{smallmatrix})(\begin{smallmatrix} +9 \\ -9 \end{smallmatrix})$
			32	$-76.0(2.3)(2.8)$	$-0.19(1)(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})$	$-0.43(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})$
			48	$-73.7(3.3)(5.1)$	$-0.18(1)(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})$	$-0.43(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})(\begin{smallmatrix} +2 \\ -2 \end{smallmatrix})$
		1	24	$-67.2(2.5)(2.5)$	$-0.17(1)(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})$	$-0.41(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})$
			32	$-70.3(2.3)(3.1)$	$-0.17(1)(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})$	$-0.42(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})$
			48	$-73.7(4.4)(7.6)$	$-0.18(1)(\begin{smallmatrix} +2 \\ -2 \end{smallmatrix})$	$-0.43(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})(\begin{smallmatrix} +2 \\ -2 \end{smallmatrix})$
		2	24	$-85.0(3.1)(4.0)$	$-0.21(1)(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})$	$-0.46(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})$
			32	$-79.6(2.6)(3.9)$	$-0.20(1)(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})$	$-0.44(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})$
			48	$-75.4(3.3)(3.3)$	$-0.19(1)(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})$	$-0.43(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})$
	<b>8<sub>A</sub></b>	0	24	$-40.1(1.7)(2.9)$	$-0.100(4)(\begin{smallmatrix} +7 \\ -7 \end{smallmatrix})$	$-0.31(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})$
			32	$-38.5(2.3)(4.4)$	$-0.10(1)(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})$	$-0.31(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})(\begin{smallmatrix} +2 \\ -2 \end{smallmatrix})$
			48	$-38.7(2.9)(2.9)$	$-0.10(1)(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})$	$-0.31(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})$
		1	24	$-26.5(1.8)(3.6)$	$-0.066(4)(\begin{smallmatrix} +9 \\ -9 \end{smallmatrix})$	$-0.25(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})(\begin{smallmatrix} +2 \\ -2 \end{smallmatrix})$
			32	$-34.0(2.6)(3.4)$	$-0.08(1)(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})$	$-0.29(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})(\begin{smallmatrix} +2 \\ -2 \end{smallmatrix})$
			48	$-34.6(2.8)(3.1)$	$-0.09(1)(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})$	$-0.29(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})$
		2	24	$-46.7(2.0)(3.2)$	$-0.116(5)(\begin{smallmatrix} +8 \\ -8 \end{smallmatrix})$	$-0.34(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})$
			32	$-45.2(3.0)(3.1)$	$-0.11(1)(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})$	$-0.33(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})$
			48	$-39.7(3.0)(2.7)$	$-0.10(1)(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})$	$-0.31(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})$
	<b>10</b>	0	24	$-11.4(1.8)(4.0)$	$-0.029(4)(\begin{smallmatrix} +10 \\ -10 \end{smallmatrix})$	$-0.10(\begin{smallmatrix} +3 \\ -3 \end{smallmatrix})(\begin{smallmatrix} +11 \\ -11 \end{smallmatrix})$
			32	$-10.5(2.5)(4.1)$	$-0.03(1)(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})$	$-0.14(\begin{smallmatrix} +3 \\ -3 \end{smallmatrix})(\begin{smallmatrix} +8 \\ -8 \end{smallmatrix})$
			48	$-6.6(3.4)(4.1)$	$-0.02(1)(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})$	$-0.12(\begin{smallmatrix} +6 \\ -6 \end{smallmatrix})(\begin{smallmatrix} +17 \\ -17 \end{smallmatrix})$
		1	24	$-6.3(1.9)(4.4)$	$-0.016(5)(\begin{smallmatrix} +11 \\ -11 \end{smallmatrix})$	$-0.12(\begin{smallmatrix} +2 \\ -2 \end{smallmatrix})(\begin{smallmatrix} +5 \\ -5 \end{smallmatrix})$
			32	$-1.1(2.4)(4.2)$	$-0.003(6)(\begin{smallmatrix} +11 \\ -11 \end{smallmatrix})$	$-0.06(\begin{smallmatrix} +4 \\ -4 \end{smallmatrix})(\begin{smallmatrix} +10 \\ -10 \end{smallmatrix})$
			48	$-2.8(3.1)(4.1)$	$-0.01(1)(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})$	$-0.08(\begin{smallmatrix} +6 \\ -6 \end{smallmatrix})(\begin{smallmatrix} +15 \\ -15 \end{smallmatrix})$
		2	24	$-15.3(2.2)(4.5)$	$-0.04(1)(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})$	$-0.15(\begin{smallmatrix} +3 \\ -3 \end{smallmatrix})(\begin{smallmatrix} +7 \\ -7 \end{smallmatrix})$
			32	$-12.9(2.6)(4.5)$	$-0.03(1)(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})$	$-0.16(\begin{smallmatrix} +3 \\ -3 \end{smallmatrix})(\begin{smallmatrix} +6 \\ -6 \end{smallmatrix})$
			48	$-7.0(3.4)(3.7)$	$-0.02(1)(\begin{smallmatrix} +1 \\ -1 \end{smallmatrix})$	$-0.13(\begin{smallmatrix} +5 \\ -5 \end{smallmatrix})(\begin{smallmatrix} +10 \\ -10 \end{smallmatrix})$

TABLE X. The same as Table V, but from NPL2015 [30]. For the data with (\*), the boost momentum  $n = 1$  is taken in this table. <sup>†</sup>Errors for  $((k/m_\pi)^2, k \cot \delta_0(k)/m_\pi)$  given in Ref. [30] seem to be inconsistent between their Table VII and their Fig. 19. In this table, we assumed their Fig. 19 is correct and reevaluated the errors for  $(k/m_\pi)^2$ . Central values are unchanged.

Label	State	$L/a$	$\Delta E$ [MeV]	$(k/m_\pi)^2$	$k \cot \delta_0(k)/m_\pi$
NPL2015	$NN(^1S_0)$	24	$-24.1(1.5)(4.5)$	$-0.15(1)(^{+3}_{-3})$	$-0.23(^{+3}_{-3})(^{+10}_{-7})$
		32	$-18.4(1.5)(3.3)$	$-0.11(1)(^{+2}_{-2})$	$-0.27(^{+2}_{-2})(^{+6}_{-5})$
		48	$-11.8(1.9)(3.1)$	$-0.07(1)(^{+2}_{-2})$	$-0.25(^{+3}_{-3})(^{+6}_{-4})$
		$\infty$	$-12.5(^{+1.9}_{-1.7})(^{+4.5}_{-2.5})$	$-0.08(1)(^{+3}_{-2})$	$-0.28(^{+2}_{-2})(^{+6}_{-3})$
		32(*)	$7.9(2.1)(^{+3.3}_{-3.3})$	$0.05(1)(^{+2}_{-2})$	$0.13(^{+10}_{-8})(^{+14}_{-8})$
		48	$33.2(1.8)(^{+4.7}_{-4.4})$	$0.21(1)(^{+3}_{-3})$	$0.87(^{+36}_{-23})(^{+379}_{-41})$
	$NN(^3S_1)$	24	$-19.6(1.2)(1.6)$	$-0.12(1)(^{+1}_{-1})$	$-0.14(^{+3}_{-3})(^{+5}_{-4})$
		32	$-17.5(1.5)(1.6)$	$-0.11(1)(^{+1}_{-1})$	$-0.25(^{+3}_{-2})(^{+3}_{-2})$
		48	$-13.3(2.0)(3.2)$	$-0.08(1)(^{+2}_{-2})$	$-0.27(^{+3}_{-2})(^{+5}_{-4})$
		$\infty$	$-14.4(^{+1.8}_{-1.6})(^{+1.8}_{-2.7})$	$-0.09(1)(^{+1}_{-2})$	$-0.30(^{+2}_{-2})(^{+2}_{-3})$
		32(*) <sup>†</sup>	$11.9(2.4)(^{+3.7}_{-5.0})$	$0.07(1)(^{+2}_{-2})$	$0.35(^{+21}_{-18})(^{+46}_{-18})$
		48 <sup>†</sup>	$29.4(5.0)(^{+0.2}_{-0.2})$	$0.18(3)(^{+1}_{-1})$	$0.44(^{+66}_{-25})(^{+42}_{-9})$

TABLE XI. The same as Table V, but from CalLat2017 [31]. <sup>†</sup>The values for the scattering states are read from the figures in Ref. [31]

Label	State	$L/a$	$\Delta E$ [MeV]	$(k/m_\pi)^2$	$k \cot \delta_0(k)/m_\pi$
CalLat2017	$NN(^1S_0)$	24	$-20.2(2.1)(1.5)$	$-0.05(1)(^{+0}_{-0})$	$-0.20(2)(^{+1}_{-1})$
		32	$-17.3(1.7)(2.3)$	$-0.043(4)(^{+6}_{-6})$	$-0.20(1)(^{+2}_{-2})$
		$\infty$	$-21.8(^{+5.1}_{-3.2})(^{+2.8}_{-0.8})$	$-0.054(^{+13}_{-8})(^{+7}_{-2})$	$-0.233(^{+29}_{-16})(^{+17}_{-4})$
		32	$-8.3(1.0)(0.5)$	$-0.021(2)(^{+1}_{-1})$	$-0.11(^{+2}_{-1})(^{+1}_{-1})$
		24 <sup>†</sup>	$49.5(1.1)(^{+1.8}_{-2.6})$	$0.125(3)(^{+5}_{-7})$	$0.19(2)(^{+4}_{-4})$
		32 <sup>†</sup>	$29.2(0.9)(^{+0.9}_{-2.1})$	$0.073(2)(^{+2}_{-5})$	$0.18(3)(^{+3}_{-5})$
	$NN(^3S_1)$	24	$-30.4(2.4)(5.1)$	$-0.08(1)(^{+1}_{-1})$	$-0.26(1)(^{+3}_{-3})$
		32	$-28.1(1.8)(2.4)$	$-0.070(4)(^{+6}_{-6})$	$-0.26(1)(^{+1}_{-1})$
		$\infty$	$-30.7(^{+2.5}_{-2.4})(^{+1.6}_{-0.5})$	$-0.077(6)(^{+4}_{-1})$	$-0.277(11)(^{+7}_{-2})$
		24	$-21.4(1.0)(0.5)$	$-0.053(2)(^{+1}_{-1})$	$-0.21(1)(^{+0}_{-0})$
		32	$-10.0(1.0)(0.4)$	$-0.025(2)(^{+1}_{-1})$	$-0.13(1)(^{+1}_{-0})$
		$\infty$	$-3.3(^{+0.9}_{-1.0})(^{+0.2}_{-0.6})$	$-0.008(^{+2}_{-3})(^{+1}_{-2})$	$-0.091(13)(^{+3}_{-7})$
		24 <sup>†</sup>	$44.3(1.1)(^{+0}_{-1.2})$	$0.112(3)(^{+1}_{-3})$	$0.10(2)(^{+0}_{-2})$
		32 <sup>†</sup>	$27.7(1.0)(^{+0}_{-1.5})$	$0.070(2)(^{+1}_{-4})$	$0.14(3)(^{+1}_{-3})$

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