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Flow equation for the scalar model in the large N expansion and its applications

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We study the flow equation of the O(N) φ^4 model in d dimensions at the next-to-leading order (NLO) in the $1/N$ expansion. Using the Schwinger–Dyson equation, we derive 2-pt and 4-pt functions of flowed fields. As the first application of the NLO calculations, we study the running coupling defined from the connected 4-pt function of flowed fields in $d+1$-dimensional theory. We show in particular that this running coupling has not only an ultraviolet fixed point but also an infrared fixed point (Wilson–Fisher fixed point) in 3-dimensional massless scalar theory. As the second application, we calculate the NLO correction to the induced metric in $d+1$ dimensions with $d=3$ in the massless limit. While the induced metric describes a 4-dimensional Euclidean Anti-de-Sitter (AdS) space at the leading order, as shown in the previous paper, the NLO corrections make the space asymptotically AdS only in the UV and IR limits. Remarkably, while the AdS radius does not receive an NLO correction in the UV limit, the AdS radius decreases at the NLO in the IR limit, which corresponds to the Wilson–Fisher fixed point in the original scalar model in 3 dimensions.

Subject Index B32, B34, B35

1. Introduction

In the previous paper (Ref. [1]), the present authors studied the proposal (Ref. [2]) that a $(d+1)$-dimensional induced metric can be constructed from a d-dimensional field theory using gradient flow (Refs. [3–6]), applying the method to the O(N) φ^4 model. We showed that in the large N limit the induced metric becomes classical and describes Euclidean Anti-de-Sitter (AdS) space in both ultraviolet (UV) and infrared (IR) limits of the flow direction. The method proposed in Ref. [2] may provide an alternative way to understand the AdS/CFT (or more generally gravity/gauge theory) correspondence (Ref. [7]), and the result in Ref. [1] might be related to the correspondence between O(N) vector models in d dimensions and (generalized) gravity theories in $d+1$ dimensions (Ref. [8]).

To further investigate a possible connection between Refs. [1] and [8] at the quantum level, one must calculate, e.g., the anomalous dimension of the O(N) invariant operator $\phi^2(x)$, which requires the next-to-leading order (NLO) of the $1/N$ expansion for the flow equation to evaluate necessary quantum corrections. Since the method employed in Refs. [1,2] is a specific one adopted for the large N limit, some systematic way to solve the flow equation in the $1/N$ expansion is needed.

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In this paper, we employ the Schwinger–Dyson equation (SDE) to solve the flow equation in the $1/N$ expansion for the $\mathrm{O}(N)$ invariant φ^4 model in d dimensions. Using this method we explicitly calculate the 2-pt and 4-pt functions at the NLO.

As the first application of the NLO calculations, we define a running coupling from the connected 4-pt function of flowed fields, which runs with the flow time t such that $t = 0$ corresponds to the UV limit while $t = \infty$ is the IR limit. This property establishes that the flow equation can be interpreted as a renormalization group transformation (see Ref. [9] as an earlier attempt). In particular at $d = 3$, we show that the running coupling so defined has not only an asymptotic free UV fixed point but also a Wilson–Fisher IR fixed point for the massless case.

As the second application, we investigate the NLO correction to the induced metric in $3 + 1$ dimensions from the massless scalar model in 3 dimensions. In the massless limit, the whole 4-dimensional space becomes AdS at the leading order, as shown in Ref. [1]. The NLO corrections give a small perturbation to the metric, which makes the space asymptotically AdS in UV ($t = 0$) and IR ($t = \infty$) limits only. A remarkable thing is that, while the NLO corrections do not change the AdS radius in the UV limit, the AdS radius is reduced by the NLO correction in the IR limit, which corresponds to the Wilson–Fisher IR fixed point of the original theory. In other words, a nontrivial fixed point in the field theory leads to a change of the AdS radius in the geometry at NLO. The induced metric at NLO describes a 4-dimensional space connecting one asymptotically AdS space at UV to another asymptotically AdS space at IR, which have different radii.

This paper is organized as follows. In Sect. 2, we first introduce the $\mathrm{O}(N)$ invariant φ^4 model in d dimensions. We then formulate the Schwinger–Dyson equation (SDE) for the flowed fields, and solve it to derive 2-pt and 4-pt functions of flowed fields at the NLO. In Sect. 3, we define a running coupling from the connected 4-pt function of flowed fields and investigate its behavior as a function of the flow time t. In Sect. 4, we study the induced metric from the 3-dimensional massless scalar model at the NLO. We finally give a summary of this paper in Sect. 5. We collect all technical details in appendices. In Appendix A, using the SDE, we present results at the NLO in the $1/N$ expansion of d-dimensional theory, necessary for the main text. We also perform the renormalization of d-dimensional theory at the NLO, and explicitly calculate renormalization constants for various d. In Appendix B, we give detailed derivations of solutions to the SDE for the flow fields at the NLO. We explicitly evaluate 2-pt and 4-pt functions of the flowed field in Appendix C while in Appendix D we derive the induced metric for the massless scalar theory in 3 dimensions.

2. $1/N$ expansion of the flow equation in $d + 1$ dimensions

2.1. *Model in d dimensions*

In this paper, we consider the N-component scalar φ^4 model in d dimensions, defined by the action

$$S(\mu^2, u) = N \int d^d x \left[\frac{1}{2} \partial^k \varphi(x) \cdot \partial_k \varphi(x) + \frac{\mu^2}{2} \varphi^2(x) + \frac{u}{4!} \left(\varphi^2(x)\right)^2 \right], \tag{1}$$

where $\varphi^a(x)$ is an N-component scalar field, $(\,\cdot\,)$ indicates an inner product of N component vectors such that $\varphi^2(x) \equiv \varphi(x) \cdot \varphi(x) = \sum_{a=1}^N \varphi^a(x)\varphi^a(x)$, μ^2 is the bare scalar mass parameter, and u is the coupling constant of the φ^4 interaction, whose canonical dimension is $4 - d$. While it is consistent to take u as N independent, as will be seen later the mass parameter μ^2 is expanded as

$$\mu^2 = \mu_0^2 + \frac{1}{N}\mu_1^2 + \cdots , \tag{2}$$

where μ_i^2 is cut-off dependent in order to make the physical mass finite, order by order, in the $1/N$ expansion.

This model describes the free massive scalar at $u = 0$, while it is equivalent to the nonlinear σ model (NLSM) in the $u \to \infty$ limit, whose action is obtained from Eq. (1) as

$$S(\lambda) = \frac{N}{2\lambda}\int d^dx\, \partial^k\sigma(x)\cdot\partial_k\sigma(x), \qquad \sigma^2(x) = 1, \tag{3}$$

with the replacement

$$\sigma^a(x) = \sqrt{\lambda}\varphi^a(x), \qquad \lambda = \lim_{u\to\infty} -\frac{u}{6\mu^2}. \tag{4}$$

Some regularization that preserves O(N) symmetry is assumed in this paper, so that we can always make formal manipulations without worrying about divergences.[1] Calculations of 2-pt and 4-pt functions at the NLO of the $1/N$ expansion in d dimensions will be given in Appendix A.

2.2. *Flow equation in the $1/N$ expansion*

In this paper, we consider the flow equation, given by

$$\frac{\partial}{\partial t}\phi^a(t,x) = -\frac{1}{N}\frac{\delta S(\mu_f^2, u_f)}{\delta\varphi^a(x)}\bigg|_{\varphi\to\phi} = \left(\Box - \mu_f^2\right)\phi^a(t,x) - \frac{u_f}{6}\phi^a(t,x)\phi^2(t,x), \tag{5}$$
$$\phi^a(0,x) = \varphi^a(x),$$

where μ_f^2 and u_f can be different from μ^2 and u in the original d-dimensional theory. As in the case of d dimensions, u_f is kept fixed and N independent, whereas μ_f^2 is adjusted as

$$\mu_f^2 = m_f^2 - \frac{u_f}{6}Z(m_f), \qquad Z(m_f) \equiv \int Dq\frac{1}{q^2 + m_f^2}, \quad Dq \equiv \frac{d^dq}{(2\pi)^d}, \tag{6}$$

where m_f is a renormalized mass. The flow with $\mu_f = \mu$ and $u_f = u$ is called gradient flow, as it is given by the gradient of the original action.

In the case of free flow ($u_f = 0$), the solution is easily given by

$$\phi^a(t,x) = \exp\left(t(\Box - \mu_f^2)\right)\varphi^a(x). \tag{7}$$

We therefore consider the interacting flow ($u_f \neq 0$) hereafter unless otherwise stated.

The above flow equation leads to the SDE (Ref. [10]) as

$$\langle D_z^f\phi^a(z)\mathcal{O}\rangle = -\frac{u_f}{6}\langle\phi^a(z)\phi^2(z)\mathcal{O}\rangle, \quad D_z^f \equiv \frac{\partial}{\partial t} - (\Box - \mu_f^2), \tag{8}$$

[1] We will call the infinite cutoff ($\Lambda \to \infty$) limit the "continuum limit".

where $z = (t, x)$, $\mathcal{O}$ is an arbitrary operator, and the expectation value $\langle \mathcal{O} \rangle$ should be calculated in d dimensions as

$$\langle \mathcal{O}(\varphi) \rangle \equiv \frac{1}{Z} \int [\mathcal{D}\varphi]\, \mathcal{O}(\varphi) \exp\left(-S(\mu^2, u)\right), \quad Z = \int [\mathcal{D}\varphi] \exp\left(-S(\mu^2, u)\right). \tag{9}$$

If we take $\mathcal{O} = \prod_{i=1}^{2n-1} \phi^{a_i}(z_i)$, the SDE becomes

$$D_z^f\, \Gamma_{2n}^{aa_1\cdots a_{2n-1}}(z, z_1, \ldots, z_{2n-1}) = -\frac{u_f}{6N^2} \sum_b \Gamma_{2n+2}^{abba_1\cdots a_{2n-1}}(z, z, z, z_1, \ldots, z_{2n-1}), \tag{10}$$

where Γ_n is the n-point function defined by

$$\Gamma_n^{a_1\cdots a_n}(z_1, \ldots, z_n) = N^{n-1} \left\langle \prod_{i=1}^{n} \phi^{a_i}(z_i) \right\rangle \equiv \Gamma_n[12\cdots n], \tag{11}$$

which is analogous to the d-dimensional counterpart in Eq. (A.3). We consider only the symmetric phase in this paper, where $\Gamma_{2n-1} = 0$ for all positive integers n.

We consider the next-to-leading order of the $1/N$ expansion, so that the following two SDEs need to be considered:

$$D_1^f\, \Gamma_2[12] = -\frac{u_f}{6N^2} \sum_b \Gamma_4[1bb2], \tag{12}$$

$$D_1^f\, \Gamma_4[1234] = -\frac{u_f}{6N^2} \sum_b \Gamma_6[1bb234], \tag{13}$$

where $z_b = z_1$, so that the sum over b runs over the O(N) index only.

The connected parts of 4-pt and 6-pt functions are introduced as

$$\Gamma_4[1234] = K_4[1234] + N\left\{\Gamma_2[12]\Gamma_2[34] + \Gamma_2[13]\Gamma_2[24] + \Gamma_2[14]\Gamma_2[23]\right\}, \tag{14}$$

$$\begin{aligned}\Gamma_6[123456] = {} & K_6[123456] + N\left\{\Gamma_2[12]K_4[3456] + 14 \text{ perms.}\right\} \\ & + N^2\left\{\Gamma_2[12]\Gamma_2[34]\Gamma_2]56] + 14 \text{ perms.}\right\}.\end{aligned} \tag{15}$$

Furthermore, decompositions in O(N) indices are given by

$$\Gamma_2[12] = \delta^{a_1a_2}\Gamma(z_1, z_2), \tag{16}$$

$$K_4[1234] = \delta^{a_1a_2}\delta^{a_3a_4}K(z_1, z_2; z_3, z_4) + 2 \text{ perms.}, \tag{17}$$

$$K_6[123456] = \delta^{a_1a_2}\delta^{a_3a_4}\delta^{a_5a_6}H(z_1, z_2; z_3, z_4; z_5, z_6) + 14 \text{ perms.}, \tag{18}$$

where $\Gamma(z_1, z_2)$, $K(z_1, z_2; z_3, z_4)$, and $H(z_1, z_2; z_3, z_4; z_5, z_6)$ are invariant under the exchange of arguments such that $z_{2i-1} \leftrightarrow z_{2i}$ or $(z_{2i-1}, z_{2i}) \leftrightarrow (z_{2j-1}, z_{2j})$.

By expanding Γ, K, and H as

$$\Gamma = \sum_{i=0}^{\infty} \frac{\Gamma_i}{N^i}, \qquad K = \sum_{i=0}^{\infty} \frac{K_i}{N^i}, \qquad H = \sum_{i=0}^{\infty} \frac{H_i}{N^i}, \tag{19}$$

the above two SDEs are reduced to

$$D_1^f \Gamma_0(12) = -\frac{u_f}{6}\Gamma_0(12)\Gamma_0(11) \tag{20}$$

at the LO of the $1/N$ expansion, and

$$D_1^f \Gamma_1(12) = -\frac{u_f}{6}\left[K_0(12;11) + \Gamma_0(12)\Gamma_1(11) + \Gamma_1(12)\Gamma_0(11) + 2\Gamma_0(12)\Gamma_0(11)\right], \quad (21)$$

$$D_1^f K_0(12;34) = -\frac{u_f}{6}\left[\Gamma_0(12)K_0(11;34) + \Gamma_0(11)K_0(12;34) + 2\Gamma_0(12)\Gamma_0(13)\Gamma_0(14)\right] \quad (22)$$

at the NLO.

2.3. *Solutions to the flowed SDE at NLO*

The solutions to the SDE at NLO are summarized below. Details of calculations can be found in Appendix B.

At the NLO, the 2-pt function is given by

$$\langle \phi^{a_1}(z_1)\phi^{a_2}(z_2)\rangle = \frac{\delta_{a_1a_2}}{N}\frac{Z(m_f)}{(\zeta(t_1)\zeta(t_2))^{1/2}}\int Dp \frac{\exp\left(-p^2(t_1+t_2)\right)\exp\left(ip(x_1-x_2)\right)}{p^2+m^2}\left[1+\frac{1}{N}G_1(t_1,t_2|p)\right], \quad (23)$$

where $\zeta(t)$ is defined in Eq. (B.7), and the NLO contribution $G_1(t_1, t_2|p)$ is given in Appendix B.3.2. In the continuum limit, $\zeta(t)$ approaches $\zeta_0(t)$ and is finite as long as $t > 0$, where

$$\zeta_0(t) \equiv \int Dp \frac{\exp\left(-2p^2t\right)}{p^2+m^2} = \frac{\exp\left(2tm^2\right)m^{d-2}}{(4\pi)^{d/2}}\Gamma(1-d/2, 2tm^2) \quad (24)$$

with the incomplete gamma function $\Gamma(a, x)$, while $Z(m_f)$ diverges at $d > 1$.

The leading contribution of the connected 4-pt function appearing at the NLO of the $1/N$ expansion can be obtained as

$$\langle \phi^{a_1}(z_1)\phi^{a_2}(z_2)\phi^{a_3}(z_3)\phi^{a_4}(z_4)\rangle_c = \frac{1}{N^3}\left[\delta_{a_1a_2}\delta_{a_3a_4}K_0(12;34) + 2\text{ perms.}\right], \quad (25)$$

where

$$K_0(12;34) = \int dP_4\, g(12;34|12;34),$$
$$dP_4 \equiv \prod_{j=1}^{4} Dp_j \left(\frac{Z(m_f)}{\zeta(t_j)}\right)^{1/2}\frac{\exp\left(ip_jx_j\right)\exp\left(-p_j^2t_j\right)}{p_j^2+m^2}, \quad (26)$$

$$g(12;34|12;34) = X(23|12;34) + X(13|21;34) + X(24|12;43) + X(14|21;43) + Y(2|12;34) + Y(1|21;34) + Y(3|43;12) + Y(4|34;12) + Z(|12;34). \quad (27)$$

Here the variables to the left of the vertical line refer to flow times and those to the right refer to momenta. Explicitly we have in the continuum or NLSM limits,

$$X(t_1, t_2|12;34) = \hat{\delta}(p_2^2 + m^2)(p_3^2 + m^2) \times \int_0^{t_1} ds_1 \int_0^{t_2} ds_2 \exp\left(s_1(p_2^2 - p_1^2)\right) \exp\left(s_2(p_3 - p_4^2)\right) \omega(s_1, s_2|p_{34}), \tag{28}$$

$$Y(t|21;34) = \hat{\delta}(p_1^2 + m^2) \int_0^t ds \exp\left(s(p_1^2 - p_2^2)\right) \psi(s|34), \tag{29}$$

$$Z(|12;34) = -\hat{\delta} \frac{2}{6/u + B(0|p_{34})}, \tag{30}$$

where $\hat{\delta} \equiv (2\pi)^d \delta(p_1 + p_2 + p_3 + p_4)$, $p_{34} = p_3 + p_4$,

$$B(t|Q) = \int Dq_1 Dq_2 \frac{\exp\left(-t(q_1^2 + q_2^2)\right)}{(q_1^2 + m^2)(q_2^2 + m^2)} (2\pi)^d \delta(q_{12} - Q), \quad q_{12} = q_1 + q_2, \tag{31}$$

and thus $B(0|Q) = B(Q^2)$, defined in Appendix A. Here ψ and ω satisfy

$$\rho(t|34) + \int_0^t ds\, K(t, s|p_{34}) \psi(s|34) = 0, \tag{32}$$

$$\rho(t_1, t_2|Q) - 2 \int_0^{t_1} ds_1\, K(t_1, s_1|Q) \int_0^{t_2} ds_2\, K(t_2, s_2|Q)\, \omega(s_1, s_2|Q) = 0, \tag{33}$$

where

$$K(t, s|Q) = \int Dq_1 Dq_2\, (2\pi)^d \delta(q_{12} - Q) \frac{\exp\left(-(t+s)q_1^2 - (t-s)q_2^2\right)}{q_1^2 + m^2}, \tag{34}$$

$$\rho(t|34) = \exp\left(-t(p_3^2 + p_4^2)\right) - \frac{B(t|p_{34})}{6/u + B(0|p_{34})}, \tag{35}$$

$$\rho(t_1, t_2|Q) = B(t_1 + t_2|Q) - \frac{B(t_1|Q)B(t_2|Q)}{6/u + B(0|Q)}. \tag{36}$$

The derivation of these results is given in Appendix B.

3. Running coupling from flowed fields

3.1. Definitions

Using the connected 4-pt functions $g \equiv \hat{\delta}\hat{g}$ for the flow fields given in Eq. (25), we define the t-dependent dimensionless coupling as

$$g(t) = -3\hat{g}(t, t; t, t|\{p\}_{\text{sym.}}) t^{2-d/2}, \tag{37}$$

where $\{p\}_{\text{sym.}}$ is given by $p_i^2 t = 3\Delta/4$ ($i = 1 \sim 4$) and $p_{12}^2 t = p_{34}^2 t = \Delta$ ($p_{ij} = p_i + p_j$), which is the symmetric point for $d > 2$, and $t^{2-d/2}$ is introduced to make the coupling dimensionless. Here Δ is an arbitrary dimensionless constant but we can take $\Delta = 1$ without loss of generality by the rescaling $t \to \Delta t$. Explicitly we have

$$\hat{g}(t, t; t, t|\{p\}_{\text{sym.}}) = 4\hat{X}(t, t|\{p\}_{\text{sym.}}) + 4\hat{Y}(t|\{p\}_{\text{sym.}}) + \hat{Z}(|\{p\}_{\text{sym.}}), \tag{38}$$

where we remove $\hat{\delta}$ by defining $\mathcal{O} = \hat{\delta}\hat{\mathcal{O}}$ for $\mathcal{O} = g, X, Y, Z$, and

$$\hat{X}(t_1, t_2|12;34) = (p_2^2 + m^2)(p_3^2 + m^2) \times \int_0^{t_1} ds_1 \int_0^{t_2} ds_2 \, \exp\left(s_1(p_2^2 - p_1^2)\right) \exp\left(s_2(p_3^2 - p_4^2)\right) \omega(s_1, s_2|p_{34}), \tag{39}$$

$$\hat{Y}(t|12;34) = (p_2^2 + m^2) \int_0^t ds \exp\left(s(p_2^2 - p_1^2)\right) \psi(s|34), \tag{40}$$

$$\hat{Z}(|12;34) = -\frac{1}{3} \frac{u}{1 + \frac{u}{6} B(0|p_{34})}. \tag{41}$$

3.2. *Free flow*

For simplicity, we first consider free flow, where $\hat{g}(t,t;t,t|\{p\}_{\mathrm{sym.}}) = \hat{Z}(|\{p\}_{\mathrm{sym.}})$. Taking $\Delta = 1$, the running coupling is given by

$$g(t) = \frac{u t^{2-d/2}}{1 + \frac{u}{6} B\,(1/t)}, \tag{42}$$

where $B(p^2) = B(0|p)$.

3.2.1. $d = 2$

In 2 dimensions, we obtain

$$g(t) = \frac{ut}{1 + \frac{ut}{6\pi\left(1 + 4m^2 t\right)^{1/2}} \tanh^{-1}\left(\frac{1}{\left(1 + 4m^2 t\right)^{1/2}}\right)}, \tag{43}$$

which behaves in the UV limit ($t \to 0$) and the IR limit ($t \to \infty$) as

$$g(t) \simeq \begin{cases} \dfrac{ut}{1 - ut\log(m^2 t)/(12\pi)} & \to 0, \qquad t = 0, \\ \dfrac{ut}{1 + u/(24\pi m^2)} & \to \infty, \qquad t = \infty. \end{cases} \tag{44}$$

In the massless limit $m^2 \to 0$, we have

$$g(t) \simeq -\frac{12\pi}{\log(m^2 t)} \to 0. \tag{45}$$

3.2.2. $d = 3$

At $d = 3$, the running coupling is given by

$$g(t) = \frac{u\sqrt{t}}{1 + \frac{u\sqrt{t}}{24\pi} \arctan\left(\frac{1}{\left(4m^2 t\right)^{1/2}}\right)}, \tag{46}$$

which behaves as

$$g(t) \simeq \begin{cases} \dfrac{u\sqrt{t}}{1+u\sqrt{t}/48} & \to 0, \qquad t=0, \\ \dfrac{u\sqrt{t}}{1+u/(48\pi m)} & \to \infty, \qquad t=\infty. \end{cases} \tag{47}$$

In the massless limit, we have

$$g(t) = \frac{u\sqrt{t}}{1+u\sqrt{t}/48} = \begin{cases} \to 0, & t\to 0, \\ \to 48, & t \to \infty, \end{cases} \tag{48}$$

which correspond to the asymptotic free UV fixed point and the Wilson–Fisher IR fixed point, respectively.

3.2.3. $d \geq 4$

Since $B(Q^2)$ diverges as Λ^{d-4} ($\log\Lambda$ at $d=4$) at $d \geq 4$, the running coupling vanishes as the cut-off is removed ($\Lambda \to \infty$). Thus the theory is trivial in the continuum limit at $d \geq 4$.

3.3. Interacting flow in the massless limit at $d = 3$

3.3.1. Massless limit

We next consider the interacting flow case, where we need to evaluate $\hat{X}$ and $\hat{Y}$, which are difficult to calculate in general. We therefore consider the massless limit.[2] In this limit, the kernel function is reduced to

$$K(t,s|\{p\}_{\text{sym.}}) = D^{d/2-1} k_0(Dt, Ds), \tag{49}$$

where

$$k_0(w,v) = \frac{e^{v-w} w^{1-d/2}}{2^{d-1}(2\pi)^{d/2}} \int_0^1 dz\, z^{d/2-2} \exp\left[\frac{(w-v)^2 z}{2w}\right], \tag{50}$$

and we regard $D \equiv Q^2 = \Delta/t$ as an independent variable. Here the z integral is convergent for $d > 2$ while the bubble integral $B(0|Q)$ is finite for $d < 4$. We thus concentrate on the $d = 3$ case hereafter.

In this limit, we obtain (see Appendix C for details)

$$\hat{Z}(|\{p\}_{\text{sym.}}) = -16\sqrt{D}\frac{\bar{u}(D)}{1+\bar{u}(D)}, \qquad \bar{u}(D) \equiv \frac{u}{48\sqrt{D}}, \tag{51}$$

$$\hat{Y}(t|\{p\}_{\text{sym.}}) = \frac{3}{4}\sqrt{D}\left\{\xi_0^{(1)}(\Delta) - 8\xi_0^{(2)}(\Delta)\frac{\bar{u}(D)}{1+\bar{u}(D)}\right\}, \tag{52}$$

$$\hat{X}(t,t|\{p\}_{\text{sym.}}) = \frac{9}{16}\sqrt{D}\left\{\Xi_0(\Delta) - 4\{\xi_0^{(2)}(\Delta)\}^2\frac{\bar{u}(D)}{1+\bar{u}(D)}\right\}, \tag{53}$$

[2] We will indicate the massless limit by the subscript $_0$.

where

$$\xi_0^{(i)}(\Delta) = \int_0^\Delta dw\, \phi_0^{(i)}(w), \quad i = 1, 2, \tag{54}$$

$$\Xi_0(\Delta) = \int_0^\Delta dw \int_0^\Delta dv\, \Omega_0(w, v), \tag{55}$$

and $\phi_0^{(i)}$ and Ω_0 are solutions to the integral equations

$$e^{-3w/2} + \int_0^w dv\, k_0(w, v)\, \phi_0^{(1)}(v) = 0, \tag{56}$$

$$b_0(w) + \int_0^w dv\, k_0(w, v)\, \phi_0^{(2)}(v) = 0, \tag{57}$$

$$b_0(w + v) - 2 \int_0^w dx\, k_0(w, x) \int_0^v dy\, k_0(v, y)\, \Omega_0(x, y) = 0, \tag{58}$$

where $b_0(w)$ is the massless bubble integral given by Eq. (C.3). These equations can be solved numerically, and at $\Delta = 1$, e.g., we have $\xi_0^{(1)}(1) = -14.8440(1)$, $\xi_0^{(2)}(1) = -1.60557(1)$ and $\Xi_0(1) = 16.6753(1)$.

3.3.2. *Running coupling and the β function*

Using the above results, the running coupling at $d = 3$ is given by

$$g_0(\mu) = G_1 + G_2 \frac{\bar{u}(\Delta)\sqrt{t}}{1 + \bar{u}(\Delta)\sqrt{t}}, \qquad \bar{u}(\Delta) = \frac{u}{48\sqrt{\Delta}}, \tag{59}$$

where $\mu = 1/\sqrt{t}$ and

$$G_1 = -9\sqrt{\Delta}\left[\xi_0^{(1)}(\Delta) + \frac{3}{4}\Xi_0(\Delta)\right], \qquad G_2 = 48\sqrt{\Delta}\left[1 + \frac{3}{4}\xi_0^{(2)}(\Delta)\right]^2 \geq 0. \tag{60}$$

With the numerical values given above we obtain $G_1 = 21.0378(1)$ and $G_2 = 2.00105(1)$ at $\Delta = 1$.[3]

We then calculate the β function for $g_0(\mu)$ as

$$\beta(g_0) \equiv \mu \frac{\partial}{\partial \mu} g_0(\mu) = \frac{(g_0(\mu) - G_1 - G_2)(g_0(\mu) - G_1)}{G_2}, \tag{61}$$

which becomes zero at $g_0(\mu) = G_1$ and $g_0(\mu) = G_1 + G_2$. The coupling $g_0(\mu)$ near G_1 behaves as

$$g_0(\mu) - G_1 \simeq C_{\mathrm{UV}} \frac{u}{\mu} \to 0, \qquad \mu \to \infty, \qquad C_{\mathrm{UV}} = \left[1 + \frac{3}{4}\xi_0^{(2)}(\Delta)\right]^2, \tag{62}$$

approaching the UV fixed point from above, while near $G_1 + G_2$ we have the IR fixed point

$$g_0(\mu) - G_1 - G_2 \simeq -C_{\mathrm{IR}} \frac{\mu}{u} \to 0, \qquad \mu \to 0, \qquad C_{\mathrm{IR}} = \left\{48\sqrt{\Delta}\left[1 + \frac{3}{4}\xi_0^{(2)}(\Delta)\right]\right\}^2, \tag{63}$$

[3] It turns out that $G_2(\Delta)$ has only one zero at $\Delta = 0.36228(1)$.

where the coupling approaches from below the Wilson–Fisher fixed point in 3-dimensional scalar theory. Note that the derivative of the β function with respect to g_0 at the fixed point becomes

$$\beta'(g_0) \equiv \frac{d\beta(g_0)}{dg_0} = \begin{cases} -1, & g_0 = G_1, \\ 1, & g_0 = G_1 + G_2, \end{cases} \tag{64}$$

which should be compared with the same quantities calculated for the standard running coupling in the 3-dimensional massless theory in Ref. [11], where $\beta'(0) = -1$ (UV) and $\beta'(48) = 1$ (IR). The derivative of the β function at the fixed point gives the anomalous dimension of the operator conjugate to the coupling in the conformal theory at the fixed point, and thus is universal. Our flow coupling indeed satisfies this condition and the derivatives at the two fixed points agree with those for the conventional definition of the coupling. This establishes that our flow coupling gives a good definition of the running coupling of the theory. The scaling dimension γ of the operator conjugate to the running coupling g_0 is given by $\gamma = d + \beta'(g_0)$, so that $\gamma_{\rm UV} = 2$ and $\gamma_{\rm IR} = 4$ in this model. Interestingly $\gamma_{\rm UV} = 2$ corresponds to the canonical dimension of the φ^4 operator in 3 dimensions, which is the interaction term in our theory.

By the redefinition of the coupling as $g(\mu) \equiv (g_0(\mu) - G_1)/G_2$, the corresponding β function is simplified as

$$\beta(g) \equiv \mu\frac{\partial}{\partial\mu}g(\mu) = g(\mu)(g(\mu) - 1). \tag{65}$$

4. NLO corrections to the induced metric

In Ref. [1], the induced metric has been calculated from the flowed scalar field in the large N limit. It has been shown that the metric from the massive scalar field describes a space that becomes the Euclidean AdS space asymptotically in both UV and IR limits, where the radius $R_{\rm IR}$ in IR is larger than the radius $R_{\rm UV}$ in UV since

$$R_{\rm UV}^2 = \frac{d-2}{2}R_{\rm IR}^2 < R_{\rm IR}^2, \tag{66}$$

while the metric describes the whole AdS space in the massless limit with the radius $R_{\rm UV}$. In this section, we consider the NLO correction to the induced metric in the $1/N$ expansion as another application of the NLO calculation, in particular, in the massless case at $d = 3$, in order to see whether the space remains AdS or not and how the radius changes at the NLO.

4.1. Induced metric at the NLO

The vev of the induced metric is defined from the normalized flowed field as (Ref. [1])

$$g_{\mu\nu}(z) = R_0^2\langle\partial_\mu\sigma^a(z)\partial_\nu\sigma^a(z)\rangle \tag{67}$$

with some length scale R_0, where $z = (\tau = 2\sqrt{t}, x)$ and $\mu, \nu = 0, 1, \ldots, d$. Here $\sigma^a(z)$ is the normalized flowed field such that $\langle\sigma^2(z)\rangle = 1$, and the corresponding 2-point function is explicitly given at the NLO as

$$\langle\sigma^{a_1}(z_1)\sigma^{a_2}(z_2)\rangle = \frac{\delta^{a_1a_2}}{N}\frac{1}{(\zeta_0(t_1)\zeta_0(t_2))^{1/2}}\left(1 - \frac{\zeta_1(t_1)+\zeta_1(t_2)}{2N}\right) \times \int Dp\frac{\exp\left(-p^2(t_1+t_2)\right)\exp\left(ip(x_1-x_2)\right)}{p^2+m^2}\left[1 + \frac{G_1(t_1,t_2|p)}{N}\right], \tag{68}$$

where

$$\zeta_1(t) = \frac{1}{\zeta_0(t)}\mathcal{H}[G_1(t,t|p)], \qquad \mathcal{H}[f(t|p)] \equiv \int Dp \frac{\exp\left(-2p^2 t\right)}{p^2+m^2} f(t|p). \tag{69}$$

After some algebra (see Appendix D), we obtain

$$g_{ij}(\tau) = \delta_{ij}\frac{R_0^2}{d}A(t) \quad (i,j=1,2,\ldots,d), \qquad g_{00}(\tau) = -\frac{R_0^2\, t}{2}\partial_t A(t), \tag{70}$$

where

$$A(t) = -\frac{1}{2}\frac{\partial_t \zeta_0(t)}{\zeta_0(t)} + \frac{1}{N}A_1(t), \tag{71}$$

and $A_1(t)$, given in Appendix D, in general is a very complicated function.

4.2. Induced metric in the massless limit at $d = 3$

In the massless limit at $d = 3$, the metric at the LO is given by

$$g_{ij}(\tau) = \delta_{ij}\frac{R_0^2}{3\tau^2}, \qquad g_{00}(\tau) = \frac{R_0^2}{2\tau^2}, \tag{72}$$

which describes the AdS space for all τ.

At the NLO, $A_1(t)$ is given by

$$A_1(t) = \frac{1}{2\sqrt{t}}\int DQ\, h_{\mathrm{total}}(Q^2)\frac{\bar{u}(Q^2)}{(1+\bar{u}(Q^2)\sqrt{t})^2}, \quad \bar{u}(Q^2) = \frac{u}{48\sqrt{Q^2}}, \tag{73}$$

$$\partial_t A_1(t) = -\frac{1}{4\sqrt{t^3}}\int DQ\, h_{\mathrm{total}}(Q^2)\frac{\bar{u}(Q^2)(1+3\bar{u}(Q^2)\sqrt{t})}{(1+\bar{u}(Q^2)\sqrt{t})^3}, \tag{74}$$

which leads to

$$g_{ij}(\tau) = \delta_{ij}\frac{R_0^2}{3\tau^2}\left[1 + \frac{\tau}{N}\int DQ\, h_{\mathrm{total}}(Q^2)\frac{\bar{u}(Q^2)}{(1+\bar{u}(Q^2)\tau/2)^2}\right], \tag{75}$$

$$g_{00}(\tau) = \frac{R_0^2}{2\tau^2}\left[1 + \frac{\tau}{2N}\int DQ\, h_{\mathrm{total}}(Q^2)\frac{\bar{u}(Q^2)(1+3\bar{u}(Q^2)\tau/2)}{(1+\bar{u}(Q^2)\tau/2)^3}\right], \tag{76}$$

where $h_{\mathrm{total}}(Q^2)$ is a function given in Appendix D.

4.3. UV and IR limits

The above expression in the UV limit ($\tau \to 0$) leads to

$$g_{ij}(\tau) \simeq \delta_{ij}\frac{R_0^2}{3\tau^2}\left[1 + \frac{\tau}{N}\int DQ\, h_{\mathrm{total}}(Q^2)\bar{u}(Q^2)\right], \quad \tau \to 0, \tag{77}$$

$$g_{00}(\tau) \simeq \frac{R_0^2}{2\tau^2}\left[1 + \frac{\tau}{2N}\int DQ\, h_{\mathrm{total}}(Q^2)\bar{u}(Q^2)\right], \quad \tau \to 0, \tag{78}$$

which shows that the NLO correction is less singular than the LO contribution. Therefore the space becomes asymptotically AdS in the UV limit at the NLO whose AdS radius is equal to that at the LO.

We cannot naively take the $\tau \to \infty$ limit in Eqs. (75) and (76), on the other hand, due to the enhancement of the UV contribution of the Q integrals. Careful evaluations of these Q integrals in Appendix D give

$$g_{ij}(\tau) \simeq \delta_{ij}\frac{R_0^2}{3\tau^2}\left[1+\frac{r}{N}\right], \quad g_{00}(\tau) \simeq \frac{R_0^2}{2\tau^2}\left[1+\frac{r}{N}\right], \quad \tau \to \infty, \tag{79}$$

where $r = -0.41869(1)$.[4] Therefore, the space becomes asymptotically AdS again in the IR limit, whose radius, however, is smaller than that in the UV limit.[5] The induced metric at the NLO describes a 4-dimensional space that is asymptotically AdS in both UV and IR regions with different radii but non-AdS in between.

It is clear that the NLO correction to the AdS radius in the IR limit is related to the Wilson–Fisher fixed point in the original 3-dimensional scalar theory, since Eqs. (75) and (76) can be written as

$$g_{ij}(\tau) = \delta_{ij}\frac{R_0^2}{3\tau^2}\left[1-\frac{1}{24N}\int DQ\, h_{\rm total}(Q^2)\beta(g(48\mu\sqrt{Q^2}))\right], \tag{80}$$

$$g_{00}(\tau) = \frac{R_0^2}{2\tau^2}\left[1-\frac{1}{24N}\int DQ\, h_{\rm total}(Q^2)\left\{1+\frac{\mu}{2}\frac{\partial}{\partial\mu}\right\}\beta(g(48\mu\sqrt{Q^2}))\right], \tag{81}$$

where $\mu = 1/\sqrt{t} = 2/\tau$, and $\beta(g(x))$ is the β function for the running coupling $g(x)$ from the free-flow field defined in the previous section with $\Delta = 1$ as

$$\beta(g) = \frac{g(g-48)}{48}, \qquad g(x) = 48\frac{u}{x+u}. \tag{82}$$

5. Summary

In this paper, we studied the flow equation of the O(N) φ^4 model in d dimensions at the NLO in the $1/N$ expansion, employing the Schwinger–Dyson equation. We calculated the 2-pt and 4-pt functions at the NLO.

As an application of the NLO calculation, we investigated the running coupling defined from the connected 4-pt function of flowed fields. In particular at $d = 3$ in the massless limit, we showed that the running coupling has two fixed points, the asymptotic free one in the UV region and the Wilson–Fisher one in the IR region. We also derived the corresponding β function. Our study suggests that the flow equation can be interpreted as a renormalization group transformation.

We also calculated the NLO correction to the $(d+1)$-dimensional metric induced from the massless scalar field theory at $d = 3$. In the massless limit, the whole 4-dimensional space becomes AdS at the LO of the $1/N$ expansion (Ref. [1]). We found that the NLO corrections give small perturbations to the metric, which make the space only asymptotically AdS in both UV ($t = 0$) and IR ($t = \infty$) limits. In addition, while the NLO corrections do not change the AdS radius at the LO in the UV limit, the AdS radius is reduced by the NLO correction in the IR limit, which corresponds to the Wilson–Fisher

[4] This is independent of $u_f \neq 0$ (the interacting flow). In the case of free flow ($u_f = 0$), however, $r = 8/3\pi^2 \simeq 0.27019$.

[5] It is interesting and also suggestive to see that the F-coefficient of the 3-dimensional $O(N)$ scalar model is given by $F_{\rm IR} = F_{\rm UV} - \zeta(3)/(8\pi^2) + O(1/N)$, where $F_{\rm UV} = NF_S$ with $F_S \simeq 0.0638$ as an example of a conjecture, the so-called "F-theorem", which claims that the F-coefficient monotonically decreases along an RG trajectory connecting two 3-dimensional CFTs. Furthermore, in the holographic dual picture, the F-coefficient is proportional to the AdS radius squared. (See Ref. [12] and references therein.)

IR fixed point of the original theory. The nontrivial fixed point in the field theory appears as a change of the AdS radius at the NLO. The induced metric at NLO describes a 4-dimensional space that connects one asymptotically AdS space at UV to the other asymptotically AdS space at IR.

This paper contains two important messages. One is that the flow equation can provide an alternative method to define a renormalization group transformation. The other is that the massless scalar field in d dimensions plus the extra dimension from the RG scale not only generates a $(d+1)$-dimensional AdS space at LO (Ref. [1]) but also gives an NLO correction, which makes the $(d+1)$-dimensional space asymptotically AdS only in the UV and IR limits at $d = 3$. In particular, the AdS radius in the IR limit, which corresponds to the Wilson–Fisher fixed point, becomes smaller than that in the UV limit, which is equal to the radius at the LO. Although the relation found in this paper between the massless scalar field theory and the induced geometry is very suggestive, further studies will be needed to establish an alternative interpretation of AdS/CFT correspondences proposed in Ref. [2] in terms of field theories.

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Appendix A. The $1/N$ expansion in d-dimensional theory

In this appendix, we consider the $1/N$ expansion in d-dimensional theory.

A.1. *Schwinger–Dyson equation (SDE)*

In order to perform the $1/N$ expansion, we consider the SDE of this model, which can be written compactly as

$$\langle \delta_x^a X[\varphi] \rangle = \langle X[\varphi] \delta_x^a S(\mu^2, u) \rangle, \tag{A.1}$$

where $\delta_x^a \varphi^b(y) = \delta^{ab} \delta^{(d)}(x-y)\epsilon$ with a small parameter ϵ, so that

$$\delta_x^a S(\mu^2, u) = N\epsilon \left[(-\Box + \mu^2)\varphi^a(x) + \frac{u}{3!}\varphi^a(x)\varphi^2(x) \right]. \tag{A.2}$$

Here the vacuum expectation value of an operator $\mathcal{O}$ is defined in Eq. (9).

We define $2n$-point functions Γ_{2n} as[6]

$$\Gamma^{a_1a_2\cdots a_{2n}}(x_1,x_2,\ldots,x_{2n}) = N^{2n-1}\left\langle \prod_{i=1}^{2n}\varphi^{a_i}(x_i)\right\rangle \equiv \Gamma_{2n}[12\cdots(2n)], \tag{A.3}$$

which can be written in terms of their connected parts K_{2n} as

$$\Gamma_4[1234] = K_4[1234] + N\left\{\Gamma_2[12]\Gamma_2[34] + \Gamma_2[13]\Gamma_2[24] + \Gamma_2[14]\Gamma_2[23]\right\}, \tag{A.4}$$

$$\Gamma_6[123456] = K_6[123456] + N\left\{\Gamma_2[12]K_4[3456] + 14\text{ perms.}\right\} + N^2\left\{\Gamma_2[12]\Gamma_2[34]\Gamma_2[56] + 14\text{ perms.}\right\}, \tag{A.5}$$

and so on. As mentioned in the main text, we assume we are working in a phase where O(N) symmetry is not broken. We therefore do not add the external source term $h\varphi(x)$ to the action, so that the action has symmetry under $\varphi \to -\varphi$, which implies $\Gamma_{2n-1} = 0$ for all positive integers n.

In terms of these, the SDE for $X(\varphi) = \varphi^{a_2}(x_2)$ becomes

$$\delta_{12} = (-\Box + \mu^2)_{x_1}\Gamma_2[12] + \frac{u}{3!N^2}\sum_b\left(K_4[bb12] + N\left\{\Gamma_2[bb]\Gamma_2[12] + 2\Gamma_2[b1]\Gamma_2[b2]\right\}\right), \tag{A.6}$$

where $\delta_{12} \equiv \delta^{a_1a_2}\delta^{(d)}(x_1 - x_2)$ and $x_b = x_1$, so that b in the summation runs over the O(N) indices only.

For $X(\varphi) = \varphi^{a_2}(x_2)\varphi^{a_3}(x_3)\varphi^{a_4}(x_4)$, on the other hand, we have

$$\begin{aligned}\delta_{12}\Gamma_2[34] + 2\text{ perms.} &= (-\Box + \mu^2)_{x_1}\frac{1}{N}\left(K_4[1234] + N\left\{\Gamma_2[12]\Gamma_2[34] + 2\text{ perms.}\right\}\right)\\ &\quad + \frac{u}{3!N^3}\sum_b\Big(K_6[bb1234] + N\left\{\Gamma_2[bb]K_4[1234] + 14\text{ perms.}\right\}\\ &\quad + N^2\left\{\Gamma_2[bb]\Gamma_2[12]\Gamma_2[34] + 14\text{ perms.}\right\}\Big),\end{aligned} \tag{A.7}$$

which can be simplified by using Eq. (A.6) to

$$\begin{aligned}0 = (-\Box + \mu^2)_{x_1}K_4[1234] &+ \frac{u}{3!N^2}\sum_b\Big(K_6[bb1234] + N\Gamma_2[bb]K_4[1234]\\ &+ 2N\left\{\Gamma_2[b1]K_4[b234] + \Gamma_2[b2]K_4[1b34] + \Gamma_2[b3]K_4[12b4] + \Gamma_2[b4]K_4[123b]\right\}\\ &+ N\left\{\Gamma_2[12]K_4[bb34] + \Gamma_2[13]K_4[b2b4] + \Gamma_2[14]K_4[b23b]\right\}\\ &+ 2N^2\left\{\Gamma_2[b2]\Gamma_2[b3]\Gamma_2[14] + \Gamma_2[b2]\Gamma_2[b4]\Gamma_2[13] + \Gamma_2[b3]\Gamma_2[b4]\Gamma_2[12]\right\}\Big).\end{aligned} \tag{A.8}$$

Using the O(N) symmetry and assuming translational invariance (e.g., infinite volume or periodic boundary condition), we can write

$$\Gamma_2[12] \equiv \delta^{a_1a_2}\Gamma(x_{12}), \qquad x_{12} \equiv x_1 - x_2 \tag{A.9}$$

$$K_4[1234] \equiv \delta^{a_1a_2}\delta^{a_3a_4}K(x_1,x_2;x_3,x_4) + 2\text{ perms.}, \tag{A.10}$$

$$K_6[123456] \equiv \delta^{a_1a_2}\delta^{a_3a_4}\delta^{a_5a_6}H(x_1,x_2;x_3,x_4;x_5,x_6) + 14\text{ perms.}, \tag{A.11}$$

[6] Note that we use the same notation Γ_{2n} for the $2n$-point functions in both d and $d+1$ dimensions, since no confusion may occur.

where $K(x_1, x_2; x_3, x_4)$ is invariant under $1 \leftrightarrow 2$ or $3 \leftrightarrow 4$ as well as $(12) \leftrightarrow (34)$, and similar invariances hold for $H(x_1, x_2; x_3, x_4; x_5, x_6)$.

We finally obtain

$$\delta^{(d)}(x_1 - x_2) = \left[(-\Box + \mu^2)_{x_1} + \frac{u}{3!}\Gamma(0)\right]\Gamma(x_{12}) + \frac{u}{3!N}\left[\left(1 + \frac{2}{N}\right)K(x_1, x_1; x_1, x_2) + 2\Gamma(0)\Gamma(x_{12})\right], \tag{A.12}$$

and

$$\begin{aligned} 0 = &\left[(-\Box + \mu^2)_{x_1} + \frac{u}{3!}\left(1 + \frac{2}{N}\right)\Gamma(0)\right]K(x_1, x_2; x_3, x_4) \\ &+ \frac{u}{3!}\Gamma(x_{12})\left[2\Gamma(x_{13})\Gamma(x_{14}) + \left(1 + \frac{2}{N}\right)K(x_1, x_1; x_3, x_4) + \frac{2}{N}K(x_1, x_3; x_1, x_4)\right] \\ &+ \frac{u}{3!N}\left[\left(1 + \frac{2}{N}\right)H(x_1, x_1; x_1, x_2; x_3, x_4) + \frac{2}{N}H(x_1, x_2; x_1, x_3; x_1, x_4)\right] \\ &+ \frac{2u}{3!N}\left[\Gamma(x_{13})K(x_1, x_2; x_1, x_4) + \Gamma(x_{14})K(x_1, x_2; x_3, x_1)\right]. \end{aligned} \tag{A.13}$$

A.2. LO in the $1/N$ expansion

We introduce the $1/N$ expansion as

$$\Gamma(x_{12}) = \sum_{i=0}^{\infty} N^{-i}\Gamma_i(x_{12}), \qquad K(x_1, x_2; x_3, x_4) = \sum_{i=0}^{\infty} N^{-i}K_i(x_1, x_2; x_3, x_4), \tag{A.14}$$

and so on, together with $\mu^2 = \sum_{i=0}^{\infty} N^{-i}\mu_i^2$.

At the LO of the $1/N$ expansion, Eq. (A.12) in momentum space becomes

$$1 = \left(p^2 + \mu_0^2 + \frac{u}{6}\int Dq\,\widetilde{\Gamma}_0(q)\right)\widetilde{\Gamma}_0(p), \qquad \Gamma_0(x) = \int Dp\,\widetilde{\Gamma}_0(p)\exp(ipx), \tag{A.15}$$

which can easily be solved as

$$\widetilde{\Gamma}_0(p) = \frac{1}{p^2 + m^2}, \qquad m^2 = \mu_0^2 + \frac{u}{6}Z(m), \tag{A.16}$$

where $m \geq 0$ is the renormalized mass and $Z(m)$ is given in Eq. (6). Thus the 2-pt function at the LO becomes

$$\langle\varphi^a(x)\varphi^b(y)\rangle = \frac{\delta^{ab}}{N}\int Dp\frac{\exp(ip(x - y))}{p^2 + m^2}. \tag{A.17}$$

Equation (A.13) at the LO leads to

$$(-\Box + m^2)_{x_1}K_0(x_1, x_2; x_3, x_4) + \frac{u}{3!}\Gamma_0(x_{12})K_0(x_1, x_1; x_3, x_4) = -\frac{2u}{3!}\Gamma_0(x_{12})\Gamma_0(x_{13})\Gamma_0(x_{14}). \tag{A.18}$$

Introducing a function $G_0(p_1,p_2,p_3,p_4)$ to rewrite $K_0(x_1,x_2,x_3,x_4)$ as

$$K_0(x_1,x_2;x_3,x_4) = \left\{\prod_{i=1}^{4}\int Dp_i \frac{\exp(ip_i x_i)}{p_i^2+m^2}\right\} G_0(p_1,p_2,p_3,p_4)(2\pi)^d\delta(p_1+p_2+p_3+p_4), \quad (A.19)$$

we obtain

$$G_0(p_1,p_2,p_3,p_4) = G_0(p_1+p_2) = -\frac{2u}{6+uB(p_{12}^2)}, \quad (A.20)$$

where $p_{12} = p_1+p_2$, and

$$B(Q^2) = \int Dq_1 Dq_2 \frac{(2\pi)^d\delta(q_1+q_2-Q)}{(q_1^2+m^2)(q_2^2+m^2)} = \int_0^1 dx \int Dq_1 \frac{\theta(\Lambda^2-q_1^2)}{(q_1^2+m^2+x(1-x)Q^2)^2}. \quad (A.21)$$

This agrees with the previous result obtained by a different method (Ref. [11]). Here we specify the way we introduce the cut-off Λ for the case where $B(Q^2)$ diverges.

A.3. NLO correction to the 2-pt functions

Let us consider the NLO correction to the 2-pt function Γ_2. At the NLO, Eq. (A.12) leads to

$$0 = (-\Box+m^2)\Gamma_1(x_{12}) + \left\{\frac{u}{6}(2Z(m)+\gamma_1)+\mu_1^2\right\}\Gamma_0(x_{12}) + \frac{u}{6}K_0(x_1,x_1;x_1,x_2), \quad (A.22)$$

$$\gamma_1 = \int Dq\, \widetilde{\Gamma}_1(q), \quad (A.23)$$

which can be solved in momentum space as

$$\widetilde{\Gamma}_1(p) = -\frac{1}{(p^2+m^2)^2}\left(\mu_1^2 + \frac{u}{6}\gamma_1 + \frac{u}{3}S(p^2)\right), \quad (A.24)$$

where

$$S(p^2) = \int \frac{DQ}{(p-Q)^2+m^2}\frac{6}{6+uB(Q^2)}, \quad (A.25)$$

and the condition for γ_1 is solved as

$$\gamma_1 = -\frac{\mu_1^2 B(0)+C_2}{1+\frac{u}{6}B(0)}, \qquad C_2 \equiv -\int \frac{DQ}{\frac{6}{u}+B(Q^2)}\frac{d}{dm^2}B(Q^2). \quad (A.26)$$

Substituting Eq. (A.26) into Eq. (A.24), we finally obtain

$$\widetilde{\Gamma}_1(p) = -\frac{1}{(p^2+m^2)^2}\left\{g(p^2)+\tilde{C}\right\}, \quad (A.27)$$

where

$$g(p^2) = \int \frac{DQ}{\frac{6}{u}+B(Q^2)}\left\{\frac{1}{(Q+p)^2+m^2}+\frac{1}{(Q-p)^2+m^2}-\frac{2}{Q^2+m^2}\right\}, \quad (A.28)$$

$$\tilde{C} = C_1 + \frac{\mu_1^2}{1+\frac{u}{6}B(0)} - \frac{C_2}{\frac{6}{u}+B(0)}, \quad C_1 = \int \frac{DQ}{\frac{6}{u}+B(Q^2)}\frac{2}{(Q^2+m^2)}, \quad (A.29)$$

and $g(p^2)$ can be expanded as

$$g(p^2) = Z_1 p^2 + \tilde{g}(p^2), \quad \tilde{g}(p^2) = O(p^4), \tag{A.30}$$

where

$$Z_1 = \frac{2}{d} \int \frac{DQ}{6/u + B(Q^2)} \left[\frac{4-d}{(Q^2+m^2)^2} - \frac{4m^2}{(Q^2+m^2)^3} \right]. \tag{A.31}$$

A.4. Renormalization

Let us now consider the renormalization of the theory.

Our renormalization condition for the renormalized 2-pt function Γ_R is given in momentum space as

$$\widetilde{\Gamma}_R^{-1}(p) \simeq p^2 + m^2, \qquad p^2 \simeq 0, \tag{A.32}$$

where m is interpreted as the renormalized mass, which is independent of both N and the cut-off. Relating the bare field to the renormalized field by the renormalization constant Z_R as $Z_R^{1/2} \varphi_R = \varphi$, we explicitly obtain

$$Z_R \widetilde{\Gamma}_R(p) = \widetilde{\Gamma}(p) = \frac{1}{p^2 + m^2 + \dfrac{1}{N} \Sigma_1(p^2)} + O\left(\frac{1}{N^2}\right), \tag{A.33}$$

where

$$\Sigma_1(p^2) = Z_1 p^2 + \tilde{C} + \tilde{g}(p^2). \tag{A.34}$$

At the LO of the $1/N$ expansion, the above condition implies

$$\mu_0^2 = m^2 - \frac{u}{6} Z(m), \qquad Z_R = 1, \tag{A.35}$$

where $Z(m)$ is potentially divergent at $d > 1$. We therefore introduce the momentum cut-off Λ to regulate the integral, and μ_0^2 is tuned to cancel the effect of $Z(m)$ including such divergences, in order to keep the renormalized mass m finite and constant. The lattice regularization or dimensional regularization is more consistent than the momentum cut-off, but calculations become much more complicated in the lattice regularization or power divergences are difficult to deal with in the dimensional regularization. Since the momentum cut-off is enough to see the leading divergences, we adopt it in this paper.

At the NLO, the renormalization condition implies

$$Z_R = 1 - \frac{Z_1}{N}, \qquad \mu_1^2 = \left(1 + \frac{u}{6} B(0)\right) Z_1 m^2 + \frac{u}{6} C, \tag{A.36}$$

where

$$C = -\int \frac{DQ}{\frac{6}{u} + B(Q^2)} \left[\frac{dB(Q^2)}{dm^2} + 2 \frac{\frac{6}{u} + B(0)}{Q^2 + m^2} \right]. \tag{A.37}$$

The renormalization condition for the coupling, which first appears at the NLO of the $1/N$ expansion, is given by $G_0(Q^2 = s) = -u_r(s)/3$, so that $u_r(s)$ is regarded as the renormalized coupling at the scale s. Equation (A.20) thus leads to

$$u_r(s) = \frac{u}{1 + \frac{u}{6}B(s)}, \tag{A.38}$$

where $B(Q^2)$ is divergent at $d \geq 4$. Therefore the renormalized coupling goes to zero as

$$u_r(s) \simeq \frac{6}{B(s)} \to 0, \quad \Lambda \to \infty \tag{A.39}$$

at $d \geq 4$. This indicates the triviality of the φ^4 theory at $d \geq 4$.

A.5. Renormalization constants

Here we explicitly evaluate the renormalization constants.

A.5.1. $d = 1$

At $d = 1$, μ_0^2 is finite since

$$Z(m) = \frac{1}{\pi m} \arctan\left(\frac{\Lambda}{m}\right) \tag{A.40}$$

is finite, and the coupling is also finite and nonzero since

$$B(Q^2) = \frac{1}{m(Q^2 + 4m^2)} \simeq \frac{1}{mQ^2} + \cdots, \qquad Q^2 \to \infty \tag{A.41}$$

has a finite limit as $\Lambda \to \infty$.

The most divergent part of Z_1 is given by

$$Z_1 \simeq \begin{cases} \displaystyle\int DQ \frac{u}{(Q^2 + m^2)^2}, & u \neq \infty, \\ \displaystyle\int \frac{DQ}{B(Q^2)} \frac{6}{(Q^2 + m^2)^2}, & u = \infty, \end{cases} \tag{A.42}$$

which shows that Z_1 is finite for all u including $u = \infty$. Equations (A.36) and (A.37) thus tell us that μ_1^2 is also finite for all u including $u = \infty$, and therefore, there is no divergence at $d = 1$ up to the NLO.

A.5.2. $d = 2$

At $d = 2$, μ_0^2 is logarithmically divergent since

$$\mu_0^2 = m^2 - \frac{u}{6} Z(m), \qquad Z(m) \simeq \frac{1}{4\pi} \log\left(\frac{\Lambda^2 + m^2}{m^2}\right). \tag{A.43}$$

On the other hand, $B(Q^2)$ is finite since

$$B(Q^2) = \frac{\tanh^{-1}\left(\left(\frac{Q^2}{Q^2+4m^2}\right)^{1/2}\right)}{\pi\left(Q^2(Q^2 + 4m^2)\right)^{1/2}} \simeq \frac{1}{2\pi Q^2} \log\frac{Q^2}{m^2} - \frac{m^2}{\pi (Q^2)^2}\left(\log\frac{Q^2}{m^2} - 1\right) + \cdots, \tag{A.44}$$

$$\frac{dB(Q^2)}{dm^2} \simeq -\frac{2B(0)}{Q^2 + 4m^2}\left[1 + \frac{2m^2}{Q^2} \log\frac{Q^2}{m^2} + \cdots\right], \quad B(0) = \frac{1}{4\pi m^2}, \tag{A.45}$$

so that the renormalized coupling becomes

$$u_r(s) = \frac{6u}{6 + u\dfrac{\tanh^{-1}\left(\left(\frac{s}{s+4m^2}\right)^{1/2}\right)}{\pi\left(s(s+4m^2)\right)^{1/2}}} \simeq \frac{12\pi us}{12\pi s + u\log(s/m^2)}, \quad s \to \infty. \tag{A.46}$$

The most singular term of Z_1 for $u \neq \infty$ becomes

$$Z_1 \simeq \frac{u}{6}\int DQ \frac{2}{(Q^2+m^2)^2}, \tag{A.47}$$

which is manifestly finite, while at $u = \infty$ we have

$$Z_1 = \int \frac{DQ}{B(Q^2)}\left[\frac{2}{(Q^2+m^2)^2} - \frac{4m^2}{(Q^2+m^2)^3}\right], \tag{A.48}$$

which diverges as $Z_1 \simeq \log\left(\log \Lambda^2\right)$.

The most divergent part of μ_1^2 is given by

$$\mu_1^2 \simeq \begin{cases} -\dfrac{u}{3}Z(m)\delta_1 \quad (\delta_1 = 1), & u \neq \infty, \\ \dfrac{u}{12\pi}\log\left(\dfrac{\Lambda^2+4m^2}{4m^2}\right), & u \to \infty. \end{cases} \tag{A.49}$$

A.5.3. $d = 3$

At $d = 3$, μ_0^2 is linearly divergent as

$$\mu_0^2 = m^2 - \frac{u}{6}Z(m), \qquad Z(m) \simeq \frac{1}{2\pi^2}\left[\Lambda - m\arctan\left(\frac{\Lambda}{m}\right)\right], \tag{A.50}$$

while $B(Q^2)$ is finite as

$$B(Q^2) = \frac{1}{4\pi\sqrt{Q^2}}\arctan\left(\left(\frac{Q^2}{4m^2}\right)^{1/2}\right) \simeq \frac{1}{8|Q|} - \frac{m}{2\pi Q^2} + \frac{2m^3}{3\pi(Q^2)^2} + \cdots, \tag{A.51}$$

$$\frac{dB(Q^2)}{dm^2} = -\frac{2B(0)}{Q^2+4m^2}, \quad B(0) = \frac{1}{8\pi m}, \tag{A.52}$$

and the renormalized coupling becomes

$$u_r(s) = \frac{6u}{6 + \dfrac{u}{4\pi\sqrt{s}}\arctan\left(\left(\dfrac{s}{4m^2}\right)^{1/2}\right)} \simeq \frac{u}{1 + \dfrac{u}{48\sqrt{s}}}, \quad s \to \infty. \tag{A.53}$$

The most singular term of Z_1 for $u \neq \infty$ becomes

$$Z_1 \simeq \frac{u}{9}\int DQ\,\frac{1}{(Q^2+m^2)^2}, \tag{A.54}$$

which is manifestly finite at $d = 3$. On the other hand, at $u = \infty$ we have

$$Z_1 = \frac{2}{3}\int \frac{DQ}{B(Q^2)}\left[\frac{1}{(Q^2+m^2)^2} - \frac{4m^2}{(Q^2+m^2)^3}\right], \tag{A.55}$$

whose divergent part becomes

$$Z_1 \simeq \frac{4}{3\pi^2}\log\Lambda^2. \tag{A.56}$$

The most divergent part of μ_1^2 becomes

$$\mu_1^2 \simeq \begin{cases} -\dfrac{u}{3}Z(m)\delta_1 \quad (\delta_1 = 1), & u \neq \infty, \\ -m\dfrac{2u}{9\pi^3}\log\Lambda^2, & u \to \infty. \end{cases} \tag{A.57}$$

A.5.4. $d = 4$

At $d = 4$, μ_0^2 is quadratically divergent as

$$\mu_0^2 = m^2 - \frac{u}{6}Z(m), \qquad Z(m) \simeq \frac{1}{16\pi^2}\left[\Lambda^2 - m^2\log\left(\frac{\Lambda^2+m^2}{m^2}\right)\right]. \tag{A.58}$$

On the other hand, at $d = 4$ we have

$$\begin{aligned} B(Q^2) = \frac{1}{(4\pi)^2}\Bigg[&\log\left(\frac{\Lambda_m^2}{m^2}\right) + 2\frac{Q^2+4\Lambda_m^2-2\Lambda^2}{\left(Q^2(Q^2+4\Lambda_m^2)\right)^{1/2}}\tanh^{-1}\left(\frac{Q^2}{Q^2+4\Lambda_m^2}\right)^{1/2} \\ &- 2\left(\frac{Q^2+4m^2}{Q^2}\right)^{1/2}\tanh^{-1}\left(\frac{Q^2}{Q^2+4m^2}\right)^{1/2}\Bigg], \end{aligned} \tag{A.59}$$

$$B(0) = \frac{1}{(4\pi)^2}\left[\log\frac{\Lambda_m^2}{m^2} - \frac{\Lambda^2}{\Lambda_m^2}\right], \quad \Lambda_m^2 \equiv \Lambda^2 + m^2, \tag{A.60}$$

which diverge logarithmically, so that $u_r(s) = 0$ as $\Lambda \to \infty$.

Since $\tanh^{-1}(x) \simeq_{x\to 1} -\frac{1}{2}\log\left(\frac{1-x}{2}\right)$, we have

$$B(Q^2) + \frac{6}{u} = \hat{B}\left(q^2, \alpha^2\right), \quad q^2 = \frac{Q^2}{\Lambda^2}, \quad \alpha = \frac{m}{\Lambda}, \tag{A.61}$$

$$\hat{B}(q^2, 0) = -c_0\log q^2 + \frac{6}{u} + c_0F(q^2), \quad c_0 = \frac{1}{(4\pi)^2}, \tag{A.62}$$

where

$$F(q^2) = \frac{2(q^2+2)}{\left(q^2(q^2+4)\right)^{1/2}}\tanh^{-1}\left(\frac{q^2}{q^2+4}\right)^{1/2}. \tag{A.63}$$

Let us now consider the continuum limit of Z_1. By rescaling the momentum, we have

$$Z_1 = -\frac{\alpha^2}{8\pi^2}\int_0^1 \frac{t\,dt}{\hat{B}(t,\alpha^2)(t+\alpha^2)^3}. \tag{A.64}$$

As $\alpha^2 \to 0$ in the $\Lambda \to \infty$ limit, we have

$$\begin{aligned} \int_0^1 \frac{t\,dt}{\hat{B}(t,\alpha^2)(t+\alpha^2)^3} &\simeq \int_0^1 \frac{t\,dt}{\hat{B}(t,0)(t+\alpha^2)^3} \\ &= \int_0^{1/2} \frac{t\,dt}{\hat{B}(t,0)(t+\alpha^2)^3} + \int_{1/2}^1 \frac{t\,dt}{\hat{B}(t,0)(t+\alpha^2)^3}, \end{aligned} \tag{A.65}$$

where the second term is finite in this limit, while the first term is bounded from above:

$$\int_0^{1/2} \frac{t\,dt}{\hat{B}(t,0)(t+\alpha^2)^3} \leq -\frac{1}{c_0}\int_0^{1/2} \frac{t\,dt}{(t+\alpha^2)^3 \log(t+\alpha^2)}$$
$$= \frac{1}{c_0}\left[\log|\log\alpha^2| + \sum_{r=1}^{\infty}\frac{(-\log\alpha^2)^r}{r\,r!} + (\text{finite terms})\right], \tag{A.66}$$

so that Z_1 in Eq. (A.64) vanishes as $\alpha^2 \to 0$.

The most divergent part of μ_1^2 becomes

$$\mu_1^2 \simeq -\frac{u}{3}\frac{\Lambda^2}{16\pi^2}\delta_1, \qquad \delta_1 = \int_0^1 dq^2 \left(\frac{c_0 T(q^2) - 6/u}{c_0\{\log q^2 - F(q^2)\} - 6/u}\right), \tag{A.67}$$
$$T(q^2) \equiv \log q^2 + 1 - \frac{q^2}{q^2+4}\left(1 + \frac{q^2+6}{q^2+2}F(q^2)\right),$$

where δ_1 is finite, but is not universal as it depends on how we regulate the integral.

A.5.5. $d > 4$

At $d > 4$, μ_0^2 is $O(\Lambda^{d-2})$ as

$$\mu_0^2 = m^2 - \frac{u}{6}Z(m), \qquad Z(m) \simeq \frac{d}{(4\pi)^{d/2}(d-2)\Gamma(1+d/2)}\Lambda^{d-2}. \tag{A.68}$$

We also write

$$B(Q^2) = \frac{d}{(4\pi)^{d/2}\Gamma(1+d/2)}\int_0^1 dx \int_0^{\Lambda} \frac{p^{d-1}dp}{\left[p^2 + m^2 + Q^2 x(1-x)\right]^2}, \tag{A.69}$$

from which we obtain

$$B(Q^2) = \Lambda^{d-4}\hat{B}\left(\frac{Q^2}{\Lambda^2}, \frac{m^2}{\Lambda^2}\right), \qquad \hat{B}(0,0) = \frac{d}{(d-4)}\frac{1}{(4\pi)^{d/2}\Gamma(1+d/2)}, \tag{A.70}$$
$$\frac{dB(Q^2)}{dm^2} = -2\Lambda^{d-6}\hat{B}_m\left(\frac{Q^2}{\Lambda^2}, \frac{m^2}{\Lambda^2}\right), \tag{A.71}$$

where

$$\hat{B}(q^2,\alpha^2) = \frac{d}{(4\pi)^{d/2}\Gamma(1+d/2)}\int_0^1 dx \int_0^1 \frac{y^{d-1}dy}{\left[y^2+\alpha^2+q^2x(1-x)\right]^2}, \tag{A.72}$$
$$\hat{B}_m(q^2,\alpha^2) = \frac{d}{(4\pi)^{d/2}\Gamma(1+d/2)}\int_0^1 dx \int_0^1 \frac{y^{d-1}dy}{\left[y^2+\alpha^2+q^2x(1-x)\right]^3} \tag{A.73}$$

so that $B(Q^2) = O(\Lambda^{d-4})$. As in the case at $d = 4$, $u_r(s) = 0$ in the limit that $\Lambda \to \infty$.

By the change of variable $Q = \Lambda q$ in Eq. (A.31) and then taking the limit $\Lambda \to \infty$, we obtain

$$Z_1 = \frac{2(4-d)}{d}\int_{q^2<1} \frac{Dq}{\hat{B}(q^2,0)}\frac{1}{(q^2)^2}. \tag{A.74}$$

The fact that $\hat{B}(0,0) \neq 0$ establishes that Z_1 is finite at $d > 4$.

The most divergent part of μ_1^2 is given by

$$\mu_1^2 \simeq -\frac{u}{3} Z(m)\delta_1, \tag{A.75}$$

where

$$\delta_1 = (d-2)\int_0^1 \frac{q^{d-1}dq}{\hat{B}(q^2,0)}\left(\frac{\hat{B}(0,0)}{q^2} - \hat{B}_m(q^2,0)\right) \tag{A.76}$$

with the change of variables as $q^2 = Q^2/\Lambda^2$. It is easy to show that δ_1 is finite.

Appendix B. Solving the SDE for the flow equation

In this appendix we explicitly solve the SDE in $d+1$ dimensions, in order to obtain the 2-pt and 4-pt functions for the flow fields at the NLO.

B.1. Solution for Γ_0

We first solve the equation at the LO for Γ_0. If we introduce one unknown function $F(t,p)$ as

$$\Gamma_0(12) = \int Dp \frac{F(t_1,p)F(t_2,p)}{p^2+m^2} \exp\left(-(p^2+\mu_f^2)(t_1+t_2)\right) \exp\left(ip(x_1-x_2)\right) \tag{B.1}$$

with the initial condition $F(0,p) = 1$, we have

$$D_1^f\, \Gamma_0(12) = \int Dp \frac{\dot{F}(t_1,p)F(t_2,p)}{p^2+m^2} \exp\left(-(p^2+\mu_f^2)(t_1+t_2)\right) \exp\left(ip(x_1-x_2)\right), \tag{B.2}$$

$$\begin{aligned} -\frac{u_f}{6}\Gamma_0(12)\Gamma_0(11) = &-\frac{u_f}{6}\int Dp \frac{F(t_1,p)F(t_2,p)}{p^2+m^2} \\ &\times \exp\left(-(p^2+\mu_f^2)(t_1+t_2)\right) \exp\left(ip(x_1-x_2)\right)\Gamma_0(t_1), \end{aligned} \tag{B.3}$$

$$\Gamma_0(t_1) = \Gamma_0(11) = \int Dp \frac{F^2(t_1,p)}{p^2+m^2} \exp\left(-2(p^2+\mu_f^2)t_1\right), \tag{B.4}$$

where $\dot{F}$ means a t-derivative of F. Then, the SDE, Eq. (20), becomes

$$\frac{\dot{F}(t,p)}{F(t,p)} = -\frac{u_f}{6}\Gamma_0(t), \tag{B.5}$$

which tells us that $F(t,p)$ is independent of p, so we put $F(t,p) = F(t)$. The above equation is thus reduced to

$$\dot{F}(t) = -\frac{u_f}{6}F^3(t)\exp\left(-2\mu_f^2 t\right)\zeta_0(t), \tag{B.6}$$

where $\zeta_0(t)$ is defined in Eq. (24), whose solution is given by

$$F^{-2}(t) = 1 + \frac{u_f}{3}\int_0^t ds\, \zeta_0(s)\exp\left(-2\mu_f^2 s\right) \equiv \exp\left(-2\mu_f^2 t\right)\frac{\zeta(t)}{Z(m_f)}, \quad \zeta(t) = \zeta_0(t) + \Delta(t) \tag{B.7}$$

where m_f is defined in Eq. (6) and

$$\Delta(t) = \exp\left(2t\mu_f^2\right)\left(Z(m_f) - Z(m)\right) + \int Dp \left(\frac{p^2+m_f^2}{p^2+m^2}\right)\frac{\exp\left(2t\mu_f^2\right) - \exp\left(-2tp^2\right)}{p^2+\mu_f^2}. \tag{B.8}$$

In the case of the interacting flow with $u_f > 0$, μ_f^2 negatively diverges as $Z(m_f) \to +\infty$ in the continuum limit at $d > 1$ or as $u_f \to +\infty$ in the NLSM limit. In these limits, $\Delta(t)$ vanishes as

$$\lim_{\mu_f^2 \to -\infty} \Delta(t) \simeq -\frac{m_f^2 \zeta_0(t) - \dot{\zeta}_0(t)/2}{\mu_f^2} + O\left(1/\mu_f^4\right) \tag{B.9}$$

for $t > 0$. In the case of free flow ($u_f = 0$), we simply have $F(t) = 1$.

We then obtain

$$\Gamma_0(12) = \begin{cases} \dfrac{Z(m_f)}{(\zeta(t_1)\zeta(t_2))^{1/2}} \displaystyle\int Dp \frac{\exp\left(-p^2(t_1+t_2)\right)\exp\left(ip(x_1-x_2)\right)}{p^2+m^2}, & u_f \neq 0, \\ \displaystyle\int Dp \frac{\exp\left(-(p^2+\mu_f^2)(t_1+t_2)\right)\exp\left(ip(x_1-x_2)\right)}{p^2+m^2}, & u_f = 0. \end{cases} \tag{B.10}$$

B.2. *Solution for K_0*

We consider K_0, which appears in the NLO. The equation for K_0 in Eq. (22) is closed, once Γ_0 is obtained. Using Eq. (26), we have

$$\begin{aligned} D_1^f K_0(12;34) &= \int dP_4 \left[\frac{\dot{F}(t_1)}{F(t_1)} + \partial_{t_1}\right] g(12;34|12;34) \\ &= \int dP_4 \left[-\frac{u_f}{6}F^2(t_1)\exp\left(-2\mu_f^2 t_1\right)\zeta_0(t_1) + \partial_{t_1}\right] g(12;34|12;34), \end{aligned} \tag{B.11}$$

$$\Gamma_0(12)\Gamma_0(13)\Gamma_0(14) = \int dP_4 \hat{\delta}\,(p_1^2+m^2)F^2(t_1)\exp\left(-2\mu_f^2 t_1\right)\exp\left((p_1^2-p_2^2-p_3^2-p_4^2)t_1\right), \tag{B.12}$$

$$\Gamma_0(11)K_0(12;34) = F^2(t_1)\exp\left(-2\mu_f^2 t_1\right)\zeta_0(t_1)\int dP_4\, g(12;34|12;34), \tag{B.13}$$

$$\begin{aligned} \Gamma_0(12)K_0(11;34) = F^2(t_1)\exp\left(-2\mu_f^2 t_1\right)\int dP_4 \hat{\delta}\,(p_1^2+m^2)\exp\left(t_1(p_1^2-p_2^2)\right) \\ \times \int Dq_1 Dq_2 \frac{\exp\left(-t_1(q_1^2+q_2^2)\right)}{(q_1^2+m^2)(q_2^2+m^2)} g(11;34|q_1q_2;34), \end{aligned} \tag{B.14}$$

so that the SDE leads to

$$\begin{aligned} \partial_{t_1} g(12;34|12;34) = -\frac{u_f}{6}F(t_1)^2\exp\left(-2\mu_f^2 t_1\right)(p_1^2+m^2)\exp\left(t_1(p_1^2-p_2^2)\right) \\ \times \hat{\delta}\Bigg[2\exp\left(-t_1(p_3^2+p_4^2)\right) \\ + \int Dq_1 Dq_2 \frac{\exp\left(-t_1(q_1^2+q_2^2)\right)}{(q_1^2+m^2)(q_2^2+m^2)} g(11;34|q_1q_2;34)\Bigg]. \end{aligned} \tag{B.15}$$

From Eq. (B.15), one can easily see $\partial_{t_2}\partial_{t_1} g(12;34|12;34) = 0$, which implies

$$\begin{aligned} g(12;34|12;34) = X(23|12;34) + X(13|21;34) + X(24|12;43) + X(14|21;43) \\ + Y(2|12;34) + Y(1|21;34) + Y(3|43;12) + Y(4|34;12) \\ + Z(|12;34), \end{aligned} \tag{B.16}$$

where we require that X and Y satisfy

$$X(\tau,\tau'|12;34) = X(\tau',\tau|43;21), \qquad X(\tau,0|12;34) = 0, \tag{B.17}$$

$$Y(\tau|12;34) = Y(\tau|12;43), \qquad Y(0|12;34) = 0. \tag{B.18}$$

Since $g(12;34|12;34)$ agrees with the amputated connected 4-pt function in d-dimensional theory at $\tau_i = 0$ $(i = 1,2,3,4)$, we obtain

$$Z(|p_1,p_2,p_3,p_4) = -\hat{\delta}\frac{2}{6/u + B(0|p_{34})}, \tag{B.19}$$

where $B(t|Q)$ is defined in Eq. (31). Then one can easily check that g satisfies the required symmetries

$$g(12;34|12;34) = g(21;34|21;34) = g(12;43|12;43) = g(34;12|34;12). \tag{B.20}$$

B.2.1. Solution for Y

Terms that depend only on t_1 in Eq. (B.15) can be written as

$$\begin{aligned}\partial_t Y(t|21;34) = &-\frac{u_f}{3}F^2(t)\exp\left(-2t\mu_f^2\right)(p_1^2+m^2)\exp\left(t(p_1^2-p_2^2)\right)\hat{\delta} \\ &\times\left[\rho(t|34) + \int Dq_1 Dq_2 \frac{\exp\left(-t(q_1^2+q_2^2)\right)Y(t|q_1,q_2;34)}{(q_1^2+m^2)(q_2^2+m^2)}\right],\end{aligned} \tag{B.21}$$

where $\rho(t|34)$ is defined in Eq. (35). To solve this equation, we set

$$Y(t|21;34) = \hat{\delta}(p_1^2+m^2)\int_0^t ds\exp\left(s(p_1^2-p_2^2)\right)\psi(s|34), \tag{B.22}$$

satisfying Eq. (B.18). Equation (B.21) is reduced to

$$\psi(t|34) = -\frac{u_f}{3}F^2(t)\exp\left(-2t\mu_f^2\right)\left[\rho(t|34) + \int_0^t ds\, K(t,s|p_{34})\psi(s|34)\right], \tag{B.23}$$

which shows ψ does not depend on p_1, p_2, where K is defined in Eq. (34). Since $u_f F^2(t)\exp\left(-2t\mu_f^2\right) = u_f Z(m_f)/\zeta(t)$ goes to infinity in the continuum limit at $t > 0$ and $d > 1$ or in the NLSM limit $u_f \to \infty$, Eq. (32) must hold in either of the two limits.

B.2.2. Solution for X

We next consider the solution for X. Terms depending on both t_1 and t_3 in Eq. (B.15), and thereafter replacing t_3 by t_2 and interchanging $p_1 \leftrightarrow p_2$, gives

$$\begin{aligned}\partial_{t_1} X(t_1,t_2|12;34) = &-\frac{u_f}{6}F^2(t_1)\exp\left(-2t_1\mu_f^2\right)(p_2^2+m^2)\exp\left(t_1(p_2^2-p_1^2)\right)\hat{\delta}\int Dq_1 Dq_2 \\ &\times\frac{\exp\left(-t_1(q_1^2+q_2^2)\right)}{(q_1^2+m^2)(q_2^2+m^2)}\left\{2X(t_1,t_2|q_1,q_2;34) + Y(t_2|43;q_1,q_2)\right\},\end{aligned} \tag{B.24}$$

where

$$Y(t|43;q_1,q_2) = (2\pi)^d\delta(p_{34}+q_{12})(p_3^2+m^2)\int_0^t ds\exp\left(s(p_3^2-p_4^3)\right)\psi(s|q_1,q_2). \tag{B.25}$$

We define

$$\begin{aligned}&\partial_{t_2}\partial_{t_1}X(t_1,t_2|12;34)\\&\quad=\hat{\delta}(p_2^2+m^2)(p_3^2+m^2)\exp\left(t_1(p_2^2-p_1^2)\right)\exp\left(t_2(p_3^2-p_4^2)\right)\beta(t_1,t_2|12;34),\end{aligned}\tag{B.26}$$

where properties of X imply $\beta(t_1,t_2|12;34)=\beta(t_2,t_1|43;21)$ and $\beta(t,0|12;34)=\beta(0,t|12;34)=0$. Then the above equation becomes

$$\begin{aligned}\beta(t_1,t_2|12;34)=&-\frac{u_f}{6}F^2(t_1)\exp\left(-2t_1\mu_f^2\right)\\&\times\left[g(t_1,t_2|p_{34})+2\int_0^{t_1}ds_1\int Dq_1Dq_2(2\pi)^d\delta(q_{12}+p_{34})\right.\\&\left.\times\frac{\exp\left(-(t_1+s_1)q_1^2-(t_1-s_1)q_2^2\right)}{q_1^2+m^2}\beta(s_1,t_2|q_1,q_2;34)\right],\end{aligned}\tag{B.27}$$

where

$$g(t_1,t_2|Q)=\int Dq_1Dq_2(2\pi)^d\delta(q_{12}+Q)\frac{\exp\left(-t_1(q_1^2+q_2^2)\right)}{(q_1^2+m^2)(q_2^2+m^2)}\psi(t_2|q_1,q_2).\tag{B.28}$$

Since the above expression tells us that β depends only on p_{34}, we can write

$$\beta(t_1,t_2|12;34)=\omega(t_1,t_2|p_{34})=\omega(t_1,t_2|-p_{34}),\tag{B.29}$$

so that we have

$$\begin{aligned}\omega(t_1,t_2|p_{34})=&-\frac{u_f}{6}F^2(t_1)\exp\left(-2t_1\mu_f^2\right)\\&\times\left[g(t_1,t_2|p_{34})+2\int_0^{t_1}ds_1K(t_1,s_1|p_{34})\omega(s_1,t_2|p_{34})\right],\end{aligned}\tag{B.30}$$

which is reduced to

$$g(t_1,t_2|Q)+2\int_0^{t_1}ds_1K(t_1,s_1|Q)\omega(s_1,t_2|Q)=0\tag{B.31}$$

in the continuum limit or NLSM limit. Equation (B.31) leads to Eq. (33) in the main text, since

$$\int_0^{t_2}ds_2\,K(t_2,s_2|Q)g(t_1,s_2|Q)=-\rho(t_1,t_2|Q).\tag{B.32}$$

B.3. Solution for Γ_1

B.3.1. SDE at NLO

The SDE for Γ_1 is a little modified as

$$\begin{aligned}&D_1^f\Gamma_1(12)+\mu_{1,f}^2\Gamma_0(12)\\&\quad=-\frac{u_f}{6}\Big[K_0(12;11)+\Gamma_0(12)\Gamma_1(11)+\Gamma_1(12)\Gamma_0(11)+2\Gamma_0(12)\Gamma_0(11)\Big],\end{aligned}\tag{B.33}$$

where we replace μ_f^2 by $\mu_f^2+\dfrac{\mu_{1,f}^2}{N}$, so that $D_1^f\to D_1^f+\dfrac{1}{N}\mu_{1,f}^2$. Here $u_{1,f}^2$ is given by Eq. (A.36) with the replacements $u,m\to u_f,m_f$.

We parametrize Γ_1 as

$$\Gamma_1(12) = F(t_1)F(t_2)\int Dp \frac{\exp\left(-(p^2+\mu_f^2)(t_1+t_2)\right)\exp\left(ip(x_1-x_2)\right)}{p^2+m^2} G_1(t_1,t_2|p) \tag{B.34}$$

with the boundary condition

$$G_1(0,0|p) \equiv b(p) = -\frac{\Sigma_1(p)}{p^2+m^2}, \tag{B.35}$$

where $\Sigma_1(p)$ is the self-energy at the NLO in d-dimensional theory.

The NLO SDE becomes

$$\partial_{t_1} G_1(t_1,t_2|p_1) + \mu_{1,f}^2 = -\frac{u_f}{6}F^2(t_1)\exp\left(-2t_1\mu_f^2\right)\mathcal{H}\left[G_1(t_1,t_1|p)\right] + \lambda(t_1,t_2|p_1), \tag{B.36}$$

where $\mathcal{H}$ is defined in Eq. (69) and

$$\lambda(t_1,t_2|p_1) \equiv -\frac{u_f}{6}F^2(t_1)\exp\left(-2t_1\mu_f^2\right)\Delta(t_1,t_2|p_1), \tag{B.37}$$

$$\begin{aligned}\Delta(t_1,t_2|p_1) = 2\zeta_0(t_1) + \exp\left(t_1p_1^2\right)\int\prod_{i=2}^{4}\frac{Dp_i\exp\left(-t_1p_i^2\right)}{p_i^2+m^2}\Big\{&Z(|21;34) + 2Y(1|34;21)\\ &+ 2X(11|12;34) + Y(1|12;34) + 2X(21|21;34) + Y(2|21;34)\Big\}.\end{aligned} \tag{B.38}$$

Using solutions X and Y, we have, in the continuum limit,

$$\begin{aligned}\lambda(t_1,t_2|p_1) = \int dp_2 \frac{\exp\left(t_1(p_1^2-p_2^2)\right)}{p_2^2+m^2}\Big[&\psi(t_1|12) + (p_2^2+m^2)\int_0^{t_1} ds\exp\left(s(p_2^2-p_1^2)\right)\omega(t_1,s|p_{12})\\ &+ (p_1^2+m^2)\int_0^{t_2} ds\exp\left(s(p_1^2-p_2^2)\right)\omega(t_1,s|p_{12})\Big].\end{aligned} \tag{B.39}$$

Since the right-hand side of Eq. (B.39) is finite, $\Delta(t_1,t_2|p) \to 0$ in the continuum limit.

B.3.2. *Solution to the SDE*

Let us define

$$G_1(t_1,t_2|p) \equiv b(p) + \kappa(t_1,t_2|p) + H(t_1) + H(t_2) \tag{B.40}$$

with $\kappa(t_1,t_2|p) = \kappa(t_2,t_1|p)$ and $\kappa(0,0|p) = H(0) = 0$, where

$$\partial_{t_1}\kappa(t_1,t_2|p) = \lambda(t_1,t_2|p), \tag{B.41}$$

$$\frac{dH(t)}{dt} = -\frac{u_f}{6}F^2(t)\exp\left(-2t\mu_f^2\right)\left[\mathcal{H}\left[G_1(t,t|p)\right] - 2\zeta(t)\delta_1\right]. \tag{B.42}$$

The second equation (B.42) can be rewritten as

$$\frac{dH(t)}{dt} = -\frac{u_f}{6}F^2(t)\exp\left(-2t\mu_f^2\right)\left[2\zeta_0(t)H(t) + b_0(t) + \kappa_0(t) - 2\zeta(t)\delta_1\right], \tag{B.43}$$

so that we have in the continuum limit

$$H(t) = -\frac{b_0(t) + \kappa_0(t)}{2\zeta_0(t)} + \delta_1, \tag{B.44}$$

where we define $b_0(t) = \mathcal{H}[b(p)]$ and $\kappa_0(t) = \mathcal{H}[\kappa(t, t|p)]$.

The first equation (B.41) can be solved as

$$\kappa(t_1, t_2|p) = k_2(t_1, t_2|p) + k_1(t_1|p) + k_1(t_2|p), \tag{B.45}$$

where

$$k_1(t|p) = \int_0^t ds\, \lambda_1(s|p), \tag{B.46}$$

$$\begin{aligned}\lambda_1(t|p) = &\int Dq \frac{\exp\left((p^2 - q^2)t\right)}{q^2 + m^2} \psi(t|p, q)\\ &+ \int_0^t ds \int Dq \exp\left((p^2 - q^2)(t - s)\right) \omega(t, s|Q),\end{aligned} \tag{B.47}$$

$$k_2(t_1, t_2|p) = \int_0^{t_1} ds_1 \int_0^{t_2} ds_2 \int Dq \frac{p^2 + m^2}{q^2 + m^2} \exp\left((p^2 - q^2)(s_1 + s_2)\right) \omega(s_1, s_2|Q), \tag{B.48}$$

with $Q = p + q$.

Appendix C. Calculations in the massless limit at $d = 3$

It can be shown that the flow bubble integral can be represented as

$$B(t|\{p\}_{\text{sym.}}) = -2 \int_0^t ds\, K(s, 0|\{p\}_{\text{sym.}}) + B(0|\{p\}_{\text{sym.}}), \quad B(0|\{p\}_{\text{sym.}}) = \frac{1}{8\sqrt{D}}, \tag{C.1}$$

which can be rescaled as

$$B(t|\{p\}_{\text{sym.}}) = \frac{1}{\sqrt{D}} b_0(Dt), \tag{C.2}$$

where

$$b_0(w) = \frac{1}{8} - \frac{\sqrt{w}}{2(2\pi)^{3/2}} \int_0^1 \frac{dx}{\sqrt{x}} e^{-wx} \int_0^1 \frac{dz}{\sqrt{z}} e^{wzx/2}. \tag{C.3}$$

Rescaling,

$$\rho(t|\{p\}_{\text{sym.}}) = R_0(Dt, D), \qquad \psi(t|\{p\}_{\text{sym.}}) = \sqrt{D}\phi_0(Dt, D), \tag{C.4}$$

the integral equation for ψ in the massless limit is written as

$$R_0(w, D) + \int_0^w dv\, k_0(w, v)\phi_0(v, D) = 0, \tag{C.5}$$

where

$$R_0(w, D) = e^{-3w/2} - 8b_0(w)\frac{\bar{u}(D)}{1 + \bar{u}(D)}, \qquad \bar{u}(D) = \frac{u}{48\sqrt{D}}. \tag{C.6}$$

Since the problem is linear, we can write

$$\phi_0(w,D) = \phi_0^{(1)}(w) - 8\phi_0^{(2)}(w)\frac{\bar{u}(D)}{1+\bar{u}(D)}, \tag{C.7}$$

where $\phi_0^{(i)}$, $i = 1,2$ solve the momentum-independent Eqs. (56) and (57). We thus finally obtain Eq. (52).

As the source term can be rescaled as

$$\rho(t,s|\{p\}_{\mathrm{sym.}}) = \frac{1}{\sqrt{D}}\left[b_0(D(t+s)) - 8b_0(Dt)b_0(Ds)\frac{\bar{u}(D)}{1+\bar{u}(D)}\right], \tag{C.8}$$

the equation for ω in the massless limit is written for $\omega(t,s|\{p\}_{\mathrm{sym.}}) = \sqrt{D}W_0(Dt,Ds,D)$ as

$$\begin{aligned} &b_0(D(t+s)) - 8b_0(Dt)b_0(Ds)\frac{\bar{u}(D)}{1+\bar{u}(D)} \\ &\qquad = 2\int_0^{Dt} du\, k_0(Dt,u)\int_0^{Ds} dv\, k_0(Ds,v)W_0(u,v,D), \end{aligned} \tag{C.9}$$

which can be solved as

$$W_0(w,v,D) = \Omega_0(w,v) - 4\phi_0^{(2)}(w)\phi_0^{(2)}(v)\frac{\bar{u}(D)}{1+\bar{u}(D)}, \tag{C.10}$$

where Ω_0 solves the momentum-independent (D-independent) equation (58). We thus obtain Eq. (53).

Appendix D. Induced metric in the massless limit at $d = 3$

D.1. Induced metric

The space component of the induced metric is given by

$$g_{ij}(z) = \delta_{ij}\frac{R_0^2}{d\zeta_0(t)}\left(1 - \frac{\zeta_1(t)}{N}\right)\mathcal{H}\left[p^2\left(1 + \frac{G_1(t,t|p)}{N}\right)\right]. \tag{D.1}$$

We then evaluate

$$\zeta_1(t) = \frac{1}{\zeta_0(t)}\mathcal{H}[G_1(t,t|p)] = 2\delta_1, \quad \mathcal{H}[1] = \zeta_0(t), \qquad \mathcal{H}[p^2] = -\frac{\partial_t\zeta_0(t)}{2}, \tag{D.2}$$

$$\mathcal{H}[p^2G_1(t,t|p)] = \mathcal{H}[\lambda(t,t|p)] + \zeta_0(t)\partial_t H(t) - \partial_t\zeta_0(t)\delta_1 = \zeta_0(t)\partial_t H(t) - \partial_t\zeta_0(t)\delta_1, \tag{D.3}$$

where in the last equation we use $\mathcal{H}[\lambda(t,t|p)] = 0$. Altogether we obtain

$$g_{ij}(z) = \delta_{ij}R_0^2\left[g^{(0)}(t) + \frac{1}{N}g^{(1)}(t)\right], \quad g^{(0)}(t) = -\frac{\partial_t\zeta_0(t)}{2d\zeta_0(t)},\ g^{(1)}(t) = \frac{\partial_t H(t)}{d}. \tag{D.4}$$

The time component is evaluated as

$$g_{00}(t) = t\partial_{t_1}\partial_{t_2}\left[\frac{R_0^2}{(\zeta_0(t_1)\zeta_0(t_2))^{1/2}}\int Dp\frac{\exp\left(-p^2(t_1+t_2)\right)}{p^2+m^2}\left(1 + \frac{\widetilde{G}_1(t_1,t_2|p)}{N}\right)\right]_{t_1=t_2=t} \tag{D.5}$$

$$= R_0^2\left\{g_{00}^{(0)}(t) + \frac{1}{N}g_{00}^{(1)}(t)\right\}, \tag{D.6}$$

where

$$\widetilde{G}_1(t_1, t_2|p) = -2\delta_1 + G_1(t_1, t_2|p). \tag{D.7}$$

The leading term is

$$g_{00}^{(0)}(t) = \frac{t}{4}\partial_t^2 \left[\log \zeta_0(t)\right] \tag{D.8}$$

and for the NLO term we have

$$\frac{1}{t}\, g_{00}^{(1)}(t) = \partial_{t_1}\partial_{t_2} \frac{I(t_1, t_2)}{(\zeta_0(t_1)\zeta_0(t_2))^{1/2}}\bigg|_{t_1=t_2=t}, \tag{D.9}$$

where

$$I(t_1, t_2) = \int Dp\, \frac{\exp\left(-p^2(t_1+t_2)\right)}{p^2+m^2} \widetilde{G}_1(t_1, t_2|p). \tag{D.10}$$

With this notation,

$$\frac{1}{t}\, g_{00}^{(1)}(t) = \frac{1}{4}\, \frac{(\partial_t \zeta_0(t))^2}{\zeta_0^3(t)}\, I(t,t) - \frac{1}{2}\, \frac{\partial_t \zeta_0(t)}{\zeta_0^2(t)}\, \partial_t I(t,t) + \frac{1}{\zeta_0(t)} \partial_{t_1}\partial_{t_2} I(t_1,t_2)\big|_{t_1=t_2=t}. \tag{D.11}$$

Since

$$I(t,t) = \mathcal{H}[\widetilde{G}_1(t,t|p)] = 0, \tag{D.12}$$

the first two terms vanish. Further,

$$\partial_{t_1}\partial_{t_2} I(t_1,t_2)\big|_{t_1=t_2=t} = \mathcal{H}\left[(p^2)^2 \widetilde{G}_1(t,t|p) - 2p^2\lambda(t,t|p) + \partial_{t_2}\lambda(t,t_2|p)\big|_{t_2=t}\right] + \partial_t H(t)\partial_t \zeta_0(t). \tag{D.13}$$

Using the identities

$$\mathcal{H}[\lambda(t,t|p)] = 0, \qquad \mathcal{H}[p^2 \widetilde{G}_1(t,t|p)] = \zeta_0(t)\, \partial_t H(t) \tag{D.14}$$

and their derivatives, this can be further simplified:

$$\partial_{t_1}\partial_{t_2} I(t_1,t_2)\big|_{t_1=t_2=t} = -\tfrac{1}{2}\, \zeta_0(t)\partial_t^2 H(t) + \mathcal{H}\left[\partial_{t_2}\lambda(t,t_2|p)\big|_{t_2=t} -\ \partial_t\lambda(t,t|p)/2\right]. \tag{D.15}$$

Here the second term vanishes and we finally obtain

$$g_{00}^{(1)}(t) = -\frac{t}{2}\, \partial_t^2 H(t). \tag{D.16}$$

D.2. Calculation of $H(t)$ in the massless limit

We recall the definition of $H(t)$ as

$$H(t) = -\frac{b_0(t) + \kappa_0(t)}{2\zeta_0(t)} + \delta_1 \tag{D.17}$$

where

$$b_0(t) = \mathcal{H}[b(p)], \qquad \kappa_0(t) = \mathcal{H}[\kappa(t,t|p)], \tag{D.18}$$

with

$$b(p) = -\frac{\Sigma_1(p)}{p^2+m^2}, \qquad \kappa(t,t|p) = k_2(t,t|p) + 2k_1(t|p). \tag{D.19}$$

Here k_1 and k_2 are given in Eqs. (B.46), (B.47), and (B.48).

Hereafter we consider the massless limit at $d=3$, where we have $\zeta_0(t)^{-1} = 2(2\pi)^{3/2}\sqrt{t}$.

D.2.1. Calculation of $b_0(t)$

We first calculate $b_0(t)$. In the massless limit, we have

$$H_b(t) \equiv -\frac{b_0(t)}{2\zeta_0(t)} = \frac{1}{2\zeta_0(t)} \int Dp \frac{\exp\left(-2p^2 t\right)}{(p^2)^2} g(p^2) \tag{D.20}$$

since $\tilde{C} = Z_1 m^2 = 0$ and

$$g(p^2) = \frac{u}{3} \int \frac{DQ}{1+\bar{u}(Q^2)} \left\{ \frac{1}{(Q+p)^2} - \frac{1}{Q^2} \right\}. \tag{D.21}$$

After rescaling, we obtain

$$H_b(t) = \int DQ\, h_b(Q^2) \frac{\bar{u}(Q^2)\sqrt{t}}{1+\bar{u}(Q^2)\sqrt{t}}, \tag{D.22}$$

where

$$h_b(Q^2) = 32\sqrt{2}\sqrt{\pi^3}\sqrt{Q^2} \int Dp\, \frac{\exp\left(-2p^2\right)}{(p^2)^2} \left\{ \frac{1}{(Q+p)^2} - \frac{1}{Q^2} \right\}. \tag{D.23}$$

D.2.2. Calculation of $\kappa_0(t)$

For this we need ψ and ω in the massless limit, which can be obtained as

$$\psi_0(t|p,q) = \sqrt{Q^2}\left[\varphi_0(Q^2 t, z) - 8\phi_0^{(2)}(Q^2 t) \frac{\bar{u}(Q^2)}{1+\bar{u}(Q^2)} \right], \tag{D.24}$$

$$\omega_0(t,s|Q) = \sqrt{Q^2}\left[\Omega_0(Q^2 t, Q^2 s) - 4\phi_0^{(2)}(Q^2 t)\phi_0^{(2)}(Q^2 s) \frac{\bar{u}(Q^2)}{1+\bar{u}(Q^2)} \right] \tag{D.25}$$

with $z = (p^2+q^2)/Q^2$, where $\phi_0^{(2)}$ and Ω_0 have already been obtained in Sect. 3, while φ_0 satisfies

$$e^{-zw} + \int_0^w dx\, k_0(w,x)\varphi_0(x,z) = 0, \tag{D.26}$$

instead of Eq. (56), and thus $\varphi_0(x, 3/2) = \phi_0^{(1)}(x)$.

Using these, we first calculate

$$H_\kappa^{(1)}(t) \equiv -\frac{1}{\zeta_0(t)}\int Dp\frac{\exp\left(-2p^2t\right)}{p^2}\int_0^t ds\int Dq\frac{\exp\left((p^2-q^2)s\right)}{q^2}\psi_0(s|p,q)$$
$$= H_\kappa^{(1)}(0) + \int DQ\int_0^1 dx\,\phi_0^{(2)}(Q^2x)h_{11}(x,Q^2)\frac{\bar{u}(Q^2)\sqrt{t}}{1+\bar{u}(Q^2)\sqrt{t}}, \tag{D.27}$$

where $H_\kappa^{(1)}(0)$ is some constant and

$$h_{11}(x,Q^2) = 32\sqrt{2}\sqrt{\pi^3}\sqrt{Q^2}\int Dp\,Dq\,(2\pi)^3\delta(q+p-Q)\frac{\exp\left(-(2-x)p^2-xq^2\right)}{p^2q^2}. \tag{D.28}$$

Similarly we have

$$H_\kappa^{(2)}(t) \equiv -\frac{1}{\zeta_0(t)}\int Dp\frac{\exp\left(-2p^2t\right)}{p^2}\int_0^t ds\int Dq\,\exp\left((p^2-q^2)s\right)$$
$$\times\int_0^s dr\,\exp\left((q^2-p^2)r\right)\omega_0(s,r|Q)$$
$$= H_\kappa^{(2)}(0) + 2\int DQ\int_0^1 dx\,\phi_0^{(2)}(Q^2x)\int_0^x dy\,\phi_0^{(2)}(Q^2y)\,h_{10}(x-y,Q^2)$$
$$\times\frac{\bar{u}(Q^2)\sqrt{t}}{1+\bar{u}(Q^2)\sqrt{t}}, \tag{D.29}$$

where

$$h_{10}(z,Q^2) = 8\sqrt{2}\sqrt{\pi^3}\sqrt{Q^2}\int Dp\,Dq\,(2\pi)^3\delta(q+p-Q)\frac{\exp\left(-(2-z)p^2-zq^2\right)}{p^2}. \tag{D.30}$$

The last contribution becomes

$$H_\kappa^{(3)}(t) \equiv -\frac{1}{2\zeta_0(t)}\int Dp\exp\left(-2p^2t\right)$$
$$\times\int_0^t ds\int Dq\,\frac{\exp\left((p^2-q^2)s\right)}{q^2}\int_0^t dr\,\exp\left((p^2-q^2)r\right)\omega_0(s,r|Q)$$
$$= H_\kappa^{(3)}(0) + \int DQ\int_0^1 dx\,\phi_0^{(2)}(Q^2x)\int_0^1 dy\,\phi_0^{(2)}(Q^2y)\,h_{10}(2-x-y,Q^2)$$
$$\times\frac{\bar{u}(Q^2)\sqrt{t}}{1+\bar{u}(Q^2)\sqrt{t}}. \tag{D.31}$$

D.3. Total contributions

We thus obtain the $H(t)$ as[7]

$$H(t) = H(0) + \int DQ\,h_{\text{total}}(Q^2)\frac{\bar{u}(Q^2)\sqrt{t}}{1+\bar{u}(Q^2)\sqrt{t}}, \tag{D.32}$$

[7] Here $H(0)$ is potentially divergent but it does not contribute to the metric.

where

$$H(0) = H_\kappa^{(1)}(0) + H_\kappa^{(2)}(0) + H_\kappa^{(3)}(0) + \delta_1, \tag{D.33}$$

$$h_{\text{total}}(Q^2) = h_b(Q^2) + \int_0^1 dx\, \phi_0^{(2)}(Q^2 x) \left\{ h_{11}(x, Q^2) + 2\int_0^x dy\, \phi_0^{(2)}(Q^2 y) h_{10}(x-y, Q^2) \right.$$
$$\left. + \int_0^1 dy\, \phi_0^{(2)}(Q^2 y) h_{10}(2-x-y, Q^2) \right\}, \tag{D.34}$$

which leads to Eqs. (73) and (74) by $A_1(t) \equiv \partial_t H(t)$ and $\partial_t A_1(t) \equiv \partial_t^2 H(t)$.

D.4. IR behaviors

D.4.1. Some definitions

We write the NLO induced metric as

$$g_{ij}(\tau) = \delta_{ij} \left\{ \frac{R_0^2}{12t} \left[1 + \frac{R(t)}{N} \right] \right\}, \qquad g_{00}(\tau) = -t\partial_t \left\{ \frac{R_0^2}{8t} \left[1 + \frac{R(t)}{N} \right] \right\}, \tag{D.35}$$

where the relative correction is a sum of four contributions:

$$R(t) = R_b(t) + \sum_{i=1}^{3} R_\kappa^{(i)}(t), \quad R_b(t) \equiv 4t\partial_t H_b(t),\ R_\kappa^{(i)}(t) \equiv 4t\partial_t H_\kappa^{(i)}(t). \tag{D.36}$$

We also introduce $G(v)$ by

$$\phi_0^{(2)}(v) = -\frac{(2\pi)^{3/2}}{\sqrt{v}} G(v), \qquad G(0) = 1/8, \quad G(v) \sim \exp(-v/2),\ v \to \infty \tag{D.37}$$

and use the time variable $T = u\sqrt{t}/48$.

In the following we will use the fact that a double 3-dimensional integral of any function depending only on the absolute values p, q, and $|Q|$, where $Q = p + q$, can be written

$$\int Dp \int Dq\, f(p,q,Q^2) = \frac{1}{(2\pi)^4} \int_0^\infty p\, dp \int_0^\infty q\, dq \int_{(q-p)^2}^{(q+p)^2} dQ^2 f(p,q,Q^2). \tag{D.38}$$

D.4.2. The R_b contribution

Here we can do the angular part of the Q^2 integral analytically and find

$$R_b(t) = \frac{32T}{\sqrt{2\pi}^5} \int_0^\infty \frac{q\, dq}{(q+T)^2} \rho_b(q), \tag{D.39}$$

where

$$\rho_b(q) = q^2 \int_0^\infty \frac{dp}{p^3} \exp\left(-2p^2\right) \left\{ \ln \frac{(p+q)^2}{(p-q)^2} - \frac{4p}{q} \right\}, \tag{D.40}$$

which behaves as $\rho_b(q) = O(q)$ for small q, while

$$\rho_b(q) \sim \frac{\sqrt{2\pi}}{3q}, \tag{D.41}$$

for large q. Thus we can establish that $R_b(t) = O(T)$ for small t, while for large t,

$$r_b \equiv R_b(\infty) = \frac{8}{3\pi^2} = 0.27019. \tag{D.42}$$

D.4.3. The $R_\kappa^{(1)}$ contribution

We have

$$R_\kappa^{(1)}(t) = -32(2\pi)^3 \int Dp \int Dq \int_0^1 \frac{dx}{\sqrt{x}} \frac{\exp\left(-p^2(2-x) - q^2 x\right)}{p^2 q^2} \frac{|Q|T}{(T+|Q|)^2} G(Q^2 x). \tag{D.43}$$

Doing the q^2 integral first and introducing $x = y^2$ we can rewrite it as

$$-\frac{32}{\pi} \int_0^\infty \frac{dp}{p} \exp\left(-2p^2\right) \int_0^\infty \frac{Q^2 T}{(T+Q)^2} dQ \int_0^1 dy\, G(Q^2 y^2) \int_{(Q-p)^2}^{(Q+p)^2} \frac{\exp\left((p^2-q^2)y^2\right)}{q^2} dq^2. \tag{D.44}$$

After some further rescaling we get

$$R_\kappa^{(1)}(t) = -\frac{64}{\pi} \int_0^\infty dQ \frac{QT}{(T+Q)^2} \rho_\kappa^{(1)}(Q), \tag{D.45}$$

where

$$\rho_\kappa^{(1)}(Q) = \int_0^\infty \frac{dp}{p} \exp\left(-2p^2\right) \int_0^Q dz\, G(z^2) Y\left(\frac{p}{Q}, z\right), \tag{D.46}$$

$$Y(\varepsilon, z) = \int_{|1-\varepsilon|}^{1+\varepsilon} \frac{d\xi}{\xi} \exp\left((\varepsilon^2 - \xi^2)z^2\right) = 2\varepsilon \exp\left(-z^2\right) + O(\varepsilon^2). \tag{D.47}$$

From this we see that $\rho_\kappa^{(1)}(Q) = O(Q)$ for small Q, while

$$\begin{aligned} \rho_\kappa^{(1)}(Q) &\sim \frac{2}{Q} \int_0^\infty dp \exp\left(-2p^2\right) \int_0^\infty dz\, G(z^2) \exp\left(-z^2\right) \\ &= \frac{1}{Q} \left(\frac{\pi}{2}\right)^{1/2} \int_0^\infty dz\, G(z^2) \exp\left(-z^2\right) \end{aligned} \tag{D.48}$$

for large Q, so that we numerically obtain

$$r_\kappa^{(1)} \equiv R_\kappa^{(1)}(\infty) = -\frac{64}{\sqrt{2\pi}} \int_0^\infty dz\, G(z^2) \exp\left(-z^2\right) = -1.14734. \tag{D.49}$$

D.4.4. The $R_\kappa^{(2)}$ contribution

Similarly

$$\begin{aligned} R_\kappa^{(2)}(t) = 16\sqrt{2\pi}^9 \int Dp \int Dq \int_0^1 \frac{dx}{\sqrt{x}} \int_0^x \frac{dy}{\sqrt{y}} \frac{T}{(T+|Q|)^2} G(Q^2 x) G(Q^2 y) \\ \times \frac{\exp\left(-2p^2\right)}{p^2} \exp\left((p^2-q^2)(x-y)\right). \end{aligned} \tag{D.50}$$

Doing the q^2 integrations first, we have

$$R_\kappa^{(2)}(t) = 64\sqrt{2\pi}\int_0^\infty dQ\frac{QT}{(T+Q)^2}\int_0^\infty \frac{dp}{p}\exp\left(-2p^2\right)$$
$$\times\int_0^1 dx\int_0^x dy\, G(Q^2x^2)\, G(Q^2y^2)\int_{(Q-p)^2}^{(Q+p)^2}\exp\left((p^2-q^2)(x^2-y^2)\right)dq^2. \quad \text{(D.51)}$$

The q^2 integral can be done analytically and we find

$$R_\kappa^{(2)}(t) = 128\sqrt{2\pi}\int_0^\infty dQ\frac{QT}{(T+Q)^2}\rho_\kappa^{(2)}(Q), \quad \text{(D.52)}$$

where

$$\rho_\kappa^{(2)}(Q) = \int_0^\infty \frac{dp}{p}\exp\left(-2p^2\right)\int_0^Q dz\int_0^z dw\, G(z^2)\, G(w^2)$$
$$\times\frac{\exp\left(w^2-z^2\right)}{z^2-w^2}\sinh\frac{2p}{Q}(z^2-w^2). \quad \text{(D.53)}$$

Thus $\rho_\kappa^{(2)} = O(Q)$ for small Q, while

$$\rho_\kappa^{(2)}(Q) \sim \frac{1}{Q}\int_{-\infty}^\infty dp\exp\left(-2p^2\right)\int_0^\infty dz\int_0^z dw\, G(z^2)\, G(w^2)\exp\left(w^2-z^2\right) \quad \text{(D.54)}$$

for large Q, and

$$r_\kappa^{(2)} \equiv R_\kappa^{(2)}(\infty) = 128\pi\int_0^\infty dz\int_0^z dw\, G(z^2)\, G(w^2)\exp\left(w^2-z^2\right) = 0.45846. \quad \text{(D.55)}$$

D.4.5. The $R_\kappa^{(3)}$ contribution

For $R_\kappa^{(3)}$ we find

$$R_\kappa^{(3)}(t) = 32\sqrt{2\pi}\int_0^\infty dQ\frac{QT}{(T+Q)^2}\rho_\kappa^{(3)}(Q) \quad \text{(D.56)}$$

with

$$\rho_\kappa^{(3)}(Q) = \int_0^1 dx\int_0^1 dy\int_0^\infty p\,dp\exp\left(-2p^2+p^2(x^2+y^2)\right)G(Q^2x^2)G(Q^2y^2)$$
$$\times\int_{(Q-p)^2}^{(Q+p)^2}\frac{\exp\left(-q^2(x^2+y^2)\right)}{q^2}dq^2. \quad \text{(D.57)}$$

After rescaling,

$$\rho_\kappa^{(3)}(Q) = \frac{1}{Q^2}\int_0^Q dz\int_0^Q dw\, G(z^2)\, G(w^2)\int_0^\infty p\,dp\exp\left(-2p^2\right)Z\left(\frac{p}{Q},z^2+w^2\right), \quad \text{(D.58)}$$

where

$$Z(\varepsilon,A) = 2\exp\left(A\varepsilon^2\right)\int_{|1-\varepsilon|}^{1+\varepsilon}\frac{\exp\left(-A\xi^2\right)}{\xi}d\xi \approx 4\varepsilon e^{-A}, \quad \varepsilon\to 0. \quad \text{(D.59)}$$

Thus $\rho_\kappa^{(3)}(Q) = O(Q)$ for small Q, while

$$\rho^{(3)}(Q) \sim \frac{4}{Q^3}\int_0^\infty dz \int_0^\infty dw\, G(z^2)\, G(w^2) \int_0^\infty p^2 dp \exp\left(-2p^2 - z^2 - w^2\right)$$
$$= \left(\frac{\pi}{8}\right)^{1/2} \left(\int_0^\infty dz\, G(z^2) \exp\left(-z^2\right)\right)^2 \frac{1}{Q^3}, \tag{D.60}$$

for large Q, which leads to

$$r_\kappa^{(3)} \equiv R_\kappa^{(3)}(\infty) = 0. \tag{D.61}$$

Thus the total relative correction is negative:

$$r = r_b + r_\kappa^{(1)} + r_\kappa^{(2)} + r_\kappa^{(3)} = -0.41869. \tag{D.62}$$

References

[1] S. Aoki, J. Balog, T. Onogi, and P. Weisz, Prog. Theor. Exp. Phys. **2016**, 083B04 (2016) [arXiv:1605.02413 [hep-th]] [Search INSPIRE].

[2] S. Aoki, K. Kikuchi, and T. Onogi, Prog. Theor. Exp. Phys. **2015**, 101B01 (2015) [arXiv:1505.00131 [hep-th]] [Search INSPIRE].

[3] R. Narayanan and H. Neuberger, J. High Energy Phys. **0603**, 064 (2006) [arXiv:hep-th/0601210] [Search INSPIRE].

[4] M. Lüscher, J. High Energy Phys. **1008**, 071 (2010); **1403**, 092 (2014) [erratum] [arXiv:1006.4518 [hep-lat]] [Search INSPIRE].

[5] M. Lüscher, Commun. Math. Phys. **293**, 899 (2010) [arXiv:0907.5491 [hep-lat]] [Search INSPIRE].

[6] M. Lüscher, PoS LATTICE **2013**, 016 (2014) [arXiv:1308.5598 [hep-lat]] [Search INSPIRE].

[7] J. M. Maldacena, Int. J. Theor. Phys. **38**, 1113 (1999); Adv. Theor. Math. Phys. **2**, 231 (1998) [arXiv:hep-th/9711200] [Search INSPIRE].

[8] I. R. Klebanov and A. M. Polyakov, Phys. Lett. B **550**, 213 (2002) [arXiv:hep-th/0210114] [Search INSPIRE].

[9] A. Kagimura, A. Tomiya, and R. Yamamura, arXiv:1508.04986 [hep-lat] [Search INSPIRE].

[10] S. Aoki, K. Kikuchi, and T. Onogi, J. High Energy Phys. **1504**, 156 (2015) [arXiv:1412.8249 [hep-th]] [Search INSPIRE].

[11] S. Aoki, J. Balog, and P. Weisz, J. High Energy Phys. **1409**, 167 (2014); **1507**, 037 (2015) [erratum] [arXiv:1407.7079 [hep-lat]] [Search INSPIRE].

[12] S. S. Pufu, arXiv:1608.02960 [hep-th] [Search INSPIRE].