Complete hypersurfaces with infinite fundamental group

Geometry of homogeneous spaces and submanifolds

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Complete hypersurfaces with infinite fundamental group*

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1. Hypersurfaces with constant scalar curvature

Let $M$ be an $n$-dimensional hypersurface in a unit sphere $S^{n+1}(1)$ of dimension $n+1$. In this section, we shall study curvature structures of complete hypersurfaces with constant scalar curvature in a unit sphere. First of all, we present several examples.

**Example 1.** For any $0 < c < 1$, by considering the standard immersions

$$S^{n-1}(c) \subset \mathbb{R}^n, \quad S^1(\sqrt{1-c^2}) \subset \mathbb{R}^2$$

and taking the Riemannian product immersion

$$S^1(\sqrt{1-c^2}) \times S^{n-1}(c) \hookrightarrow \mathbb{R}^2 \times \mathbb{R}^n,$$

we obtain a compact hypersurface $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$ in $S^{n+1}(1)$ with constant scalar curvature $n(n-1)r$, where $r = \frac{n-2}{nc^2} > 1 - \frac{2}{n}$.

We know that this hypersurface $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$ has the following characterizations:

1. $r > 1 - \frac{2}{n}$,

2. the number of its distinct principal curvatures is two.

3. its fundamental group is infinity.

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Example 2. By make using of the same construction as in example 1, we can obtain a compact hypersurface $S^k(c_1) \times S^{n-k}(c_2), 1 < k < n - 1$, in $S^{n+1}(1)$ with constant scalar curvature $n(n-1)r$. This hypersurface has finite fundamental group and the number of its distinct principal curvatures is two.

Example 3. We consider an isoparametric hypersurface $M^6$ in $S^7(1)$ with principal curvatures $\lambda_1 = \lambda_2 = \theta, \lambda_3 = \frac{\theta + 1}{1 + \theta}, \lambda_4 = \lambda_5 = -\frac{1 - \theta}{1 + \theta}, \lambda_6 = -\frac{1 - \theta}{1 + \theta},$ where $\theta = \sqrt{\frac{21 + \sqrt{30}}{2}}$. This hypersurface $M^6$ satisfies $r = 1$ and the number of its distinct principal curvatures is four.

In 1977, S.Y. Cheng and Yau [4] characterized compact hypersurfaces with constant scalar curvature in $S^{n+1}(1)$. They proved

Theorem 1. Let $M$ be an $n$-dimensional compact hypersurface with constant scalar curvature $n(n-1)r$. If $r \geq 1$ and the sectional curvatures of $M$ are non-negative, then $M$ is isometric to the totally umbilical hypersurface $S^n(c)$ or the Riemannian product $S^k(c_1) \times S^{n-k}(c_2)$, $1 \leq k \leq n - 1$, where $S^k(c)$ denote the sphere of radius $c$.

Proof. For a $C^2$-function $f$ on $M$, we consider a differential operator $\Box$ defined by

$$\Box f = \sum_{i,j=1}^{n} (nHh_{ij} - h_{ij}) \nabla_i \nabla_j f,$$  \hspace{1cm} (1.1)

where $h_{ij}$ and $H$ are components of the second fundamental form and the mean curvature of $M$, respectively. Thus, we have

$$\Box nH = \sum_{i,j,k=1}^{n} h_{ijk}^2 - n^2||\text{grad}H|| + \sum_{i,j=1}^{n} (\lambda_i - \lambda_j)^2 K_{ij},$$  \hspace{1cm} (1.2)

where $\lambda_i$'s are principal curvatures and $h_{ijk}$'s denote components of the covariant differentiation of the second fundamental form. From $r \geq 1$, we can prove

$$\sum_{i,j,k=1}^{n} h_{ijk}^2 \geq n^3||\text{grad}H||^2.$$  \hspace{1cm} (1.3)

Since $M$ has non-negative sectional curvature, we have $K_{ij} \geq 0$. Hence, we infer

$$\Box nH \geq 0.$$  \hspace{1cm} (1.4)

According to Stokes theorem, we know that $H$ is constant and the number of distinct principal curvatures is at most two. Therefore, $M$ is an isoparametric hypersurface with at most two distinct principal curvatures. From a theorem of Cartan, we know that theorem 1 is true.

Further, by making use of the similar method which was used by Nakagawa and the author in [3] and the differential operator (1.1) introduced by S.Y. Cheng and Yau, Li [5] proved
**Theorem 2.** Let $M$ be an $n$-dimensional compact hypersurface with constant scalar curvature $n(n-1)r$. If $r \geq 1$ and $S \leq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$, then $M$ is isometric to either the totally umbilical hypersurface or the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$ with $c^2 = \frac{n-2}{n} \leq \frac{n-2}{n}$, where $S$ is the squared norm of the second fundamental form of $M$.

**Proof.** Since $S \leq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$ holds, we can prove

$$\sum_{i,j=1}^{n} (\lambda_i - \lambda_j)^2 K_{ij} \geq 0.$$  

From $r \geq 1$, we know that (1.3) is satisfies. Thus, we infer that the inequality (1.4) is true. Hence, theorem 2 is true by using the same assertion as in theorem 1. \(\square\)

**Remark 1.** In proofs of theorems 1 and 2, the estimate $\sum_{i,j,k=1}^{n} h_{ijk}^2 \geq n^2|\text{grad}H|^2$ is necessary. In order to prove it, the condition of $r \geq 1$ and the assumption of constant scalar curvature is essential. Hence, the condition $r \geq 1$ and the assumption of constant scalar curvature play an essential role in theorems 1 and 2.

**Remark 2.** From example 1, we know that some of $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$ does not appear in these results of theorems 1 and 2 because some of them does not satisfy the condition $r \geq 1$.

Moreover, Cheng [2] researched the inversed problem of example 1. The following was proved.

**Theorem 3.** Let $M$ be an $n$-dimensional complete hypersurface with constant scalar curvature $n(n-1)r$ in $S^{n+1}(1)$. If $M$ has only two distinct principal curvatures one of which is simple, then $r > 1 - \frac{2}{n}$ holds and $M$ is isometric to $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$ if $r \neq \frac{n-2}{n-1}$ and $S \geq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$, where $c^2 = \frac{n-2}{n} r^2$.

From the assertions above, it is natural and interesting to study the following:

**Problem 1.** Let $M$ be an $n$-dimensional compact hypersurface with constant scalar curvature $n(n-1)r$ in $S^{n+1}(1)$. If $r > 1 - \frac{2}{n}$ and $S \leq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$, then is $M$ isometric to the totally umbilical hypersurface or the Riemannian product $S^1(\sqrt{1-c^2}) \times S^{n-1}(c)$?

From theorem 2, we know that if $r \geq 1$, then the problem 1 was solved affirmatively. In [2], the author gave an affirmative answer for this problem when $r = \frac{n-2}{n-1}$. But for the other case, this problem seems to be a very hard problem.

**Problem 2.** Let $M$ be an $n$-dimensional compact hypersurface with constant scalar curvature $n(n-1)r$ in $S^{n+1}(1)$. If $r > 1 - \frac{2}{n}$ and the sectional curvature is non-negative, then is $M$ isometric to the totally umbilical hypersurface or the Riemannian product $S^k(\sqrt{c_1}) \times S^{n-k}(c_2)$, $1 \leq k \leq n-1$?
2. Compact hypersurfaces with infinite fundamental group

In this section, we shall try to solve problems 1 and 2 introduced in the section 1. From example 1, we know that \(S^1(\sqrt{1-c^2}) \times S^{n-1}(c)\) has infinite fundamental group. We shall consider these problems under a topological condition. The following theorems will be proved.

**Theorem 4.** Let \(M\) be an \(n\)-dimensional compact hypersurface with infinite fundamental group in \(S^{n+1}(1)\). If \(r \geq \frac{n-2}{n-1}\) and \(S \leq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}\), then \(M\) is isometric to the Riemannian product \(S^1(\sqrt{1-c^2}) \times S^{n-1}(c)\), where \(n(n-1)r\) is the scalar curvature of \(M\) and \(c^2 = \frac{n-2}{n(r-1)}\).

**Proof.** Since \(r \geq \frac{n-2}{n-1}\) and \(S \leq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}\), we infer

\[
n + 2nH^2 - S \geq \frac{n-2}{\sqrt{n(n-1)}} \sqrt{n^2H^2(S-nH^2)}.
\]

(2.1)

For any point \(p\) and any unit vector \(\vec{u} \in T_p M\), we choose a local orthonormal frame field \(\{e_1, \ldots, e_n\}\) such that \(e_n = \vec{u}\). From Gauss equation, we have

\[
\text{Ric}(\vec{u}) = (n-1) + nHh_{nn} - \sum_{i=1}^{n} h_{in}^2
\]

(2.2)

and we can prove

\[
\text{Ric}(\vec{u}) \geq \frac{n-1}{n}(n + 2nH^2 - S - \frac{n-2}{\sqrt{n(n-1)}} \sqrt{n^2H^2(S-nH^2)}).
\]

(2.3)

From (2.1), we have \(\text{Ric}(\vec{u}) \geq 0\). In particular, we can show that if \(S < (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}\) holds, then \(\text{Ric}(\vec{u}) > 0\). Thus, if there exists a point \(p\) in \(M\) such that \(S < (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}\), then, at the point \(p\), the Ricci curvature is positive. From the following Lemma 1 due to Aubin [1], we know that there exists a metric on \(M\) such that the Ricci curvature is positive on \(M\). According to Myers theorem, we know that the fundamental group is finite. This is impossible because \(M\) has infinite fundamental group.

**Lemma 1.** (cf. Aubin [1, p. 344]). *If the Ricci curvature of a compact Riemannian manifold is non-negative and positive at somewhere, then the manifold carries a metric with positive Ricci curvature.*

Thus, we must have \(S = (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}\). And at each point, there exists a unit vector \(\vec{u}\) such that \(\text{Ric}(\vec{u}) = 0\). Thus, we can conclude that \(M\) has only two distinct principal curvatures one of which is simple. Let \(\{e_1, \ldots, e_n\}\) be a local orthonormal frame field such that \(h_{ij} = \lambda_i \delta_{ij}\), where \(\lambda_i\)'s are principal curvatures on \(M\). Without loss of generality, we can assume \(\mu = \lambda_n, \lambda = \lambda_1 = \cdots = \lambda_{n-1}\). From Gauss equation (2.2) and the definition of the Ricci curvature, we have \(1 + \mu \lambda = 0\)
because of $1 + \lambda_i \lambda_j = 1 + \lambda^2 > 0$, for any $i, j = 1, \cdots, n - 1$. From Gauss equation, we have

$$\mu = \frac{n(r - 1)}{2\lambda} - \frac{n - 2}{2} \lambda.$$  

Hence $\lambda^2 = \frac{n(r - 1)^2 + 2}{n - 2}$ and $\mu^2 = \frac{n - 2}{n(r - 1)^2 + 2}$.

We consider the integral submanifold of the corresponding distribution of the space of principal vectors corresponding to the principal curvature $\lambda$. Since the multiplicity of the principal curvature $\lambda$ is greater than 1, we know that the principal curvature $\lambda$ is constant on this integral submanifold (cf. Otsuki [6]). From $\lambda^2 = \frac{n(r - 1)^2 + 2}{n - 2}$ and $\mu^2 = \frac{n - 2}{n(r - 1)^2 + 2}$, we know that the scalar curvature $n(n - 1)r$ and the principal curvature $\mu$ are constant. Thus, we obtain that $M$ is isoparametric. Therefore, $M$ is isometric to the Riemannian product $S^1(\sqrt{1 - c^2}) \times S^{n-1}(c)$ because $S = (n - 1) \frac{n(r - 1)^2 + 2}{n - 2} + \frac{n - 2}{n(r - 1)^2 + 2}$ holds. This completes the proof of Theorem 4.

**Theorem 5.** Let $M$ be an $n$-dimensional compact hypersurface with infinite fundamental group in $S^{n+1}(1)$. If the sectional curvatures are non-negative, then $M$ is isometric to the Riemannian product $S^1(\sqrt{1 - c^2}) \times S^{n-1}(c)$.

**Proof.** Since the sectional curvatures are non-negative, we have that the Ricci curvature is non-negative. From the arguments in the proof of theorem 4, we infer that at each point, there exists a unit vector $\mathbf{u}$ such that $\text{Ric}(\mathbf{u}) = 0$.

Let $\{e_1, \cdots, e_n\}$ be a local orthonormal frame field such that $h_{ij} = \lambda_i \delta_{ij}$, where $\lambda_i$'s are principal curvatures on $M$. Then, from Gauss equation, we have $1 + \lambda_i \lambda_j \geq 0$ for $i \neq j$. Further, there exists an $i$ such that $\sum_{j \neq i}(1 + \lambda_i \lambda_j) = 0$ from the definition of Ricci curvature. Hence, we must have $1 + \lambda_i \lambda_j = 0$ for $j \neq i$. Therefore, $M$ has only two distinct principal curvatures one of which is simple. Let $\mu = \lambda_i$ and $\lambda = \lambda_j$ for $j \neq i$. From Gauss equation, we have

$$\mu = \frac{n(r - 1)}{2\lambda} - \frac{n - 2}{2} \lambda. \quad (2.4)$$

Since $1 = \mu \lambda = 0$ and (2.4) hold, we have $\lambda^2 = \frac{n(r - 1)^2 + 2}{n - 2}$ and $\mu^2 = \frac{n - 2}{n(r - 1)^2 + 2}$. Hence, we have

$$S = (n - 1) \lambda^2 + \mu^2 = (n - 1) \frac{n(r - 1)^2 + 2}{n - 2} + \frac{n - 2}{n(r - 1)^2 + 2}.$$ 

By making use of the same assertion as in the proof of theorem 4, we infer that $M$ is isometric to the Riemannian product $S^1(\sqrt{1 - c^2}) \times S^{n-1}(c)$. This completes the proof of Theorem 5.

**Remark 3.** In our theorems 4 and 5, we do not assume that the scalar curvature is constant. And in our theorem 5, we do not assume any condition on scalar curvature.

**References**


