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On the construction and classification of almost complex curves in a nearly Kähler 6-sphere

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This report is based on a joint work with Prof. H. Hashimoto (Meijo University), Dr. T. Taniguchi (Tohoku University) (cf. [HTU]).

It is well known that a standard 6-dimensional sphere $S^6$ has a nearly Kähler structure. We denote it by $J$. Although a submanifold whose tangent space is invariant under the action of $J$ is 2-dimensional or 4-dimensional, there is no 4-dimensional $J$-invariant submanifold by the result of A. Gray ([Gr]). Therefore, the only possible case of $J$-invariant submanifold is that of immersed surface. We express the $J$-invariant surface as the image of almost complex conformal immersion of some Riemann surface $M$. In this case, we denote it by $f : M \rightarrow S^6$. On the other hand, (totally real or CR) 3-dimensional submanifold can be often constructed as a tube of some radius over some almost complex curve (cf. Mashimo’s article in this volume). For example, Ejiri immersion : $S^3(\frac{1}{16}) \rightarrow S^6$ can be realized as a tube of radius $\frac{\pi}{2}$ in the direction of second normal space over almost complex curve $S^2(\frac{1}{6}) \rightarrow S^6$. Almost complex curves of $S^6$ is divided into the four types of the following : (I) linearly full and superminimal in $S^6$, (II) linearly full and non-superminimal in $S^6$, (III) linearly fully immersed in some totally geodesic 5-dimensional sphere $S^5$ (which is necessarily non-superminimal), (IV) totally geodesic almost complex 2-sphere.

Since the automorphism group of the nearly Kähler structure is the exceptional Lie group $G_2$, $S^6$ can be expressed as a homogeneous space $S^6 = G_2/SU(3)$, which is a 3-symmetric space. For Type (I), it can be lifted to a horizontal holomorphic curve in $Q^5 = G_2/U(2)$ which is the twister space over $S^6 = G_2/SU(3)$. Bryant ([Br]) gave the representation formula for almost complex curve of type (I) using this twister space. For types (II) and (III), Bolton-Pedit-Woodward ([BPW]) showed that $f$ has a Toda-framing into a 6-symmetric space $\tilde{f} : M \rightarrow G_2/T^2$, where $T^2$ is the maximal torus of $SU(3)$. From these points of views, we may consider the following problem: “Construct and classify the cases of type (II) and (III)”. For the classification of type (II) and (III), there are some pioneering works by Bolton-Vrancken-Woodward ([BVW]). In this note, we present some construction and classification of almost complex 2-tori of type (III).

1. Primitive map of finite type into 6-symmetric space

Theorem 1.1 ([BPW]). Any almost complex 2-torus of type (II) or (III) $f :
$T^2 \rightarrow S^6$ can be lifted to a primitive map of finite type into 6-symmetric space $G_2/T^2$.

This theorem, together with the results of McIntosh([Mc1], [Mc2]), says that any almost complex 2-torus can be constructed from the spectral data. However, since the spectral data describes the moduli space of primitive map of finite type from 2-tori, it is not so easy to pick up only the data for almost complex 2-tori.

In the following, we consider only the almost complex curves of type (III). We denote by $V_6$ the hyperplane of $\mathbb{R}^7$ whose sixth coordinate is identically zero. We then consider the following correspondence.

(*) \quad V_6 \ni x_R = ^{t}(x_1, x_2, x_3, x_4, x_5, 0, x_7) \leftrightarrow x_C = \begin{pmatrix} x_1 + ix_7 \\ x_2 + ix_4 \\ -x_3 + ix_6 \end{pmatrix} \in \mathbb{C}^3.

By this correspondence (*), we identify an almost complex curve of type (III) $f : M \rightarrow S^6 \cap V_6(= S^5_N)$ with a conformal immersion $f_C : M \rightarrow S^5 \subset \mathbb{C}^3$. We have the following theorem.

**Theorem 1.2.** Let $f_C : T^2 \rightarrow S^5 \subset \mathbb{C}^3$ be the one corresponding to an almost complex 2-torus $f : T^2 \rightarrow S^5_N$ by (*). Then, $f_C$ may be lifted to a primitive map of finite type into 6-symmetric space $SU(3)/T^1$. Moreover, its homogeneous projection into $S^6$, $f_C : T^2 \rightarrow S^6 = G_2/SU(3)$, is a harmonic map of finite type.

**Remark.** (1) If a harmonic map into a symmetric space has a lift $\tilde{f}$ into a generalized flag manifold so that $\tilde{f}$ is a primitive map of finite type, then its homogeneous projection into some symmetric space is a harmonic map of finite type in very many cases (cf. [OU]).

(2) Arbitrary non-isotropic harmonic map of 2-torus $\varphi : T^2 \rightarrow S^6 = SO(7)/SO(6)$ can be lifted to a primitive map of finite type $\tilde{\varphi} : T^2 \rightarrow SO(7)/T^2$, where $SO(7)/T^2$ is a 6-symmetric space and the above $G_2/T^2$ is a 6-symmetric submanifold of $SO(7)/T^2$.

We then have an diagram:

$$
\begin{array}{ccc}
SO(7)/T^2 & \supset & G_2/T^2 & \supset & SU(3)/T^1 \\
\downarrow & & \downarrow & & \downarrow \\
S^6 \cong SO(7)/SO(6) & \cong & G_2/SU(3) & \supset & S^5 \cong SU(3)/SU(2)
\end{array}
$$

The last inclusion is due to the correspondence (*).

2. Kähler angle of conformal immersion into $S^5$ and examples

Since an almost complex curve of type (III) $f_C : M \rightarrow S^5$ is a horizontal curve with respect to the Hopf fibration $S^5 \rightarrow \mathbb{C}P^2$, $f_C$ can be realized as a horizontal
lift of a totally real minimal surface in $\mathbb{C}P^2$. However, to construct an almost complex curve as a horizontal lift of a totally real minimal surface in $\mathbb{C}P^2$, we need to know the Kähler angle of the lift with respect to the nearly Kähler structure $J$. Once we come to know the Kähler angle, we may make the lifted horizontal curve into an almost complex curve by rotating it in $\mathbb{C}^3$. The latter fact is due to [BVW]. Hence, we need to know the Kähler angle of a horizontal surface in $S^5$ and the following proposition gives the answer.

**Proposition 2.1.** Suppose that $f : M \rightarrow S^5_N \subset S^6$ is a conformal immersion. Let $\theta$ be the Kähler angle of $f$ with respect to the nearly Kähler structure $J$ on $S^6$. If $f_c : M \rightarrow S^5 \subset \mathbb{C}^3$ is horizontal with respect to the Hopf fibration $S^5 \rightarrow \mathbb{C}P^2$, then

$$\cos \theta = \Re \left\{ i \det \left( f_c e^{-\frac{\partial f_c}{\partial z}} e^{-\frac{\partial f_c}{\partial \bar{z}}} \right) \right\}$$

Proposition 2.1 yields the machinery method of constructing almost complex curves of type (III) as follows.

**Theorem 2.3.** Let $s_0 : M \rightarrow \mathbb{C}^3$ be a smooth map and $\omega : M \rightarrow \mathbb{R}$ a smooth function. Set $s_1 = e^{-\frac{\partial s_0}{\partial z}}$, $s_2 = e^{-\frac{\partial s_0}{\partial \bar{z}}}$. If $S = (s_0, s_1, s_2)$ has values in $U(3)$ and satisfies $\det S = -i$, then $f : M \rightarrow S^5_N \subset S^6$ corresponding to $f_c : M \rightarrow S^5 \subset \mathbb{C}^3$ defined by $f_c = s_0$ is a conformal immersion and an almost complex curve with respect to $J$. The converse is also true.

Using Theorem 2.3, we may construct a 1-parameter family of almost complex curve of type (III) in terms of Jacobi elliptic functions based on the known example of totally real minimal surface due to Castro-Urbano([CU]).

[Example] Define $f : \mathbb{R}^2 \rightarrow S^6$ by

$$f = \left( \sqrt{\frac{r_2}{r_1 + r_2}} \cos \left( \frac{\pi}{6} - r_3 y \right) \operatorname{sn}(rx,p), \sqrt{\frac{r_1}{r_1 + r_2}} \cos \left( \frac{\pi}{6} - r_2 y \right) \operatorname{cn}(rx,p), \right.$$

$$- \sqrt{\frac{r_2 p^2 + r_1}{r_1 + r_2}} \cos \left( \frac{\pi}{6} - r_3 y \right) \operatorname{sn}(rx,p), \sqrt{\frac{r_1}{r_1 + r_2}} \sin \left( \frac{\pi}{6} - r_2 y \right) \operatorname{cn}(rx,p),$$

$$\left. \sqrt{\frac{r_2 p^2 + r_1}{r_1 + r_2}} \sin \left( \frac{\pi}{6} - r_3 y \right) \operatorname{sn}(rx,p), 0, \sqrt{\frac{r_2}{r_1 + r_2}} \sin \left( \frac{\pi}{6} + r_1 y \right) \operatorname{cn}(rx,p) \right),$$

where

$$\begin{cases} r^2 = \frac{\alpha^3 - 2 + 2\sqrt{\alpha^3 + 1}}{\alpha}, & p^2 = \frac{\alpha^3 - 2 - 2\sqrt{\alpha^3 + 1}}{\alpha^3 - 2 + 2\sqrt{\alpha^3 + 1}}, \\ r_1 = \frac{\sqrt{\alpha^3 + 1} + 1}{\alpha}, & r_2 = \frac{\sqrt{\alpha^3 + 1} - 1}{\alpha}, & r_3 = \frac{2}{\alpha}, \end{cases}$$
and \( \alpha (\geq 2) \) is a real number. Then, it follows from the correspondence (*) and Theorem 2.3 that \( f \) gives a 1-parameter family of almost complex curves of type (III).

### 3. Spectral data and representation formula in term of Prym-theta function

The spectral data for constructing almost complex 2-torus of type (III), \( f_{C} : T^{2} \rightarrow S^{5} \) is given by the triplet \((\hat{C}, \hat{\mathcal{L}}, \pi)\) which satisfies the following four conditions: For \( d \equiv 1 \text{ mod } 6 \)

1. \( \hat{C} \): compact Riemann surface of genus \( \hat{g} = 2d \) admitting an anti-holomorphic involution \( \rho \) and a holomorphic involution \( \sigma \), which satisfy \( \rho \sigma = \sigma \rho \).
2. \( \pi : \hat{C} \rightarrow \mathbb{C}P^{1} \) is a three-fold holomorphic covering map with \( \pi \circ \rho = \pi^{-1} \).

Moreover, the divisor \( (\pi) \) and the ramification divisor \( R \) of \( \pi \) are given by

\[
(\pi) = 3P_{0} - 3P_{\infty}, \quad R = 2P_{0} + 2P_{\infty} + D_{0} + \rho D_{0},
\]

where two points \( P_{0} \) and \( P_{\infty} \) satisfy \( \sigma(P_{0}) = P_{0}, \sigma(P_{\infty}) = P_{\infty}, \rho(P_{0}) = P_{\infty} \) and \( D_{0} \) is defined by

\[
\left\{ \begin{array}{l}
D_{0} = \{ P_{1}, \ldots, P_{g}, P_{g+1}, \ldots, P_{g} \}, \\
P_{g+j} = \sigma \rho(P_{j}) \quad \text{for } j = 1, \ldots, g,
\end{array} \right.
\]

and arbitrary two points of \( D_{0} \) are distinct each other.

3. A complex line bundle \( \hat{\mathcal{L}} = \mathcal{O}_{\hat{C}}(2P_{0} + D_{0}) \) of degree \( (\hat{g} + 2) \) over \( \hat{C} \).

4. \( \pi \) has no branch points over \( \mathcal{S}_{\lambda}^{1} = \{ \lambda \in P^{1}(C) : |\lambda| = 1 \} \) and \( \rho \) fixes each point of \( \pi^{-1}(\mathcal{S}_{\lambda}^{1}) \).

If we define a complex line bundle \( \hat{\mathcal{L}}_{0} \) over \( \hat{C} \) by \( \hat{\mathcal{L}}_{0} = \mathcal{O}_{\hat{C}}(2P_{0} - 2P_{\infty} + D_{0}) \), then we have \( \deg(\hat{\mathcal{L}}_{0}) = \hat{g} \). It is known ([Mc1]) that \( \hat{\mathcal{L}}_{0} \) is non-special, hence \( H^{0}(\hat{C}, \hat{\mathcal{L}}_{0}) \) is of 1-dimension. Therefore, if we express a non-trivial section \( \psi \) of \( \hat{\mathcal{L}}_{0} \) explicitly, then we can write down \( f_{C} : T^{2} \rightarrow S^{5} \) explicitly. The section \( \psi \) is given in terms of Prym-theta function \( \eta \) (cf. [Fa], [MaMa], [HTU]) as follows:

\[
(3.1) \quad \psi(P) = \frac{\eta(B(P) + iUz + iV\bar{z} - e)}{\eta(B(P) - e)} \nu^{-2} \exp \left( \int_{P_{0}}^{P} i\nu \Omega_{\infty} + \int_{P_{\infty}}^{P} i\nu \Omega_{0} \right),
\]

where \( \Omega_{0} \) and \( \Omega_{\infty} \) are normalized ("zero \( \mathcal{A} \)-periods") Abelian differentials of second kind. They satisfy the relations \( \rho^{*}\Omega_{\infty} = \overline{\Omega_{0}}, \sigma^{*}\Omega_{0} = -\Omega_{0}, \sigma^{*}\Omega_{\infty} = -\Omega_{\infty} \) and have the asymptotic behaviors as follows:

\[
\left\{ \begin{array}{l}
\int_{P_{0}}^{P} \Omega_{\infty} = c\nu^{-1} + o(\nu^{-2}) \\
\int_{P_{\infty}}^{P} \Omega_{\infty} = \nu + o(\nu^{-2}) \quad \text{around } P_{\infty}, \\
\int_{P_{0}}^{P} \Omega_{0} = \nu^{-1} + o(\nu^{2}) \\
\int_{P_{\infty}}^{P} \Omega_{0} = c\nu + o(\nu^{2}) \quad \text{around } P_{0},
\end{array} \right.
\]
where $c \in \mathbb{R}$ and $U = (U_1, \cdots, U_g), V = (V_1, \cdots, V_g)$ are $B$-periods given by $U_j = \int_B \Omega_\infty, V_j = \int_B \Omega_0$. Let $e^\omega$ be the one defined by

\begin{equation}
(3.2) \quad e^\omega = 2 \partial_z \partial_{\overline{z}} \log \eta (iUz + iV\overline{z} - e) + c.
\end{equation}

This gives a finite gap solution for Tzitzeica equation, which is the integrability condition for almost complex curve of type (III). The induced metric on $T^2$ may be expressed as $2e^\omega dzd\overline{z}$. That $e^\omega$ is $\mathbb{R}$-valued follows from the relation $\psi(\sigma(P)) = \nu^{-4}\overline{\psi(\rho(P))}$.

If we define $\hat{\psi}_0, \hat{\psi}_1, \hat{\psi}_2$ by

\begin{equation}
(3.3) \quad \hat{\psi}_0 = \psi, \quad \hat{\psi}_1 = \partial_z \psi, \quad \hat{\psi}_2 = i\lambda e^{-\omega} \partial_{\overline{z}} \psi
\end{equation}

then the expression (3.1) of $\psi$ follows from the integrability condition of the system of differential equations formed by $\hat{\psi}_0, \hat{\psi}_1, \hat{\psi}_2$.

Now, we obtain the following theorem.

**Theorem 3.4.** Given the spectral data $(\hat{C}, \hat{L}, \pi)$, we define $\hat{\psi}_0, \hat{\psi}_1, \hat{\psi}_2$ by (3.3). Moreover, define $\hat{S}$ by

\begin{equation}
(3.5) \quad \hat{S} = \frac{1}{\sqrt{3}} \begin{pmatrix}
\hat{\psi}_0(Q_1) & e^{-\frac{i\pi}{6}} \hat{\psi}_1(Q_1) & e^{\frac{i\pi}{6}} \hat{\psi}_2(Q_1) \\
\hat{\psi}_0(Q_2) & e^{-\frac{i\pi}{6}} \hat{\psi}_1(Q_2) & e^{\frac{i\pi}{6}} \hat{\psi}_2(Q_2) \\
\hat{\psi}_0(Q_3) & e^{-\frac{i\pi}{6}} \hat{\psi}_1(Q_3) & e^{\frac{i\pi}{6}} \hat{\psi}_2(Q_3)
\end{pmatrix}.
\end{equation}

Then, $S = \exp(\frac{\pi i}{2} + \frac{2n \pi i}{3} \text{det}(\hat{S}))^{-\frac{1}{3}} \hat{S}, (n = 0, 1, 2)$, gives a Toda-framing for almost complex curve of type (III). Thus, the first column vector of $S$ gives an almost complex curve of type (III), $f : \mathbb{R}^2 \to S_N^5$. Moreover, the necessary condition for $f$ be doubly-periodic is that there are two complex numbers $c_1, c_2$ satisfying $c_1 \neq \overline{c_2}$ such that

$$\text{Re}(c_k U_j), \text{Re}(c_k \int_{P_0}^{Q_l} \Omega_\infty) \in \pi \mathbb{Z} \quad \text{for} \quad j = 1, \cdots, g; k = 1, 2; l = 1, 2, 3.$$ 

**Remark.** There are some overlaps in fundamental concepts of the works between [Mc3] and [HTU]. Although the paper [Mc3] has been already published, our work had been announced in [U].

**References**


