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ASYMPTOTICS OF DRESSED CMC SURFACES

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ABSTRACT. This paper is a description of last section of the paper [KKRS]. In this paper, we show that ends of CMC (constant mean curvature) surfaces, which have Delaunay (CMC surface of revolution) ends, with bubbletons converge to Delaunay surfaces. We recall that a CMC surface with bubbletons is defined by the simple factor dressing matrices which solve the period problems. As examples, we can construct a cylinder with bubbletons, a Delaunay surface with bubbletons (see [Ki], [Ko]) and a n-noid, which have embedded Delaunay ends, with bubbletons (see [KSS]) using this simple factor dressing matrices. Particulary, we prove that ends of these surfaces converge to Delaunay surfaces.

1. INTRODUCTION

The classical Bianchi-Bäcklund transformation transforms a CMC surface into an another CMC surface and can be described as the action of the simple factor on the loop group using integrable systems methods (see [TU]). As a classical example, it is well known that the bubbleton surface is a Bianchi-Bäcklund transformation of a round cylinder ([Sie] and [SW]).

There exist many CMC surfaces with Delaunay ends. Korevaar, Kusner and Solomon (see [KKS]) proved that embedded finite topology ends of CMC surfaces converge to Delaunay surfaces. N-noids with Delaunay ends were first constructed by Kapouleas ([Kap]). Also Grosse-Brauckmann, Kusner and Sullivan classified all (Alexandrov) embedded CMC trinoid surfaces (see [GKS]). Recently Kilian, Schmitt and Sterling (see [KSS]) discussed dressing on CMC trinoids and n-noids which have embedded Delaunay ends. They proved the existence of CMC trinoids and n-nodis with bubbletons using the simple factor dressing matrices which solve the period problem.

In the present paper, we consider the CMC srufaces with Delaunay ends, and cosider the CMC surfaces with bubbletons of these surfaces. Naturally we call the classical example bubbletons as cylinders with bubbletons, and call the examples by [KSS] as n-noids with bubbletons.

We prove that ends of these CMC surfaces with bubbletons converge to Delaunay surfaces.

This paper organized as follows: In Section 2, we give the basic notations and results for DPW (Dorfmeister-Pedit-Wu) method, which constructs constant mean curvature surfaces in all three space forms. In particular we discuss the period problems, and give the explict formula for Delaunay surfaces using the elliptic functions in Theorem 2.3 in all three space forms. In Section 3, we show the main result in the present paper as follows. We define the simple factor dressing matrices using the Terng-Uhlenbeck approach, and define CMC surfaces with bubbletons by restriction of this simple factor dressing matrices to solve the period problems in Definition 3.2. Then we show that ends of these CMC surfaces, which have Delaunay ends, with bubbletons converge to Delaunay surfaces in Theorem 3.3. In Section 4, we apply Theorem 3.3 to concrete objects. We discuss ends of Delaunay surfaces with bubbbletons and Delaunay surfaces with multi-bubbletons using the diagnoal simple factor dressing matrices. We also discuss ends of trinoids with bubbletons and n-noids with bubbletons.

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2. Preliminaries

2.1. **Loop groups.** Let $C_r := \{\lambda \in \mathbb{C} \mid |\lambda| = r\}$ be the radius r circle, and let $D_r := \{\lambda \in \mathbb{C} \mid |\lambda| < r\}$ be the radius r open disk. We denote the closure of D_r by $\overline{D_r} := \{\lambda \in \mathbb{C} \mid |\lambda| \leq r\}$. We also denote $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Definition 2.1. *For any $r \in (0,1] \subset \mathbb{R}$, we define the following loop groups:*

(i) *The twisted* $\mathrm{sl}_2\mathbb{C}$ *r-loop algebra is*

$$\Lambda_r \mathrm{sl}_2\mathbb{C} = \{A : C_r \to^{C^\infty} \mathrm{sl}_2\mathbb{C} \mid A(-\lambda) = \sigma_3 A(\lambda) \sigma_3 \},$$

(ii) *The twisted* $\mathrm{SL}_2\mathbb{C}$ *r-loop group is*

$$\Lambda_r \mathrm{SL}_2\mathbb{C} = \{\Phi : C_r \to^{C^\infty} \mathrm{SL}_2\mathbb{C} \mid \Phi(-\lambda) = \sigma_3 \Phi(\lambda) \sigma_3 \}.$$

(iii) *The twisted* SU_2 *r-loop group is*

$$\Lambda_r \mathrm{SU}_2 = \{F \in \Lambda_r \mathrm{SL}_2\mathbb{C} \mid F(1/\bar{\lambda})^* = (F(\lambda))^{-1},$$
$$F(\lambda) \text{ extends holomorphically to } D_1 \setminus \overline{D_r} \}.$$

When $r = 1$, we abbreviate $\Lambda_1 \mathrm{SU}_2$ to $\Lambda \mathrm{SU}_2$, and in this case the condition that F extends holomorphically to $D_1 \setminus \overline{D_r}$ is vacuous.

(iv) *The twisted plus r-loop group with $\mathbb{R}^+$ constant terms is*

$$\Lambda_{+,r}\mathrm{SL}_2\mathbb{C} = \{B \in \Lambda_r\mathrm{SL}_2\mathbb{C} \mid B(\lambda) \text{ extends holomorphically to } D_r,$$

$$\text{and } B|_{\lambda=0} = \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} \text{ with } \rho > 0\,\} \,.$$

When $r = 1$, we abbreviate $\Lambda_{+,1}\mathrm{SL}_2\mathbb{C}$ to $\Lambda_+\mathrm{SL}_2\mathbb{C}$.

Lemma 2.1. (Iwasawa decomposition) *For any $r \in (0, 1]$, we have the following real-analytic diffeomorphism globally defined from $\Lambda_r\mathrm{SL}_2\mathbb{C}$ to $\Lambda_r\mathrm{SU}_2 \times \Lambda_{+,r}\mathrm{SL}_2\mathbb{C}$ (with respect to the natural smooth manifold structure, as in Chapter 3 of* [PrS]*): For any $\Phi \in \Lambda_r\mathrm{SL}_2\mathbb{C}$, there exist unique $F \in \Lambda_r\mathrm{SU}_2$ and $B \in \Lambda_{+,r}\mathrm{SL}_2\mathbb{C}$ so that*

$$\Phi = FB \,.$$

We call this r-Iwasawa splitting *of Φ. When $r = 1$, we call it simply* Iwasawa splitting. *Because the diffeomorphism is real-analytic, if Φ depends real-analytically (resp. smoothly) on some parameter z, then F and B do as well.*

2.2. **DPW method.** We now describe the DPW method. Let

$$\xi = A(z,\lambda)dz \,, \quad A(z,\lambda) \in \Lambda_r\mathrm{sl}_2\mathbb{C} \,, \tag{2.1}$$

where $A := A(z, \lambda)$ is holomorphic in both z and λ for $z \in \Sigma$ and $\lambda \in \mathbb{C} \setminus \{0\}$. Furthermore, we assume that A has a pole of order at most 1 at $\lambda = 0$, and the upper-right entry of A has a pole of order exactly 1 at $\lambda = 0$. We call ξ a *holomorphic potential.*

Let Φ be the solution to

$$d\Phi = \Phi\xi \,, \qquad \Phi(z_*) = \mathrm{id}$$

for some base point $z_* \in \Sigma$. Then Φ has the same holomorphicity properties as A, and

$$\Phi \in \Lambda_r\mathrm{SL}_2\mathbb{C} \,.$$

By Lemma 2.1 above, we can perform r-Iwasawa splitting, and write the result as

$$\Phi = FB \,. \tag{2.2}$$

Let $F \in \Lambda_r\mathrm{SU}_2$ be above F, and we define Sym-Bobenko formula in three-dimensional space forms as follows:

(2.3)

$$f_{\mathbb{R}^3} = \left.\left[\frac{-i}{2}F\sigma_3F^{-1} - i\lambda(\partial_\lambda F)\cdot F^{-1}\right]\right|_{\lambda=1} \,, \quad N_{\mathbb{R}^3} = \frac{-i}{2}F\sigma_3F^{-1} \,.$$

$$f_{\mathbb{S}^3} = F|_{\lambda=e^{i\gamma_1}}AF^{-1}|_{\lambda=e^{i\gamma_2}} \,, \quad N_{\mathbb{S}^3} = iF_1A\sigma_3F_2^{-1}, \tag{2.4}$$

$$\text{with } A = \begin{pmatrix} e^{\frac{i(\gamma_1-\gamma_2)}{2}} & 0 \\ 0 & e^{\frac{i(\gamma_2-\gamma_1)}{2}} \end{pmatrix} \text{ and } \gamma_1, \gamma_2 \in \mathbb{R}.$$

$$f_{\mathbb{H}^3} = FAF^*|_{\lambda=e^{q/2i\psi}} \quad , \quad N_{\mathbb{H}^3} = FA\sigma_3 F^* \ , \tag{2.5}$$

$$\text{with } A = \begin{pmatrix} e^{q/2} & 0 \\ 0 & e^{-q/2} \end{pmatrix}, \ F^* := \overline{F}^t \text{ and } q, \psi \in \mathbb{R}.$$

Then $f_{\mathbb{R}^3}$ (resp. $f_{\mathbb{S}^3}$, $f_{\mathbb{H}^3}$) is conformal parametrizations of CMC $H = 1/2$ (resp. $H = \cot(\gamma_2 - \gamma_1)$, $H = \coth(-q) > 1$) surfaces in $\mathbb{R}^3$ (resp. $\mathbb{S}^3$, $\mathbb{H}^3$). We call this F as extended frame of CMC surface f.

2.3. **Dressing.** Let $\mathcal{R}$ be a connected Riemann surface and let Φ be a solution to $d\Phi = \Phi\xi$ on $\mathcal{R}$. If we define

$$\hat{\Phi} = h_+ \cdot \Phi \ ,$$

for $h_+ = h_+(\lambda) \in \Lambda_{+,r}\mathrm{SL}_2\mathbb{C}$ depending only on λ, then this multiplication on the left by h_+ is called a dressing.

2.4. **Period problems.** We consider the universal cover $\tilde{\mathcal{R}}$ of $\mathcal{R}$ and let Δ denote the group of deck transformations. For each $\tau \in \Delta$, we define the monodromy matrix M_τ of Φ as $M_\tau(\lambda) := (\Phi \circ \tau) \cdot \Phi^{-1}$. We introduce the following theorem to solve the period problems in $\mathbb{R}^3$, $\mathbb{S}^3$ or $\mathbb{H}^3$, respectively, as in [KKRS].

Theorem 2.2. *Assume $M_\tau \in \Lambda_r\mathrm{SU}_2$ and thus M_τ is also the monodromy matrix of F about τ, where F is as in* (2.2). *Let f be one of the Sym-Bobenko formulas* (2.3), (2.4) *or* (2.5) *for F, respectively. Then*

(i) $\mathbb{R}^3$ *case: $f \circ \tau = f$ holds if and only if*

$$M_\tau|_{\lambda=1} = \pm\,\mathrm{id} \quad \textit{and} \quad \partial_\lambda M_\tau|_{\lambda=1} = 0 \ , \tag{2.6}$$

(ii) $\mathbb{S}^3$ *case: $f \circ \tau = f$ holds if and only if*

$$M_\tau|_{\lambda=e^{i\gamma_1}} = M_\tau|_{\lambda=e^{i\gamma_2}} = \pm\,\mathrm{id} \ , \tag{2.7}$$

(iii) $\mathbb{H}^3$ *case: $f \circ \tau = f$ holds if and only if*

$$M_\tau|_{\lambda=e^{q/2}e^{i\psi}} = \pm\,\mathrm{id} \ . \tag{2.8}$$

2.5. **Delaunay surfaces.** Delaunay surfaces are constant mean curvature surfaces of revolution in $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$ are described via the DPW in detail in [Ki] and [Ko]. And we also give the extended frame for a Delaunay surface in explicit form in Theorem 2.3.

Let $\mathcal{R}$ be $\mathbb{C} \setminus \{0\}$. Define

$$\xi = \frac{D}{z} dz \,, \quad \text{where } D = \begin{pmatrix} l & s\lambda^{-1} + t\lambda \\ s\lambda + t\lambda^{-1} & -l \end{pmatrix} \,, \tag{2.9}$$

with $l, s, t \in \mathbb{R}$. One solution of $d\Phi = \Phi\xi$ is

$$\Phi = \exp\left(\log z \cdot D\right) \,. \tag{2.10}$$

This Φ can be split (this is not r-Iwasawa splitting) in the following way:

$$\Phi = \tilde{F}\tilde{B} \,, \quad \tilde{F} = \exp\left(i\theta \cdot D\right) \,, \quad \tilde{B} = \exp\left(\log\rho \cdot D\right) \,,$$

where $z = \rho e^{i\theta}$, with $\rho, \theta \in \mathbb{R}$. We note that $\tilde{F} \in \Lambda_r \mathrm{SU}_2$.

Since $D^2 = X^2 id$, where $X = \sqrt{l^2 + (s+t)^2 + st(\lambda - \lambda^{-1})^2}$, we see that

$$\tilde{F} = \begin{pmatrix} \cos(\theta X) + ilX^{-1}\sin(\theta X) & iX^{-1}\sin(\theta X)(s\lambda^{-1} + t\lambda) \\ iX^{-1}\sin(\theta X)(s\lambda + t\lambda^{-1}) & \cos(\theta X) - ilX^{-1}\sin(\theta X) \end{pmatrix} \,, \tag{2.11}$$

We can now r-Iwasawa split on $\tilde{B}$, i.e. $\tilde{B} = \hat{F} \cdot B$, where $\hat{F} \in \Lambda_r \mathrm{SU}_2$ and $B \in \Lambda_{r,+}\mathrm{SL}_2\mathbb{C}$. We define $F = \tilde{F} \cdot \hat{F}$. Thus $\Phi = FB$ is the r-Iwasawa splitting of Φ (for any choice of $r \in (0,1]$).

Because, for each fixed λ, $\hat{F}$ and B depend only on ρ and $\tilde{F}$ depends only on θ, we have that, under the rotation of the domain $z \to R_{\theta_0}(z) = e^{i\theta}z$, $\theta_0 \in \mathbb{R}$, the following transformations occur:

$$F \to M_{\theta_0}F \quad \text{and} \quad B \to B \,, \quad \text{where } M_{\theta_0} = \exp(i\theta_0 D) \,.$$

We note that $M_{\theta_0} \in \Lambda_r \mathrm{SU}_2$, and that M_{θ_0} is of the same explicit form as $\tilde{F}$ in (2.11), but evaluated at $\theta = \theta_0$. When $\theta = 2\pi$, $M_\tau = M_{2\pi}$ is the monodromy matrix of the generating counterclockwise deck transformation $\tau \in \Delta$ of the universal cover of $\mathbb{C} \setminus \{0\}$.

Now we consider the closing conditions in each of the three space forms. If we choose l, s and t satisfy the following equation:

$$l^2 + (s+t)^2 + a = 1/4 \,, \tag{2.12}$$

$$\text{with } \begin{cases} \mathbb{R}^3 \text{case: } a = 0 \\ \mathbb{S}^3 \text{ case: } a = -4st\sin^2(\gamma) \ , \gamma = \gamma_1 = -\gamma_2 \in \mathbb{R} \\ \mathbb{H}^3 \text{ case: } a = 4st\sinh^2(q/2) \ , q \in \mathbb{R}^+ \end{cases} .$$

then $M_{2\pi}$ satisfies (2.6) (resp. (2.7), (2.8)) for $\mathbb{R}^3$ case (resp. $\mathbb{S}^3$case, $\mathbb{H}^3$case). With these conditions, Delaunay surfaces are produced in $\mathbb{R}^3$,

$\mathbb{S}^3$ and $\mathbb{H}^3$, and this can be seen as follows: In the case of $\mathbb{R}^3$, under the mapping $z \to R_{\theta_0}(z)$, we have that f as in (2.3) changes as

$$f \to M_{\theta_0} f M_{\theta_0}^{-1} - i(\partial_\lambda M_{\theta_0})|_{\lambda=1} M_{\theta_0}^{-1} \,. \tag{2.13}$$

One can check that Equation (2.13) represents a rotation of angle θ_0 about the line

$$\{x \cdot (-s-t, 0, l) + 2(s-t) \cdot (2l, 0, 2s+2t) \,|\, x \in \mathbb{R}\} \,,$$

hence f is a surface of revolution, and thus a Delaunay surface. Analogous computations imply that the immersions f into $\mathbb{S}^3$ and $\mathbb{H}^3$ are also Delaunay surfaces.

Theorem 2.3 gives the extended frame of the Delaunay surface explicitly in terms of elliptic functions and elliptic integrals, proven in [KKRS] in different notation. We restrict to the case that D defined in Equation (2.9) is off-diagonal (i.e. $l = 0$) in Theorem 2.3.

Theorem 2.3. *Let $\Phi : \mathbb{C} \setminus \{0\} \to \Lambda_r \mathrm{SL}_2\mathbb{C}$ be the solution in Equation* (2.10) *of $d\Phi = \Phi\xi$, where ξ is defined in Equation* (2.9) *with $l = 0$, and where s and t satisfy one of the closing conditions* (2.12). *Then*

(1) the Iwasawa factorization of Φ is $\Phi = FB$, where

$$F = \Phi \exp(-\ell D) B_1^{-1}, \quad B = B_1 \exp(\ell D), \tag{2.14}$$

and the functions $v = v(\rho)$, $\ell = \ell(\rho)$ and the matrices B_0, B_1 satisfy

$$\rho^2 v'^2 = -\left(v^2 - 4s^2\right)\left(v^2 - 4t^2\right), \quad v(1) = 2t, \tag{2.15}$$

where $'$ denotes differentiation with respect to ρ.

$$\ell = \int_1^\rho \frac{2\,d\mathbf{t}}{(1 + (4st\lambda^2)^{-1} v^2(\mathbf{t}))\mathbf{t}},$$

$$B_0 = \begin{pmatrix} 2v(t + s\lambda^2) & -\rho v'\lambda \\ 0 & 4st\lambda^2 + v^2 \end{pmatrix}, \quad B_1 = (\det B_0)^{-1/2} B_0.$$

(2) F is the extended frame for a Delaunay associate family with mean curvature H, $\sqrt{H^2-1}$, $\sqrt{H^2+1}$ respectively in $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$, with conformal factor $v_1^2 = H^{-2}v^2$ and Hopf differential $Q = -2stH^{-1}\lambda^{-1}$.

Remark 2.4. *We can explicitly solve the first Equation in* (2.15) *using elliptic functions. When $0 < s < t$, we have*

$$v(\rho) = \frac{2s}{\mathrm{dn}\left(2t \log \rho, 1 - \frac{s^2}{t^2}\right)} \,, \tag{2.16}$$

where $\mathrm{dn}(2t \log \rho, 1 - \frac{s^2}{t^2})$ is the Jacobi elliptic dn *function.*

3. Asymptotic of dressed CMC surfaces

3.1. **CMC surfaces with bubbletons.** First we define the hermitian projection onto a line $L \in \mathbb{P}$ by $\pi_L : \mathbb{C}^2 \to L$, and define simple factor as follows:

$$\psi_{L,\alpha}(\lambda) = \left(\pi_L + \frac{\lambda^2 - \alpha^2}{1 - \bar{\alpha}^2\lambda^2}\pi_L^{\perp}\right) , \tag{3.1}$$

where for $L = \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} \in \mathbb{P}^1$,

$$\pi_L = \frac{1}{|a_0|^2 + |b_0|^2}\begin{pmatrix} |a_0|^2 & a_0\bar{b}_0 \\ \bar{a}_0 b_0 & |b_0|^2 \end{pmatrix} \tag{3.2}$$

$$\text{and } \pi_L^{\perp} = \frac{1}{|a_0|^2 + |b_0|^2}\begin{pmatrix} |b_0|^2 & -a_0\bar{b}_0 \\ -\bar{a}_0 b_0 & |a_0|^2 \end{pmatrix} . \tag{3.3}$$

We consider the QR decompositon of $\psi_{L,\alpha}(0)$ to obtain elements of $\Lambda_{r,+}\mathrm{SL}_2\mathbb{C}$,

$$(\det \psi_{L,\alpha}(0))^{-1/2}\psi_{L,\alpha}(0) = Q_{L,\alpha}R \tag{3.4}$$

with $Q_{L,\alpha} \in \mathrm{SU}_2, R \in \mathcal{B} = \{\begin{pmatrix} a & c \\ 0 & 1/a \end{pmatrix} | a \in \mathbb{R}_+^*, c \in \mathbb{C}\}$ and restrict to $0 < |\alpha| < 1$. We define the simple factor as follows:

Definition 3.1. *Let $L \in \mathbb{P}^1$ and $\alpha \in \mathbb{C}$ with $0 < |\alpha| < 1$. A simple factor of $\Lambda_{r,+}\mathrm{SL}_2\mathbb{C}$ with $r < |\alpha|$, is a loop the form*

$$h_{L,\alpha} = (det\psi_{L,\alpha}(\lambda))^{-1/2}\Lambda(\lambda)Q_{L,\alpha}^{-1}\psi_{L,\alpha}(\lambda)\Lambda(\lambda)^{-1} , \tag{3.5}$$

$$\textit{with } \Lambda(\lambda) = \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \sqrt{\lambda}^{-1} \end{pmatrix} ,$$

where $\psi_{L,\alpha}$ and $Q_{L,\alpha}$ defined as in (3.1) *respectively* (3.4). *The set $\mathcal{G}_\alpha = \{h_{L,\alpha} | L \in \mathbb{P}^1\}$ can be identified with $\mathbb{P}^1$.*

If we choose $r > |\alpha|$, then $h_{L,\alpha} \in \Lambda_r\mathrm{SU}_2$. Thus simple factor is explicit. From [KSS] and [TU], we have the following theorem.

Theorem 3.1. *Let Σ be a simply connected domain and let $\xi \in \Lambda_r\mathrm{sl}_2\mathbb{C}$ be as in Equation* (2.1). *Let Φ be the solution of $d\Phi = \Phi\xi$ with $\Phi(z_*) = \mathrm{id}$ for some base point $z_* \in \Sigma$, and let $\Phi = FB$ be the r-*Iwasawa splitting, *and let $h_{L,\alpha}$ be a simple factor with α, $r < |\alpha| < 1$ and $L \in \mathbb{P}^1$. Then*

$$h\Phi = hF\tilde{h}^{-1} \cdot \tilde{h}B \in \Lambda_r\mathrm{SU}_2 \times \Lambda_{+,r}\mathrm{SL}_2\mathbb{C}, \tag{3.6}$$

i.e., $hF\tilde{h}^{-1} \in \Lambda_r\mathrm{SU}_2$ and $\tilde{h}B \in \Lambda_{+,r}\mathrm{SL}_2\mathbb{C}$, where $h = h_{L,\alpha}, \tilde{h} = h_{L',\alpha}$ with $L' = \Lambda(\alpha)\overline{F(z,\alpha)}^t\Lambda(\alpha)^{-1}L$.

We introduce the following definition of CMC surfaces with bubbletons.

Definition 3.2. *Let $\mathcal{R}$ be a connected Riemann surface with universal cover $\tilde{\mathcal{R}}$, and let Δ donote the group of deck transformations on this cover. And let F be a extended frame defined on $\tilde{\mathcal{R}}$, and let M_τ be the monodromy matrix of F for each $\tau \in \Delta$, and let h be a simple factor dressing matrix. We consider f (resp. $\tilde{f}$) as the Sym-Bobenko formula defined from F (resp. $hF\tilde{h}^{-1}$). Then $\tilde{f}$* is the CMC surface with bubbletons of f *if*

$$(3.7) \qquad hMh^{-1} \in \Lambda_r \mathrm{SU}_2 \ .$$

Remark 3.2. *Let $\tilde{f}$ be the CMC surface with bubbletons of f, and let Δ be the group of deck transformation. Then*

$$\tilde{f} \circ \tau = \tilde{f}, \ \textit{for all} \ \tau \in \Delta \ .$$

3.2. **Asymptotics of dressed CMC surfaces.** In this subsection, we show that ends of CMC surfaces, which have the Delaunay ends, with bubbletons converge to Delaunay surfaces.

Theorem 3.3. *Let F be the extended frame on a connected Riemann surface $\mathcal{R}$. And we assume $\lim_{z\to 0} \| F - UF_{\mathrm{Del}} \| = 0$, where F_{Del} is the Delaunay extended frame defined from Equation* (2.14) *and $U \in \Lambda_r \mathrm{SU}_2$ depends only λ. And let f_{bubble} be CMC surface with bubbletons Sym-Bobenko formula* (2.3), (2.4) *or* (2.5) *defined using the frame $hF\tilde{h}^{-1}$ in Equation* (3.6), *i.e., we assume that Equation* (3.7) *holds. Then each end of f_{bubble} converges to the Delaunay surface $f_{\mathrm{Del,limit}}$ defined using the frame $hUF_{\mathrm{Del}}\tilde{h}^{-1}_{\mathrm{Del,limit}}$, where entries of $\tilde{h}_{\mathrm{Del,limit}}$ are bounded functions and independent of θ.*

Proof. We show that $\lim_{z\to 0} \| hF\tilde{h}^{-1} - hUF_{\mathrm{Del}}\tilde{h}^{-1}_{\mathrm{Del,limit}} \| = 0$.

$$(3.8) \quad \begin{aligned} \| hF\tilde{h}^{-1} - hUF_{\mathrm{Del}}\tilde{h}^{-1}_{\mathrm{Del,limit}} \| \leq & \| hF\tilde{h}^{-1} - hUF_{\mathrm{Del}}\tilde{h}^{-1} \| \\ & + \| hUF_{\mathrm{Del}}\tilde{h}^{-1} - hUF_{\mathrm{Del}}\tilde{h}^{-1}_{\mathrm{Del}} \| \\ & + \| hUF_{\mathrm{Del}}\tilde{h}^{-1}_{\mathrm{Del}} - hUF_{\mathrm{Del}}\tilde{h}^{-1}_{\mathrm{Del,limit}} \| \quad . \end{aligned}$$

By assumption, $\lim_{z\to 0} \| hF\tilde{h}^{-1} - hUF_{\mathrm{Del}}\tilde{h}^{-1} \| = 0$ and $\lim_{z\to 0} \| hUF_{\mathrm{Del}}\tilde{h}^{-1} - hUF_{\mathrm{Del}}\tilde{h}^{-1}_{\mathrm{Del}} \| = 0$. We show that $\tilde{h}_{\mathrm{Del}}$ converges to $\tilde{h}_{\mathrm{Del,limit}}$. We expand the Delaunay frame F_{Del} of Equation (2.14) to the following matrix form:

$$(3.9) \quad F_{\mathrm{Del}} = \frac{1}{L} \begin{pmatrix} 2v(t+s\lambda^2)C & -\rho v'\lambda C + (4st\lambda^2 + v^2)(s\lambda^{-1} + t\lambda)X^{-1}S \\ 2v(t+s\lambda^2)(s\lambda + t\lambda^{-1})X^{-1}S & -\rho v'\lambda(s\lambda + t\lambda^{-1})X^{-1}S + (4st\lambda^2 + v^2)C \end{pmatrix} \ ,$$

where $L = \sqrt{2v(t+s\lambda^2)(4st\lambda^2+v^2)}$, $C = \cosh(\log\rho + i\theta - \ell)X$, $S = \sinh(\log\rho + i\theta - \ell)X$, $X = \sqrt{1/4 + a + st(\lambda - \lambda^{-1})^2}$, and s, t and a are defined in Equation (2.12). We set
(3.10)

$$F_{\mathrm{Del}}|_{\lambda=\alpha} = \begin{pmatrix} A & B \\ -B^* & A^* \end{pmatrix} \quad , \text{ where } A^* = \overline{A(\bar{\lambda}^{-1})}^t \text{ and } B^* = \overline{B(\bar{\lambda}^{-1})}^t.$$

The upper-left entry of F_{Del} defined from Equation (3.9) has no zero points at $\lambda = \alpha$, so A defined in Equation (3.10) has no zero points. From Equations (3.2) and (3.3), entries of $\psi_{L',\alpha}$ represent in terms of polinomials $B/A, B^*/A$, A^*/A and their conjugations. Thus entries of $\tilde{h} = h_{L',\alpha}$ represent in terms of $B/A, B^*/A$, A^*/A and their conjugations. Lemma 3.4 implies that B/A (resp. $B^*/A, A^*/A$) converges to bounded function P (resp. P_2, P_3). Let $\tilde{h}_{\mathrm{Del,limit}}$ be $\tilde{h}_{\mathrm{Del}}$ with B/A (resp. $B^*/A, A^*/A$) replaced by P (resp. P_2, P_3). Thus $\lim_{z\to 0} \| \tilde{h}_{\mathrm{Del}} - \tilde{h}_{\mathrm{Del,limit}} \| = 0$. We note that $\tilde{h}_{\mathrm{Del,limit}}$ is independ of θ. $\tilde{h}_{\mathrm{Del,limit}}$ is also in $\Lambda_r\mathrm{SU}_2$, and $\partial_\lambda \tilde{h}$ converges to $\partial_\lambda \tilde{h}_{\mathrm{Del,limit}}$, where ∂_λ denotes the derivative with respect to λ. Therefore

$$\lim_{z\to 0} \| f_{\mathrm{bubble}} - f_{\mathrm{Del,limit}} \| = 0 \ .$$

Finally, Lemma 3.5 shows that f_{limit} is a Delaunay surface. □

Lemma 3.4. *Let F_{Del} be the Delaunay frame defined from Equation (3.9), and let A, B, A^* and B^* be the functions defined from Equation (3.10). Then B/A (resp. B^*/A, A^*/A) converges to the following bounded function P (resp. P_2, P_3) as $z \to 0$.*

$$P = \frac{-\rho v'\alpha - (4st\alpha^2+v^2)(s\alpha^{-1}+t\alpha)X_\alpha^{-1}}{2v(t+s\alpha^2)} \ , \tag{3.11}$$

$$P_2 = \frac{2v(t+s\alpha^2)(s\alpha+t\alpha^{-1})X_\alpha^{-1}}{2v(t+s\alpha^2)} \ , \tag{3.12}$$

$$P_3 = \frac{\rho v'\alpha(s\alpha+t\alpha^{-1})X_\alpha^{-1} + 4st\alpha^2+v^2}{2v(t+s\alpha^2)} \ , \tag{3.13}$$

where X_α denotes the value X at $\lambda = \alpha$.

Proof. We show only B/A converges to P. Convergences of B^*/A and A^*/A are the same as convergence of B/A.

$$\begin{aligned}\frac{B}{A} &= \frac{-\rho v'\alpha + (4st\alpha^2+v^2)(s\alpha^{-1}+t\alpha)X_\alpha^{-1}\frac{S_\alpha}{C_\alpha}}{2v(t+s\alpha^2)} \\ &= \frac{-\rho v'\alpha + (4st\alpha^2+v^2)(s\alpha^{-1}+t\alpha)X_\alpha^{-1}\frac{e^{2(\log\rho+i\theta-\ell_\alpha)X_\alpha}-1}{e^{2(\log\rho+i\theta-\ell_\alpha)X_\alpha}+1}}{2v(t+s\alpha^2)} \ ,\end{aligned}$$

where C_α (resp. S_α, ℓ_α) denotes the value C (resp. S, ℓ) at $\lambda = \alpha$. Thus the following claims prove that B/A converges to P:

$$\lim_{z\to 0}(\log\rho + i\theta - \ell_\alpha)X_\alpha = -\infty \ . \tag{3.14}$$

We consider the derivative of $(\log\rho+i\theta-\ell_\alpha)X_\alpha$ with respect to ρ. This function $\partial_\rho(\log\rho + i\theta - \ell_\alpha)X_\alpha$ is always non-negative for all $\rho \in \mathbb{R}^+$. Thus $(\log\rho + i\theta - \ell_\alpha)X_\alpha$ is minus infinite at $z = \rho e^{i\theta} = 0$. □

Lemma 3.5. *$f_{\mathrm{Del,limit}}$ is a Delaunay surface.*

Proof. We show only $\mathbb{R}^3$ case. $\mathbb{S}^3$ and $\mathbb{H}^3$ cases are the same arguments as $\mathbb{R}^3$ case. We consider the translation of the domain

$$z \to e^{i\theta}z \ , \text{with } \ \theta \in \mathbb{R} \ ,$$

Because $\tilde{h}_{\mathrm{Del,limit}}$ depends only on ρ, the following transformations occur:

$$U^{-1}\left(f_{\mathrm{Del,limit}} + i(hU)_\lambda(hU)^{-1}\right)U \to$$
$$[\exp(i\theta D)U^{-1}(f_{\mathrm{Del,limit}} + i(hU)_\lambda(hU)^{-1})U\exp(-i\theta D) - i(\partial_\lambda(\exp(i\theta D)))\exp(-i\theta D)]_{\lambda=1} \ .$$

From Section 2.5, $U^{-1}(f_{\mathrm{Del,limit}} - i(hU)_\lambda(hU)^{-1})U$ is a Delaunay surface, and therefore f_{limit} is also a Delaunay surface. □

4. Examples

In this section, we give the two examples of Theorem 3.3.

4.1. **Asymptotics of Delaunay surfaces with bubbletons.** We consider ξ to be a Delaunay potential as defined in Equation (2.9), and we fix Riemann surface $\mathcal{R}$ to be $\mathbb{C}\setminus\{0\}$. First we introduce the dressing matrix h, the projection matrix $\tilde{h}$ and the matrix $\mathcal{C} \in \Lambda_r\mathrm{SU}_2$ as follows (see also [Ko]):

$$h = \begin{pmatrix} \sqrt{\frac{1-\bar\alpha^2\lambda^2}{\lambda^2-\alpha^2}} & 0 \\ 0 & \sqrt{\frac{\lambda^2-\alpha^2}{1-\bar\alpha^2\lambda^2}} \end{pmatrix} \ , \tag{4.1}$$

$$\tilde{h} = g^{-1/2}\tilde{\pi}_1 + g^{1/2}\tilde{\pi}_2 \ , \tag{4.2}$$

where

$$g = \frac{\lambda^2-\alpha^2}{1-\bar\alpha^2\lambda^2} \ , \tag{4.3}$$

$$\tilde{\pi}_1 = \frac{1}{|A|^2+|\alpha|^{-2}|B|^2}\begin{pmatrix} |A|^2 & \lambda\alpha^{-1}\bar{A}B \\ \lambda^{-1}\bar\alpha^{-1}A\bar{B} & |\alpha|^{-2}|B|^2 \end{pmatrix}$$

$$\text{and } \ \tilde{\pi}_2 = \frac{1}{|A|^2+|\alpha|^{-2}|B|^2}\begin{pmatrix} |\alpha|^{-2}|B|^2 & -\lambda\alpha^{-1}\bar{A}B \\ -\lambda^{-1}\bar\alpha^{-1}A\bar{B} & |A|^2 \end{pmatrix} \ .$$

We define α, A and B as follows:
(4.4)

$$\alpha=\frac{\sqrt{\delta+4}-\sqrt{\delta}}{2}\in\mathbb{R}\cup i\mathbb{R}\backslash\{0,\pm1,\pm i\}\ \text{ with }\ \delta=\frac{1}{st}\left(\frac{k^2-1}{4}+a\right),$$

where $k^2\geq\max\{-16st-4a+1,-4a+1,4\}$ and $k\in\mathbb{N}$, and a is defined in Equation (2.12).

Remark 4.1. *α is real (resp. pure imaginary) when the Delaunay surface is of unduloid type (resp. nodoid type).*

If we consider the case $|\alpha|<r<1$, then the resulting surface by dressing differs from the original surface by only a rigid motion. Therefore we assume $0<r<|\alpha|$. We take α as in Equation (4.4), depending on the weight of a Delaunay surface and the positive integer k, then we apply the dressing h for Φ.

Theorem 4.2. *We choose α as in Equation (4.4), then there exist cylinder bubbleton and Delaunay bubbleton surfaces in all three space forms.*

We prove the asymptotic of Delaunay single-bubbletons using Theorem 3.3.

Corollary 4.3. *Let F be the Delaunay frame defined in Equation (2.14), and let f_{bubble} be the Delaunay bubbleton Sym-Bobenko formula (2.3), (2.4) or (2.5) defined using the frame $hF\tilde{h}^{-1}$ in Theorem 3.1. Then each end of f_{bubble} converges to the Delaunay surface f_{limit} defined using the frame $hF\tilde{h}^{-1}_{\text{limit}}$, where $\tilde{h}^{-1}_{\text{limit}}$ is defined as in Equation (4.5).*

Proof. We show that $\tilde{h}^{-1}$ converges to the following matrix $\tilde{h}^{-1}_{\text{limit}}$, whose entries are bounded functions independent of θ:
(4.5)

$$\tilde{h}^{-1}_{\text{limit}}=\frac{1}{1+|\alpha|^{-2}|B|^2|P|^2}\begin{pmatrix}g^{1/2}+g^{-1/2}|\alpha|^{-2}|P|^2 & \lambda\alpha^{-1}P(g^{1/2}-g^{-1/2})\\ \lambda^{-1}\bar{\alpha}^{-1}\bar{P}(g^{1/2}-g^{-1/2}) & g^{1/2}|\alpha|^{-2}|P|^2+g^{-1/2}\end{pmatrix},$$

where P is the bounded function defined in Equation (3.11). □

4.2. **Asymptotic of trinoids with bubbletons in $\mathbb{R}^3$.** From [Sc1] and [KSS], we introduce the family of trinoid potentials in $\mathbb{R}^3$.

Definition 4.1. *The family of trinoid potentials $\mathcal{T}_{\text{trinoid}}$ is the set of potentials ξ_{w_0,w_1,w_∞} defined as follows. Let $w_0,w_1,w_\infty\in(-\infty,1]\setminus\{0\}$, let $n_k=1/2\sqrt{1-w_k}\in(-\infty,1/2]\setminus\{0\}$ $(k\in\{0,1,\infty\})$ and suppose the following inequalities are satisfied:*

$$|n_0|+|n_1|+|n_\infty|\leq 1 \qquad (4.6)$$
$$|n_i|\leq|n_j|+|n_k|,\quad \{i,j,k\}=\{0,1,\infty\}$$

$$|w_i|\leq|w_j|+|w_k|,\quad \{i,j,k\}=\{0,1,\infty\}. \qquad (4.7)$$

Let $\Sigma = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ *and let* $\xi_{w_0,w_1,w_\infty} \in \Lambda \mathrm{sl}_2\mathbb{C} \otimes dz$ *be defined by*

$$\xi_{w_0,w_1,w_\infty} = \begin{pmatrix} 0 & \lambda^{-1} \\ \frac{w_\infty z^2 - (w_0 - w_1 + w_\infty)z + w_0}{16z^2(z-1)^2}(1-\lambda)^2 & 0 \end{pmatrix} dz \ . \tag{4.8}$$

Theorem 4.4. *Let* $\mathcal{R}$ *be* $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, *and let* $\tilde{\mathcal{R}}$ *be its universal cover, and let* Δ *be the group of deck transformations for this cover. Let* $\xi_{w_0,w_1,w_\infty} \in \mathcal{T}_{\mathrm{trinoid}}$ *(Definition 4.1), and let* Φ *be a solution of* $d\Phi = \Phi\xi_{w_0,w_1,w_\infty}$ *with some initial condition* $\Phi(z_*, \lambda) \in \Lambda_r \mathrm{SL}_2\mathbb{C}$ *at* z_*. *Then there exists an analytic map* $h_{\mathrm{close}} : \{0 \le |\lambda| < 1\} \to \mathrm{GL}_2\mathbb{C}$ *such that for any* $r \in (0, 1)$, *the CMC immersion* f *defined from the frame in Equation* (4.9) *satisfies* $\tau^* f = f$ *for all* $\tau \in \Delta$ *and has three asymptotically Delaunay ends with weights* w_0, w_1, w_∞.

For later use, we define the Iwasawa splitting of $h_{\mathrm{close}} \cdot \Phi$ is as follows:

$$h_{\mathrm{close}} \cdot \Phi = F_{\mathrm{tri}} \cdot B_{\mathrm{tri}} \ . \tag{4.9}$$

The following Lemma by [Scl] implies that we can assume UF as the dressing Delaunay frame by the dressing matrix C.

Lemma 4.5. *Let* Φ *be the solution in Equation* (2.10) *of* $d\Phi = \Phi\xi$, *and let* f *be the Sym-Bobenko formula defined from extended frame* F *in Equation* (2.14). *Let* $C \in \Lambda_r \mathrm{SL}_2\mathbb{C}$, *and suppose that* C *is the boundary of an analytic map* $C : \{r < |\lambda| < 1 + \epsilon\} \to M_{2\times 2}$ *for some* $\epsilon \in \mathbb{R}^+$ *such that* $\{\det C = 0\} \subset \mathbb{S}^1$. *Suppose that* $C \exp(2\pi i D) C^{-1}$ *satisfies the reality condition on* $\mathbb{S}^1 \setminus \{\det C = 0\}$. *Let* $C\Phi = \tilde{F}\tilde{B}$ *be the r-Iwasawa splitting of* $C\Phi$, *and let* $\tilde{f}$ *be the Sym-Bobenko formula defined from extended frame* $\tilde{F}$. *Then*

(i) *There exists* $\tilde{D}$ *of the form* (2.9), $U_0 \in \Lambda_r \mathrm{SU}_2$ *and* $C_+ \in \Lambda_{r,+}\mathrm{GL}_2\mathbb{C}$ *such that* $C\Phi = U_0 \exp(\log z D) C_+$.
(ii) *Then there exists* $c \in \mathbb{R}^+$ *and an isometry* T *of* su_2 *such that* $\tilde{f}(z) = T(f(cz))$.

¿From Theorem 3.6 in [KSS], we define the particular class of trinoid potentials and have the following closed dressing trinoid surface.

Definition 4.2. *We define* μ_{w_k} *as follows:*

$$\mu_{w_k} = \frac{1}{2}\sqrt{1 + w_k \frac{(1-\lambda)^2}{4\lambda}}, \tag{4.10}$$

where $w_k \subset (-\infty, 1] \setminus \{0\}$. *And we define the set*

$$\Lambda_{w_0,w_1,w_\infty} = \{\lambda \in \{0 < |\lambda| < 1\} | \tfrac{1}{2} \pm \mu_0 \pm \mu_1 \pm \mu_\infty \in \mathbb{Z}^{\le 0}\}, \tag{4.11}$$

i. e., $\frac{1}{2} \pm \mu_0 \pm \mu_1 \pm \mu_\infty \in \mathbb{Z}^{\le 0}$ *holds for some choice of signs.*

Theorem 4.6. *Let $\tilde{\mathcal{R}}, \mathcal{R}$ and Δ be as in Theorem 4.4 and let ξ_{w_0,w_1,w_∞} be a trinoid potential. We choose Φ and h_{close} are as in Theorem 4.4. Let $\Lambda_{w_0,w_1,w_\infty}$ be as in Definition 4.2. Then*

(i) *For every $\alpha \in \Lambda_{w_0,w_1,w_\infty}$, the monodromy $\mathcal{M}_{h_{\text{close}}\Phi}(\alpha)$ is reducible. Hence there exists a simple factor matrix $h \in \mathcal{G}_\alpha$ such that for any $r \in (0, |\alpha|)$, the CMC immersion f satisfies $\tau^* f = f$ for all $\tau \in \Delta$.*

(ii) *If $\mu = \mu_0 = \mu_1 = \mu_\infty \in \frac{1}{2} + \mathbb{Z}$, then $\mathcal{M}_{h_{\text{close}}\Phi}(\alpha)$ takes values in $\{\pm \mathrm{I}\}$. Hence for every simple factor matrix $h \in \mathcal{G}_\alpha$, for any $r \in (0, |\alpha|)$, the CMC immersion f satisfies $\tau^* f = f$ for all $\tau \in \Delta$.*

We apply Theorem 3.3, then we have the following asymptotics of a trinoid with bubbletons in $\mathbb{R}^3$.

Corollary 4.7. *Let h_{close} and Φ be as in Theorem 4.6, and let F_{tri} be the extended frame defined in Equation (4.9), and let $f_{\text{tri,bubble}}$ be it's bubbleton Sym-Bobenko formula defined using the frame $hF_{\text{tri}}\tilde{h}_{\text{tri}}^{-1}$ in Equation (3.5) and Theorem 4.6. Then each end of $f_{\text{tri,bubble}}$ converges to the Delaunay surface.*

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