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Letter

Flow equation, conformal symmetry, and anti-de Sitter geometry

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We argue that the anti-de Sitter (AdS) geometry in $d+1$ dimensions naturally emerges from an arbitrary conformal field theory in d dimensions using the free flow equation. We first show that an induced metric defined from the flowed field generally corresponds to the quantum information metric, called the Bures or Helstrom metric, if the flowed field is normalized appropriately. We next verify that the induced metric computed explicitly with the free flow equation always becomes the AdS metric when the theory is conformal. We finally prove that the conformal symmetry in d dimensions converts to the AdS isometry in $d+1$ dimensions after d-dimensional quantum averaging. This guarantees the emergence of AdS geometry without explicit calculation.

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1. *Introduction* The anti-de Sitter/conformal field theory (AdS/CFT) (or gravity/gauge theory) correspondence [1] is a promising tool to crack a hard problem in strongly coupled gauge theories (see Refs. [2–4] for some reviews), but is still mysterious even after many pieces of evidence and application appeared after the first proposal. Although the correspondence may be a manifestation of the closed string/open string duality, an alternative understanding might exist due to its holographic property.

One important mystery is the precise mechanism by which the AdS radial direction emerges from CFT. It may be common sense that the AdS radial direction is emergent as a renormalization scale of dual CFT as both behave similarly under dilatational symmetry [1], and this viewpoint works for renormalization group flows triggered by relevant deformations [5,6]. However, a direct approach to search for the Wilsonian cutoff corresponding to the sharp cutoff on the AdS radial direction [7] is still far from a clear answer, since the ordinary Wilsonian renormalization gives rise to non-local interaction in the bulk interpretation (see also Ref. [8]).[1]

While several approaches have been developed to *grab the tail* of the AdS radial direction and construct bulk dynamics from CFT [10–13] (see also Refs. [14,15]), one of the present authors, together with his collaborators, has proposed an alternative method to define a geometry from a quantum field theory and explicitly calculated the metric from several quantum field theories [16–18]. In Ref. [16], the method was proposed and applied to the O(n) nonlinear sigma model in

[1] This situation may drastically change in the AdS_3/CFT_2 correspondence. A clear dictionary was proposed in Ref. [9].

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2 dimensions, and the 3D metric in the large-n limit was shown to describe an AdS space in the massless limit. In Ref. [17], it was shown in the large-n limit that the induced metric describes an AdS_{d+1} space with $d \geq 3$ in the UV limit and with $d \geq 1$ in the IR limit, if the method is applied to the massive $\mathrm{O}(n)$ φ^4 model and an appropriate normalization is introduced for the flowed field. In Ref. [18], the large-n expansion was performed for the massless $\mathrm{O}(n)$ φ^4 model in 3 dimensions. While the induced metric describes the AdS_4 space at the leading order, the next-leading-order corrections make the space asymptotically AdS only in the UV and IR limits with different radii.

By observing that the induced metric always gives the AdS metric when the theory is conformal, it is natural to expect that symmetry plays a key role behind these results. In this letter, we investigate a direct relation between CFT and the AdS metric in this framework. The goal of this letter is to generalize the previous results to an arbitrary conformal field theory incorporating the symmetry argument.

The rest of this letter is organized as follows. In Sect. 1, after a brief explanation of the proposal of Ref. [16], we first show that our metric corresponds to the information metric, stressing the importance of the field normalization introduced in Ref. [17]. In Sect. 2, we explicitly derive the AdS metric directly from the CFT. We then prove that the induced metric in $d+1$ dimensions possesses the isometry of the AdS space as a consequence of the conformal symmetry in d dimensions. Section 6 is devoted to the summary and discussion.

2. *Gradient flow and information metric* In this section, we briefly review the proposal in Refs. [16,17] to define a $d+1$-dimensional induced metric from a d-dimensional quantum field theory. We also show that the metric defined in this way with appropriate normalization can be interpreted as a quantum information metric, called the Bures or Helstrom metric.

Gradient flow and induced metric We consider an n real component scalar field $\varphi(x)$ in d dimensions, whose quantum dynamics is controlled by the action functional $S(\varphi)$. The flowed field ϕ is defined from φ with the initial condition $\phi(x;0) = \varphi(x)$ through the flow equation as

$$\frac{\partial \phi^a(x;t)}{\partial t} = - \left.\frac{\delta S_f(\varphi)}{\delta \varphi^a(x)}\right|_{\varphi(x)\to\phi(x;t)}, \tag{1}$$

where the flow time t has the $(\text{length})^2$ dimension, x is the d-dimensional coordinate system, $a = 1, 2, \dots, n$ labels a component of the scalar field, and $S_f(\varphi)$ is an appropriate action for φ, which is not necessarily related to the original action $S(\varphi)$ in general. In particular, when they coincide, the flow is called the gradient flow [19–22]. In the case of the free flow (i.e., S_f is the free action), the flow equation becomes the heat equation. Thus the flow equation defines a procedure to smear the original field φ into a smeared field ϕ, the correlation functions of which are all finite at $t > 0$.

A $d+1$-dimensional metric operator is given by

$$\hat{g}_{MN}(x;t) := R^2 \sum_{a=1}^{n} \frac{\partial \sigma^a(x;t)}{\partial z^M} \frac{\partial \sigma^a(x;t)}{\partial z^N}, \tag{2}$$

where R is a constant with the length dimension, and $z^M = (x^\mu, \tau)$ with $\tau = \sqrt{2dt}$, which is regarded as the $d+1$-dimensional coordinates after d-dimensional quantum averaging, and $\sigma^a(x;t)$ is the (dimensionless) normalized flowed field defined as

$$\sigma^a(x;t) := \frac{\phi^a(x;t)}{\sqrt{\langle \sum_{a=1}^n \phi^a(x;t)^2 \rangle_S}}. \tag{3}$$

Here $\langle O(\varphi)\rangle_S$ denotes the quantum average with the d-dimensional action S, given by

$$\langle O(\varphi)\rangle_S := \frac{1}{Z}\int \mathcal{D}\varphi\, O(\varphi)\, e^{-S(\varphi)},\ Z = \int \mathcal{D}\varphi\, e^{-S(\varphi)}. \tag{4}$$

Since $\sigma^a(x;t)$ always satisfies $\sum_{a=1}^n \left\langle \sigma^a(x;t)\sigma^a(x;t)\right\rangle_S = 1$, we call this normalization condition the nonlinear sigma model (NLSM) normalization, the importance of which will become clear in the next subsection. The vacuum expectation value (VEV) of the metric is denoted as $g_{MN}(z) := \langle \hat{g}_{MN}(x;t)\rangle_S$, the fluctuations of which are suppressed in the large-n limit (see Refs. [16–18] for more details).

Information metric In this subsection, we show that $g_{MN}(z)$ defined in the previous subsection is equivalent to a quantum information metric, called the Bures (or Helstrom) metric. The NLSM normalization is important to show this.

The Bures metric for the density matrix is defined from the infinitesimal distance between two density matrices ρ and $\rho + d\rho$ as

$$D(\rho, \rho + d\rho)^2 = \frac{1}{2}\mathrm{tr}(d\rho\, G), \tag{5}$$

where G is the Hermitian 1-form operator implicitly given by $\rho\, G + G\,\rho = d\rho$. In particular, for the density matrix ρ of a pure state, the Hermitian operator is determined as $G = d\rho$ since $\rho^2 = \rho$.

In order to apply to our case, we consider an eigenstate of the position operator as well as the flow time one denoted by $|z\rangle = |(x, \tau)\rangle$ and define the inner product of the state as

$$\langle z|w\rangle := \sum_{a=1}^n \langle \sigma^a(x;t)\sigma^a(y;s)\rangle_S, \quad w = (y, \sqrt{2ds}). \tag{6}$$

Notice that the NLSM normalization guarantees $\langle z|z\rangle = 1$. Then the information metric for this pure state is computed as

$$R^2 D(\rho_z, \rho_{z+dz})^2 = g_{MN}(z)dz^M dz^N, \tag{7}$$

where $\rho_z = |z\rangle\langle z|$, and we used $\langle \sigma(x;t)\cdot \partial_M \sigma(x;t)\rangle_S = 0$ and

$$\langle \sigma(x;t)\cdot \partial_M\partial_N \sigma(x;t)\rangle_S = -\langle \partial_M\sigma(x;t)\cdot \partial_N\sigma(x;t)\rangle_S$$

with $A\cdot B := \sum_{a=1}^n A^a B^a$. Our metric $g_{MN}(z)$ defines a distance in the space of the density matrices made of the pure states (in units of R). In particular, when the action S of the original theory has O(n) symmetry, such pure states defined above in an abstract way may be given by $\sum_a \sigma^a(x;t)|0\rangle$.[2]

[2] In the case of the $O(n)$ invariant mixed state $\rho_z := \sum_a \sigma^a(x;t)|0\rangle\langle 0|\sigma^a(x;t)$, we have $G = n d\rho_z$ from $\rho_z^2 = \rho_z/n$.

3. *AdS isometry from conformal symmetry* In this section, we directly relate arbitrary conformal field theory in the flat d-dimensional space-time with the AdS metric in $d+1$ dimensions.[3] We here assume that the CFT contains a real scalar primary operator $\varphi(x)$ of a general conformal dimension Δ without specifying any concrete models for CFT. In this section, we are interested in the VEV of the metric operator, which determines the classical geometry induced from this primary operator, though its "quantum" fluctuations around the VEV exist for $n=1$ in this case.

AdS metric from CFT We start with the free generally massive flow equation with $n=1$:

$$\frac{\partial \phi(x;t)}{\partial t} = (\partial^2 - m^2)\phi(x;t), \tag{8}$$

where $\partial^2 = \eta^{\mu\nu}\partial_\mu\partial_\nu$. This is easily solved as

$$\phi(x;t) = e^{t(\partial^2 - m^2)}\varphi(x). \tag{9}$$

The two-point function of ϕ is evaluated as

$$\begin{aligned} G_0(x;t|y;s) &:= \langle \phi(x;t)\phi(y;s)\rangle_{\text{CFT}} \\ &= e^{-(t+s)m^2} e^{(t\partial_x^2 + s\partial_y^2)} \langle \varphi(x)\varphi(y)\rangle_{\text{CFT}}, \end{aligned} \tag{10}$$

where we have used the subscript CFT instead of S, since the action is not specified. The Poincaré invariance and the scale transformation of φ with $\varphi(\lambda x) = \lambda^{-\Delta}\varphi(x)$ fix the form of G_0 such that

$$G_0(x;t|y;s) = \frac{e^{-(t+s)m^2}}{(t+s)^\Delta} F_0\left(\frac{(x-y)^2}{t+s}\right), \tag{11}$$

where F_0 is a certain smooth function. Furthermore, the flow equation (8) implies

$$\frac{\partial}{\partial t} G_0(x;t|y;s) = (\partial_x^2 - m^2) G_0(x;t|y;s), \tag{12}$$

which leads to $\Delta F_0(0) = -2dF_0'(0)$. Thus the two-point function of the normalized field σ becomes m independent as

$$\begin{aligned} G(x;t|y;s) &:= \langle \sigma(x;t)\sigma(y;s)\rangle_{\text{CFT}} \\ &= \left(\frac{2\sqrt{ts}}{t+s}\right)^\Delta F\left(\frac{(x-y)^2}{t+s}\right), \end{aligned} \tag{13}$$

where $F(x) \equiv F_0(x)/F_0(0)$. Hence $F(0)=1$ and $2dF'(0) = -\Delta$.[4] Taking the $d+1$-dimensional coordinates as $z = (x, \tau = \sqrt{2dt})$, the vacuum expectation value of the induced metric $\hat{g}_{MN}$ is

$$g_{MN}(z) = R^2 \langle \partial_M \sigma(x;t)\partial_N\sigma(x;t)\rangle_{\text{CFT}}. \tag{14}$$

[3] The argument and calculation below hold just by changing the signature suitably when we consider the Euclidean flat space.

[4] Explicitly, $F(x)$ is computed as

$$F(x) = \frac{\Gamma(d/2)}{\Gamma(\Delta)\Gamma(d/2-\Delta)} \int_0^1 dv\, v^{\Delta-1}(1-v)^{d/2-\Delta-1} e^{-xv/4}.$$

This is computed as ($g_{\mu\tau}(z) = g_{\tau\nu}(z) = 0$)

$$g_{\mu\nu}(z) = \eta_{\mu\nu}\frac{R^2\Delta}{\tau^2}, \quad g_{\tau\tau}(z) = \frac{R^2\Delta}{\tau^2}, \tag{15}$$

which is nothing but the AdS metric with $R\sqrt{\Delta}$ as its radius.

Isometry from conformal transformation In the previous subsection, we have explicitly calculated the induced metric g_{MN} from CFT, and have shown that it becomes the AdS metric in the Poincaré patch. In this subsection, we discuss the relation between the AdS metric and the CFT only from the symmetry. Namely, we show that the induced metric $g_{MN}(z) = \langle \hat{g}_{MN}(x;t)\rangle_{\text{CFT}}$ possesses the isometry of the AdS space. This is necessary and sufficient since AdS is a maximally symmetric space, so that the metric is completely fixed by the isometry group $SO(2,d)$ up to an overall constant.

We first relate the conformal transformation to the isometry of AdS.[5] The infinitesimal conformal transformation and the response of the primary scalar operator are given by

$$\begin{aligned}\delta x^\mu &= a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + b^\mu x^2 - 2x^\mu(b_\nu x^\nu),\\ \delta^{\text{conf}}\varphi(x) &= -\delta x^\mu\partial_\mu\varphi(x) - \frac{\Delta}{d}(\partial_\mu\delta x^\mu)\varphi(x).\end{aligned} \tag{16}$$

Here a^μ, $\omega^{\mu\nu}$, λ, b^μ are parameters of the transformation. Since the infinitesimal conformal transformation is quadratic in the coordinate x, the normalized field σ, given in terms of φ by

$$\sigma(x;t) = \frac{\left(\sqrt{2t}\right)^\Delta}{\sqrt{F_0(0)}}\, e^{t\partial^2}\,\varphi(x), \tag{17}$$

is transformed as

$$\begin{aligned}\delta^{\text{conf}}\sigma(x;t) = &-\left\{t(\partial^2\delta x^\mu) + 2t^2(\partial^\nu\partial^\rho\delta x^\mu)\partial_\nu\partial_\rho + 2t(\partial^\nu\delta x^\mu)\partial_\nu + \delta x^\mu\right\}\partial_\mu\sigma(x;t)\\ &- \frac{\Delta}{d}\left\{2t(\partial^\nu\partial_\mu\delta x^\mu)\partial_\nu + (\partial_\mu\delta x^\mu)\right\}\sigma(x;t).\end{aligned} \tag{18}$$

Plugging Eq. (16) into Eq. (18), we obtain

$$\delta^{\text{conf}}\sigma(x;t) = -\{2\lambda - 4(b_\mu x^\mu)\}t\partial_t\sigma(x;t) - \{\delta x^\mu + 2t(d-2-\Delta)b^\mu\}\partial_\mu\sigma(x;t) + 4t^2 b^\mu\partial_\mu\partial_t\sigma(x;t), \tag{19}$$

where we have used $\partial_t\sigma(x;t) = (\frac{\Delta}{2t} + \partial^2)\sigma(x;t)$. Notice that a special conformal transformation of the normalized flow operator contains a higher-derivative term as the last term in Eq. (19), which cannot be interpreted as an infinitesimal diffeomorphism in the bulk.[6] To deal with this we rewrite

[5] This question was addressed in the early stage after Maldacena's proposal in Ref. [23], in which was studied the mechanism of how the special conformal transformation of the adjoint scalar fields in $\mathcal{N} = 4$ super Yang–Mills theory was *metamorphosed* into the corresponding isometry transformation in the bulk. We would like to thank T. Yoneya for his valuable comment given in the KEK Theory workshop 2017 "East Asia Joint Workshop on Fields and Strings 2017".

[6] Difficulty in the bulk interpretation of the special conformal transformation of a key operator was also observed in the bilocal field approach by using a vector model [24], where the special conformal transformation of the collective field mixes up fields with different spin. The authors speculated on the necessity of a suitable field redefinition to resolve this issue, though our answer may be different in the flow field approach, as shown below. We would like to thank S. Das for his valuable comment given in the KIAS-YITP joint workshop 2017 "Strings, Gravity and Cosmology".

Eq. (19) as

$$\delta^{\rm conf}\sigma(x;t) = \delta^{\rm diff}\sigma(x;t) + \delta^{\rm extra}\sigma(x;t), \tag{20}$$

$$\delta^{\rm diff}\sigma(x;t) = -(\bar\delta t\partial_t + \bar\delta x^\mu\partial_\mu)\sigma(x;t), \quad \delta^{\rm extra}\sigma(x;t) = 4t^2 b^\nu\partial_\nu\left(\partial_t + \frac{\Delta+2}{2t}\right)\sigma(x;t), \tag{21}$$

with $\bar\delta x^\mu = \delta x^\mu + 2dtb^\mu, \bar\delta t = (2\lambda - 4(b_\mu x^\mu))t$. By setting $\tau^2 = 2dt$, the transformation $\delta^{\rm diff}$ can be rewritten as

$$\bar\delta x^\mu = \delta x^\mu + \tau^2 b^\mu, \quad \bar\delta\tau = (\lambda - 2(b_\mu x^\mu))\tau. \tag{22}$$

This is nothing but the isometry transformation of the AdS space whose metric is given by $ds^2_{\rm AdS} \propto \dfrac{d\tau^2 + dx^{\mu 2}}{\tau^2}$.

The conformal transformation of the induced metric operator is computed as

$$\delta^{\rm conf}\hat g_{MN}(x;t) = \delta^{\rm diff}\hat g_{MN}(x;t) + R^2 \lim_{(y;s)\to(x;t)} \frac{\partial}{\partial z^M}\frac{\partial}{\partial w^N}\left\{\delta^{\rm extra}\sigma(x;t)\sigma(y;s) + \sigma(x;t)\delta^{\rm extra}\sigma(y;s)\right\}. \tag{23}$$

The first term is

$$\delta^{\rm diff}\hat g_{MN}(x;t) = -\bar\delta z^K\partial_K\hat g_{MN}(x;t) - \partial_M\bar\delta z^K\hat g_{KN}(x;t) - \partial_N\bar\delta z^K\hat g_{MK}(x;t), \tag{24}$$

which is nothing but the diffeomorphism of the metric tensor in $d+1$ dimensions.

Thus our task is to show that the second term in Eq. (23) vanishes in the vacuum expectation value. By using Eqs. (13) and (21), the term is computed as

$$\begin{aligned}&\left\langle\delta^{\rm extra}\sigma(x;t)\sigma(y;s) + \sigma(x;t)\delta^{\rm extra}\sigma(y;s)\right\rangle\\ &\quad= -8\frac{(\sqrt{4ts})^\Delta}{(t+s)^{\Delta+2}}(t-s)b_\mu(x-y)^\mu(x-y)^2 F''\left(\frac{(x-y)^2}{t+s}\right).\end{aligned} \tag{25}$$

Then it is easy to see

$$\lim_{(y;s)\to(x;t)}\frac{\partial}{\partial z^M}\frac{\partial}{\partial w^N}\left\langle\delta^{\rm extra}\sigma(x;t)\sigma(y;s) + \sigma(x;t)\delta^{\rm extra}\sigma(y;s)\right\rangle = 0. \tag{26}$$

We stress that this happens only when the conformal transformation is decomposed as Eq. (21). Note that the quantum averaging and the differentiation commute since all correlation functions of σ are finite as long as the flow time is non-zero. Therefore we obtain

$$\left\langle\delta^{\rm conf}\hat g_{MN}(x;t)\right\rangle = \left\langle\delta^{\rm diff}\hat g_{MN}(x;t)\right\rangle. \tag{27}$$

Since the conformal invariance of the two-point function of the primary scalar operator implies that $\langle\delta^{\rm conf}\hat g_{MN}(x;t)\rangle = 0$, it follows that $\langle\delta^{\rm diff}\hat g_{MN}(x;t)\rangle = 0$. This means that the induced metric $g_{MN}(z)$ satisfies the Killing equation of the AdS space as

$$\delta^{\rm diff}g_{MN}(z) = -\bar\delta z^K\partial_K g_{MN}(z) - \partial_M\bar\delta z^K g_{KN}(z) - \partial_N\bar\delta z^K g_{MK}(z) = 0, \tag{28}$$

which implies that the induced metric must be the AdS metric up to an overall constant. This completes the proof of our claim.

4. *Discussion* In this letter an induced geometry by a flow equation from a quantum field theory was investigated. The induced metric was shown to appear as a quantum information metric, which measures a distance in the space of the pure states constructed by scalar fields in a general quantum field theory. In a conformally symmetric situation, it was shown that the induced metric matches the AdS one when the flow equation is free.[7] This agreement was confirmed only by using symmetry without any explicit computations of the metric. An appearance of the AdS metric from CFT in our method relies on the following two facts. (a) The field σ used to define the metric operator is dimensionless thanks to the NLSM normalization. If ϕ were used instead, one would not obtain the AdS metric. (b) The VEV of the metric operator is UV finite thanks to the free flow equation. If the VEV were UV divergent, one would not obtain the AdS metric due to additional dimensionful quantities introduced through renormalization.

So far any relation between the induced metric formalism presented in this letter and the other approaches to seeing dual geometry mentioned in the introduction is not known. It may be reasonable to think that there is no relation between them since, e.g., the procedure to renormalize fields in quantum field theory and that to smear operators are generally independent. Still, we expect that the results and technique developed in this letter, particularly the symmetry argument, will become useful in studying the AdS geometry from CFT by other methods. For example, it may be possible to define an induced metric similarly in the Wilsonian renormalization approach to dual gravity. Then it would be interesting to see whether the metric becomes the AdS one or not.

In this letter we assume that the background of quantum field theory or conformal field theory is flat. It would be interesting to extend the presented calculation to curved backgrounds. In particular, it would be interesting to check whether the induced metric from CFT on a curved space-time is still of AdS form in a different coordinate system from the Poincaré patch.

A challenging but important issue is whether this formalism encodes the gravitational dynamics or not. The first step toward this goal may be to see how a linearized Einstein gravity is encoded in this formulation, as shown in a different method to derive dual bulk dynamics by using entanglement entropy [25,26]. For this analysis it will be necessary to specify a concrete model to test the proposal such as an O(n) sigma model, since bulk dynamics is dependent on each CFT. Note that the $1/n$ expansion becomes important to see the dual bulk dynamics beyond the geometry in the AdS/CFT correspondence. A virtue of this formulation is that observables in this formalism are correlation functions of scalar fields in quantum field theory, which admit analytic computation by the ordinary technique of $1/n$ expansion [18], so that one can proceed by checking one's guesswork explicitly by hand.

We hope to report on resolutions on these issues in the near future.

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[7] This result does not depend on mass m in the flow equation, which violates the conformal symmetry, thanks to the NLSM normalization.

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References

[1] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998) [arXiv:hep-th/9711200] [Search INSPIRE].
[2] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri, and Y. Oz, Phys. Rept. **323**, 183 (2000) [arXiv:hep-th/9905111] [Search INSPIRE].
[3] I. R. Klebanov, Strings, Branes and Gravity: Proc. Theoretical Advanced Study Institute (TASI'99), p. 615 (2000) [arXiv:hep-th/0009139] [Search INSPIRE].
[4] E. D'Hoker and D. Z. Freedman, Strings, Branes and Extra Dimensions: Proc. TASI 2001, p. 3 (2002) [arXiv:hep-th/0201253] [Search INSPIRE].
[5] L. Girardello, M. Petrini, M. Porrati, and A. Zaffaroni, J. High Energy Phys. **12**, 022 (1998) [arXiv:hep-th/9810126] [Search INSPIRE].
[6] J. Distler and F. Zamora, Adv. Theor. Math. Phys. **2**, 1405 (1999) [arXiv:hep-th/9810206] [Search INSPIRE].
[7] I. Heemskerk and J. Polchinski, J. High Energy Phys. **06**, 031 (2011) [arXiv:1010.1264 [hep-th]] [Search INSPIRE].
[8] S.-S. Lee, Nucl. Phys. B **851**, 143 (2011) [arXiv:1011.1474 [hep-th]] [Search INSPIRE].
[9] L. McGough, M. Mezei, and H. Verlinde, arXiv:1611.03470 [hep-th] [Search INSPIRE].
[10] T. Banks, M. R. Douglas, G. T. Horowitz, and E. J. Martinec, arXiv:hep-th/9808016 [Search INSPIRE].
[11] V. Balasubramanian, S. B. Giddings, and A. Lawrence, J. High Energy Phys. **03**, 001 (1999) [arXiv:hep-th/9902052] [Search INSPIRE].
[12] I. Bena, Phys. Rev. D **62**, 066007 (2000) [arXiv:hep-th/9905186] [Search INSPIRE].
[13] G. Lifschytz and V. Periwal, J. High Energy Phys. **04**, 026 (2000) [arXiv:hep-th/0003179] [Search INSPIRE].
[14] M. Van Raamsdonk, arXiv:0907.2939 [hep-th] [Search INSPIRE].
[15] M. Nozaki, S. Ryu, and T. Takayanagi, J. High Energy Phys. **10**, 193 (2012) [arXiv:1208.3469 [hep-th]] [Search INSPIRE].
[16] S. Aoki, K. Kikuchi, and T. Onogi, Prog. Theor. Exp. Phys. **2015**, 101B01 (2015) [arXiv:1505.00131 [hep-th]] [Search INSPIRE].
[17] S. Aoki, J. Balog, T. Onogi, and P. Weisz, Prog. Theor. Exp. Phys. **2016**, 083B04 (2016) [arXiv:1605.02413 [hep-th]] [Search INSPIRE].
[18] S. Aoki, J. Balog, T. Onogi, and P. Weisz, Prog. Theor. Exp. Phys. **2017**, 043B01 (2017) [arXiv:1701.00046 [hep-th]] [Search INSPIRE].
[19] R. Narayanan and H. Neuberger, J. High Energy Phys. **03**, 064 (2006) [arXiv:hep-th/0601210] [Search INSPIRE].
[20] M. Lüscher, J. High Energy Phys. **08**, 071 (2010); **03**, 092 (2014) [erratum] [arXiv:1006.4518 [hep-lat]] [Search INSPIRE].
[21] M. Lüscher, Commun. Math. Phys. **293**, 899 (2010) [arXiv:0907.5491 [hep-lat]] [Search INSPIRE].
[22] M. Lüscher, PoS **LATTICE2013**, 016 (2014) [arXiv:1308.5598 [hep-lat]] [Search INSPIRE].
[23] A. Jevicki, Y. Kazama, and T. Yoneya, Phys. Rev. Lett. **81**, 5072 (1998) [arXiv:hep-th/9808039] [Search INSPIRE].
[24] S. R. Das and A. Jevicki, Phys. Rev. D **68**, 044011 (2003) [arXiv:hep-th/0304093] [Search INSPIRE].
[25] T. Faulkner, M. Guica, T. Hartman, R. C. Myers, and M. Van Raamsdonk, J. High Energy Phys. **03**, 051 (2014) [arXiv:1312.7856 [hep-th]] [Search INSPIRE].
[26] J. Bhattacharya and T. Takayanagi, J. High Energy Phys. **10**, 219 (2013) [arXiv:1308.3792 [hep-th]] [Search INSPIRE].