Title: Approximation of a Reaction-Diffusion Equation with a Nonlocal Term (Variational Problems and Related Topics)

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Approximation of a Reaction-Diffusion Equation with a Nonlocal Term

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1 Introduction.

We consider a scalar bistable reaction-diffusion equation

\[ \epsilon u_t = \epsilon^2 \Delta u + f(u) - v, \quad t > 0, \ x \in \Omega, \]

under the Neumann boundary condition

\[ \frac{\partial u}{\partial n} = 0, \quad t > 0, \ x \in \partial \Omega. \]

Here \( u \) is an order parameter while \( v \) an additional parameter (acting as inhibitors). \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^N \ (N \geq 2) \) and \( n \) stands for the outward unit normal vector on the boundary \( \partial \Omega \). The nonlinear term \( f \) is assumed to be the negative derivative of a smooth double-well potential \( W: f(u) = -W'(u) \). A typical example is \( f(u) = u - u^3 \).

The parameter \( \epsilon > 0 \) is supposed to be very small, and we intend to study the problem above as the singular perturbation problem.

We will treat in this paper a situation in which the spacial average of the order parameter is preserved:

\[ \frac{1}{|\Omega|} \int_{\Omega} u(t, x) \, dx \equiv m \quad (\text{constant}), \quad t \geq 0, \]

i.e., a case where \( v \) in (RD) is given by

\[ v(\cdot) = \frac{1}{|\Omega|} \int_{\Omega} f(u(\cdot, x)) \, dx. \]

When \( \epsilon > 0 \) is very small, the solution \( u(t, x) \) of (RD) with an appropreate initial condition creates a sharp transition layer with width of \( O(\epsilon) \) and it is expected to move according to some motion laws, called interface equations. Our purpose of this paper is (1) to derive interface equations from (RD); and (2) to investigate how solutions of interface equations evolve.

Remark 1. From a variational point of view, the equation (RD) is characterized as the \( L^2(\Omega) \)-gradient system for the energy functional of van der Waals type

\[ E^\epsilon(u) := \int_{\Omega} \frac{\epsilon}{2} |\nabla u|^2 + \frac{1}{\epsilon} W(u) \, dx \]

subject to the constraint (PP), and the nonlocal term \( v \) is regarded as the Lagrange multiplier (see [2] for example).
2 Derivation of interface equations.

Throughout the remaining part of this paper, an interface is meant to be a smooth, closed, $N - 1$ dimensional hypersurface embedded in $\Omega \subset \mathbb{R}^N$. We will derive some interface equations from (RD) by the method of matched asymptotic expansions (see [9] for more details).

2.1 Preliminaries.

We now present precise assumptions on $f$ and prepare some notations for our problem.

(A1) The function $f$ is $C^\infty$ on $\mathbb{R}$ and the curve $f(u) - v = 0$ consists of three sub-branches of solutions

$$ C^- = \{(u, v) | u = h^-(v), v \in I^- := (\underline{v}, \infty)\}, $$

$$ C^+ = \{(u, v) | u = h^+(v), v \in I^+ := (-\infty, \overline{v})\}, $$

and

$$ C^0 = \{(u, v) | u = h^0(v), v \in I^0 := I^- \cap I^+ = (\underline{v}, \overline{v})\}, $$

satisfying $f'(h^\pm(v)) < 0$ (or equivalently $h^\pm_U(v) < 0$) on $I^\pm$.

(A2) For each $v \in I^0$, it holds that $h^-(v) < h^0(v) < h^+(v)$.

(A3) For each $v \in I^0$, we define

$$ S(v) := \int_{h^-(v)}^{h^+(v)} f(u) - v \, du. $$

Then there exists a unique point $v^* \in I^0$ such that $S(v^*) = 0$ and $S'(v^*) < 0$.

Remark 2. We may regard the point $(h^0(v^*), v^*)$ as the origin $(0, 0)$ by appropriate translations.
An unknown interface $\Gamma(t)$, which is to be determined, is expressed as a smooth embedding from a fixed $N - 1$ dimensional reference manifold $\mathcal{M}$ to $\mathbb{R}^N$:

\begin{equation}
(2.1) \quad \gamma(t, \cdot) : \mathcal{M} \to \Gamma(t) \subset \Omega, \quad \mathcal{M} \ni y \mapsto x = \gamma(t, y) \in \Gamma(t).
\end{equation}

Let $\Omega^\pm(t)$ be subregions (called bulk regions) in $\Omega$ decomposed by $\Gamma(t)$ such as

$$\Omega = \Omega^-(t) \cup \Gamma(t) \cup \Omega^+(t),$$

and $\nu(t, y) \in \mathbb{R}^N$ the unit normal vector on $\Gamma(t)$ at $x = \gamma(t, y)$ pointing into the interior of the bulk region $\Omega^+(t)$. In advance we standardize the parametrization as in (2.1) in such a way that $\gamma(t, y)$ is always parallel to $\nu(t, y)$ [3]. For sufficiently small $\delta > 0$, a point $x$ in a neighborhood $\{x \in \Omega \mid \text{dist}(x; \Gamma(t)) < \delta\}$ is uniquely represented as

\begin{equation}
(2.2) \quad x = \gamma(t, y) + r\nu(t, y),
\end{equation}

which gives us a new coordinate system $(t, r, y)$. We denote by $J(t, r, y)$ Jacobian associated with (2.2). Namely,

$$J(t, r, y) = \prod_{i=1}^{N-1} (1 + r\kappa_i(t, y)) = 1 + \sum_{i=1}^{N-1} H_i(t, y)r^i,$$

where $\kappa_i(t, y) (i = 1, \cdots, N - 1)$ stand for the principal curvatures of $\Gamma(t)$ at $x = \gamma(t, y)$.

Let $u^\epsilon$ be a solution of (RD) for an appropriate initial condition:

\begin{equation}
(2.3) \quad \epsilon u_t^\epsilon(t, x) = \epsilon^2 \Delta u^\epsilon(t, x) + f(u^\epsilon(t, x)) - v^\epsilon(t), \quad t > 0, \ x \in \Omega,
\end{equation}

\begin{equation}
(2.4) \quad v^\epsilon(t) = \frac{1}{|\Omega|} \int_{\Omega} f(u^\epsilon(t, x)) \, dx, \quad t > 0.
\end{equation}

We define an interface $\Gamma^\epsilon(t)$ as a level set of the solution $u^\epsilon$ to (RD). Since transition layers are expected to develop in regions $\{x \in \Omega \mid u^\epsilon(t, x) \approx h^0(v^\star)\}$, we set (cf. Remark 2)

\begin{equation}
(2.5) \quad \Gamma^\epsilon(t) := \{x \in \Omega \mid u^\epsilon(t, x) = 0\}.
\end{equation}

On the other hand, $\Gamma^\epsilon(t)$ is also assumed to be expressed as a graph of a smooth function over the interface $\Gamma(t)$:

\begin{equation}
(2.6) \quad \Gamma^\epsilon(t) = \{x \in \Omega \mid x = \gamma(t, y) + \epsilon R^\epsilon(t, y)\nu(t, y), \ y \in \mathcal{M}\}.
\end{equation}

$R^\epsilon$, of course, is a priori unknown and is to be determined.

### 2.2 Outer expansion.

We separate the whole domain $\Omega$ into two components $\Omega^\epsilon \pm(t)$ by the interface $\Gamma^\epsilon(t)$ such as $\Omega = \Omega^\epsilon -, (t) \cup \Gamma^\epsilon(t) \cup \Omega^\epsilon + (t)$, and substitute the formal expansions

\begin{equation}
(2.7) \quad U^\epsilon(t, x) = U^\epsilon \pm(t, x) = \sum_{j \geq 0} \epsilon^j U^\epsilon j \pm(t, x), \quad v^\epsilon(t) = \sum_{j \geq 0} \epsilon^j v^j(t)
\end{equation}
into (2.3) in order to see the profile of solutions away from layer regions. Equating to zero the coefficient of each power of \( \epsilon \) in the resulting equation, we obtain the following series of equations:

\[
(2.8) \quad f(U^0_{\pm}) - v^0 = 0, \\
(2.9) \quad f'(U^0_{\pm})U^{j,\pm}_0 = v^j + F^\pm_j, \quad j \geq 1.
\]

Here \( F^\pm_j \) stand for functions depending on \( U^k_{\pm} (0 \leq k < j) \) only.

As the solution of (2.8), noting that (A1), we choose

\[
(2.10) \quad U^0_{\pm}(t, x) := h^\pm(v^0(t)).
\]

Once we make this choice, \( U^{j,\pm} (j \geq 1) \) can be successively expressed by (2.9) as

\[
(2.11) \quad U^{j,\pm}(t, x) = h^\pm(v^0(t))v^j(t) + V^\pm_j(t)
\]

with \( V^\pm_j \) being some functions depending on \( v^k (0 \leq k < j) \). Therefore once \( v^j \) is known, \( U^{j,\pm} \) are determined completely. \( v^j (j \geq 0) \) will be determined later so that the \( C^1 \)-matching conditions are satisfied (cf. subsection 2.5). We note, in particular, that the outer solution \( U^\epsilon(t, x) \) is independent of \( x \), and therefore is denoted simply as \( U^\epsilon(t) \) in the sequel.

### 2.3 Inner expansion.

To deal with layer phenomena near \( r = \epsilon R^\epsilon(t, y) \) (cf. (2.2), (2.6)), we use a stretched variable \( z := \epsilon^{-1}[r - \epsilon R^\epsilon(t, y)] \) and recast our problem (2.3) in terms of \( (t, z, y) \):

\[
(2.12) \quad \tilde{u}_z^\epsilon + (\gamma_t \cdot \nu)\tilde{u}_z^\epsilon + f(\tilde{u}^\epsilon) + \epsilon R^\epsilon \tilde{u}_z^\epsilon - v^\epsilon + D^\epsilon \tilde{u}^\epsilon = 0, \quad z \in (-\delta/\epsilon - R^\epsilon, \delta/\epsilon - R^\epsilon),
\]

where \( D^\epsilon \) stands for a differential operator including \( R^\epsilon \).

We will seek an asymptotic solution to (2.12) of the form

\[
(2.13) \quad \tilde{u}^\epsilon(t, z, y) = U^\epsilon_j(t, z, y)|_{x=\gamma(t,y)+\epsilon R^\epsilon(t,y)} + \phi^\epsilon_j(t, z, y) = U^\epsilon(t) + \phi^\epsilon(t, z, y),
\]

i.e., we will determine \( \phi^\epsilon \) in such a way that \( \tilde{u}^\epsilon \) in (2.13) asymptotically satisfies (2.12) for \( z \in (-\infty, \infty) \). We substitute the formal expansions

\[
(2.14) \quad R^\epsilon(t, y) = R^1(t, y) + \epsilon R^2(t, y) + \epsilon^2 R^3(t, y) + \cdots, \\
(2.15) \quad \tilde{u}^\epsilon(t, z, y) = \tilde{u}^{j,\pm}(t, z, y) = U^{j,\pm}_\epsilon(t) + \phi^{j,\pm}(t, z, y)
\]

\[
= \sum_{j \geq 0} \epsilon^j U^{j,\pm}_\epsilon(t) + \sum_{j \geq 0} \epsilon^j \phi^{j,\pm}(t, z, y) =: \sum_{j \geq 0} \epsilon^j \tilde{u}^{j,\pm}(t, z, y)
\]

together with the expansion for \( v^\epsilon \) into (2.12) to obtain some series of equations for \( \tilde{u}^{j,\pm} \) and \( \phi^{j,\pm} \) in \( \pm z \in (0, \infty) \). We now exhibit equations for \( \tilde{u}^{j,\pm} \) only:

\[
(2.16) \quad \tilde{u}^{0,\pm}_{zz} + (\gamma_t \cdot \nu)\tilde{u}^{0,\pm}_z + f(\tilde{u}^{0,\pm}) - v^0 = 0, \\
(2.17) \quad \tilde{u}^{j,\pm}_{zz} + (\gamma_t \cdot \nu)\tilde{u}^{j,\pm}_z + f'(\tilde{u}^{0,\pm})\tilde{u}^{j,\pm} = v^j - R^\epsilon \tilde{u}^{0,\pm}_z + F^\pm_j, \quad j \geq 1.
\]

Here \( F^\pm_j \) stand for functions depending on \( R^k, v^k \), \( \tilde{u}^{k,\pm} (0 \leq k < j) \) with \( R^0 := \gamma \).

We impose the following conditions:
2.4 Expansion of nonlocal term.

(2.4) is recast as follows:

\[
\dot{U}^{\epsilon,-}|\Omega^-| + \dot{U}^{\epsilon,+}|\Omega^+| = (\dot{U}^{\epsilon,+} - \dot{U}^{\epsilon,-}) \sum_{i \geq 0} \int_{\mathcal{M}} \frac{H_i(t, y)}{i+1} \left( \epsilon R^\epsilon(t, y) \right)^{i+1} dS_y^\gamma(t, \cdot),
\]

\[
+ \int_{\mathcal{M}} \int_{-\infty}^{0} \left[ \phi^{0,-}_z + (\gamma_t \cdot \nu) \phi^{0,-}_z + \epsilon R^\epsilon(t, y) \right] dz \int_{\mathcal{M}} \int_{0}^{\infty} \left[ \phi^{0,+}_z + (\gamma_t \cdot \nu) \phi^{0,+}_z \right] \wedge \kappa R^j dz dS_y + O(\epsilon^{-1} \epsilon^{-\delta/\epsilon}).
\]

Here \( J^\epsilon(t, z, y) := J(t, r, y)|_{r = \epsilon z + \epsilon R^\epsilon(t, y)} \) and \( dS_y^\gamma(t, \cdot) \) stands for the volume element on \( \mathcal{M} \) induced from \( dS_x^\gamma(t) \), the surface element on \( \Gamma(t) \) at \( x \), by the embedding \( \gamma(t, \cdot) \). These are denoted simply as \( dS_x \) and \( dS_y \) in the sequel.

We substitute the outer and inner expansions into (2.21) to obtain some series of equations:

\[
(2.22) \quad \dot{U}^{0,-}|\Omega^-| + \dot{U}^{0,+}|\Omega^+| = \int_{\mathcal{M}} \int_{-\infty}^{0} \left[ \phi^{0,-}_z + (\gamma_t \cdot \nu) \phi^{0,-}_z \right] dz dS_y,
\]

\[
(2.23) \quad \dot{U}^{j,-}|\Omega^-| + \dot{U}^{j,+}|\Omega^+| = (\dot{U}^{0,+} - \dot{U}^{0,-}) \int_{\mathcal{M}} R^j dS_y
\]

\[
+ \int_{\mathcal{M}} \int_{-\infty}^{0} \left[ \phi^{0,-}_z + (\gamma_t \cdot \nu) \phi^{0,-}_z \right] \kappa R^j dz dS_y
\]

\[
+ \int_{\mathcal{M}} \int_{0}^{\infty} \left[ \phi^{0,+}_z + (\gamma_t \cdot \nu) \phi^{0,+}_z \right] \kappa R^j dz dS_y
\]

\[
+ \int_{\mathcal{M}} \int_{-\infty}^{0} \left[ \phi^{0,-}_z + (\gamma_t \cdot \nu) \phi^{0,-}_z + R^j(t, y) \phi^{0,+}_z \right] dz dS_y
\]

\[
+ \int_{\mathcal{M}} \int_{0}^{\infty} \left[ \phi^{0,+}_z + (\gamma_t \cdot \nu) \phi^{0,+}_z + R^j(t, y) \phi^{0,+}_z \right] dz dS_y + I_j, \quad j \geq 1.
\]
Here \( \kappa := \kappa_1 + \cdots + \kappa_{N-1} \) is the mean curvature of \( \Gamma(t) \) at \( x = \gamma(t, y) \), and \( I^j(t) \) stands for a function calculated by using functions \( R^k, U^k,\pm \) and \( \phi^k,\pm \) (\( 0 \leq k < j \)).

2.5 \( C^1 \)-matching.

We note that the following problem

\[
\begin{align*}
Q_{zz} + c Q_z + f(Q) - v &= 0, \quad z \in (-\infty, \infty), \\
Q(\pm \infty) &= \hat{h}^\pm(v), \quad Q(0) = 0,
\end{align*}
\]

(2.24)

has a unique solution pair \( (Q(z; v), c(v)) \) for each \( v \in I^0 \). Then (2.16) with (2.18)-(2.20) have unique solutions if and only if

\[
\gamma_t(t, y) \nu(t, y) = c(v^0(t)) \\
\hat{u}^0(t, z, y) = Q(z; v^0(t)), \quad \pm z \in (0, \infty).
\]

(2.25)

(2.26)

Once (2.25) is satisfied and we have (2.26), we can successively show the existence and uniqueness of \( \phi^{j,\pm} \) satisfying (2.19) for all \( j \geq 0 \).

As for \( \tilde{u}^j,\pm \) (\( j \geq 1 \)), equations (2.17) with (2.18)-(2.20) have unique solutions if and only if a solvability condition of (2.17)

\[
\int_{-\infty}^{\infty} e^{cz} Q_z(v) - R_i^j Q_z + F_j dz = 0
\]

is satisfied, which is equivalent to

\[
R_i^j(t, y) = c'(v^0(t)) \nu^i(t) + \rho_j(t, y)
\]

(2.27)

with \( \rho_j \) being a function calculated by using \( R^k, v^k \) and \( \tilde{u}^k \) (\( 0 \leq k < j \)). For instance, \( \rho_1 \) is given by

\[
\rho_1 = -\kappa + \frac{\int_{-\infty}^{\infty} e^{c(v^0(z))} Q_z(z; v^0) Q_v(z; v^0) dz}{\int_{-\infty}^{\infty} e^{c(v^0(z))} [Q_z(z; v^0)]^2 dz} v^0.
\]

(2.28)

On the other hand, (2.22) and (2.23) with (2.18)-(2.20) respectively yield

\[
\dot{v}^0(t) = \frac{h^+(v^0(t)) - h^-(v^0(t))}{h_v^-(v^0(t)) \Omega^- - h_v^+(v^0(t)) \Omega^+} c(v^0(t)) \Gamma(t),
\]

(2.29)

\[
\dot{v}^j(t) = \int_M a(t, y) R^j(t, y) dS_y + b(t) v^j(t) + \sigma_j(t).
\]

(2.30)

Here \( a \) and \( b \) are some functions depending only on \( (\Gamma, v^0) \) given by

\[
a := \frac{[h^+(v^0) - h^-(v^0)] c(v^0) \kappa + [h^+_v(v^0) - h^-_v(v^0)] \dot{v}^0}{h_v^-(v^0) \Omega^- + h_v^+(v^0) \Omega^+},
\]

(2.31)
\[ b := \frac{h^+(v^0) - h^-(v^0)}{h_v^-(v^0)|\Omega^-| + h_v^+(v^0)|\Omega^+|} c'(v^0) |\Gamma| \]

\begin{equation}
(2.32) + \frac{(h_v^+(v^0) - h_v^-(v^0)) c(v^0) |\Gamma|}{h_v^-(v^0)|\Omega^-| + h_v^+(v^0)|\Omega^+|} \frac{(h_{uv}^-(v^0)|\Omega^-| + h_{vU}^+(v^0)|\Omega^+|)}{c'(v^0) |\Gamma|},
\end{equation}

while \( \sigma_j \) stands for a function computed by employing \( R^k, v^k \) and \( \phi^{k,\pm} (0 \leq k < j) \). For instance, \( \sigma_1 \) is given by

\[
\sigma_1 = - \frac{h^+(u^0) - h^-(v^0)}{h_v^-(v^0)|\Omega^-| + h_v^+(v^0)|\Omega^+|} \int_{\mathcal{M}} \kappa dS_y \]

\[
+ \left[ h_v^-(v^0)|\Omega^-| + h_v^+(v^0)|\Omega^+| \right]^{-1} \times \left[ \int_{-\infty}^\infty zQ_z(z;v^0) dz \right] \int_{\mathcal{M}} \kappa dS_y
\]

\[
- \frac{d}{dt} \left( h_v^-(v^0)\dot{\nu}^0 \right) |\Omega^-| - \frac{d}{dt} \left( h_v^+(v^0)\dot{\nu}^0 \right) |\Omega^+| \]

\[
- \left( \int_{-\infty}^0 (Q_v(z;v^0) - h_v^-(v^0)) dz + \int_0^\infty (Q_v(z;v^0) - h_v^+(v^0)) dz \right) \dot{v}^0 |\Gamma|
\]

\[
+ \left( h_v^+(v^0)^2 - h_v^-(v^0)^2 \right) c(v^0) (\dot{v}^0)^2 |\Gamma|
\]

\[
+ \left( h^+(v^0) - h^-(v^0) \right) \int_{-\infty}^\infty e^{c(v^0)z} Q_z(z;v^0)Q_v(z;v^0) dz \int_{-\infty}^\infty e^{c(v^0)z} [Q_z(z;v^0)]^2 dz \dot{v}^0 |\Gamma|.
\]

We finally arrived at the following interface equations:

\[ (\text{IE}^0) \quad \gamma \cdot \nu = c(v^0), \quad \dot{v}^0 = \frac{h^+(v^0) - h^-(v^0)}{h_v^-(v^0)|\Omega^-| + h_v^+(v^0)|\Omega^+|} c(v^0) |\Gamma|, \]

\[ (\text{IE}^j) \quad R^j_i = c'(v^0) v^j + \rho_j, \quad \dot{v}^j = \int_{\mathcal{M}} a R^j dS_y + b v^j + \sigma_j, \quad j \geq 1. \]

3 Analysis of interface equations.

We are now ready to study the interface equations. Let us begin with the 0-th order equation \((\text{IE}^0)\).

3.1 0-th order equation.

The equation is as follows:

\[ (\text{IE}^0-a) \quad v(x; \Gamma(t)) = c(v(t)), \quad t > 0, \quad x \in \Gamma(t), \]

\[ (\text{IE}^0-b) \quad \dot{v}(t) = \frac{h^+(v(t)) - h^-(v(t))}{h_v^-(v(t))|\Omega^-| + h_v^+(v(t))|\Omega^+|} c(v(t)) |\Gamma(t)|, \quad t > 0, \]

\[ (\text{IE}^0-c) \quad \Gamma(0) = \Gamma_0, \quad v(0) = v_0. \]
Here \( v(x; \Gamma(t)) := \gamma(t, y) \cdot \nu(t, y) \) is the normal velocity of \( \Gamma(t) \) at \( x = \gamma(t, y) \). We note that the superscript '0' in \( v^0(t) \) has been suppressed.

It immediately turns out, due to \((\text{IE}^0\text{-a})\), that the normal speed is independent of the position \( x \in \Gamma(t) \) and is regulated by the (0-th order) nonlocal term \( v \). Thanks to the identity

\[
\frac{d}{dt} |\Omega^-(t)| = -\frac{d}{dt} |\Omega^+(t)| = \int_{\Gamma(t)} v(x; \Gamma(t)) dS_x,
\]

the interface equation \((\text{IE}^0)\) implies

\[
h^-(v(t)) \frac{|\Omega^-(t)|}{|\Omega|} + h^+(v(t)) \frac{|\Omega^+(t)|}{|\Omega|} \equiv m_0, \quad t \geq 0,
\]

where \( m_0 = m_0(\Gamma_0, v_0) \) is given by

\[
m_0 := h^-(v_0) \frac{|\Omega_0^-|}{|\Omega|} + h^+(v_0) \frac{|\Omega_0^+|}{|\Omega|}
\]

with \( \Omega_0^\pm \) being initial bulk regions such that \( \Omega = \Omega_0^- \cup \Gamma_0 \cup \Omega_0^+ \). We note that (3.2) corresponds to \((\text{PP})\) for \((\text{RD})\) as \( \epsilon \to 0 \) (cf. (2.10)).

We recast \((\text{IE}^0)\) as a system of ordinary differential equations after the manner of Sakamoto [11]. For a given initial interface \( \Gamma_0 \) we express \( \Gamma(t) \) as the graph of a function \( r(t, y) \) over \( \Gamma_0 \): \( \gamma(t, y) = \gamma(0, y) + r(t, y) \nu(0, y) \). Then some elementary calculations yield \( \nu(t, y) \equiv \nu(0, y) \) and \( r(t, y) \equiv r(t) \), and therefore \((\text{IE}^0\text{-a})\) is recast as \( \dot{r}(t) = c(v(t)) \). On the other hand, the surface area of an interface \( \{ x \in \Omega | x = \gamma(0, y) + r \nu(0, y), y \in \mathcal{M} \} \) is given by

\[
g(r) := \int_{\mathcal{M}} J(0, r, y) dS_y^0 = |\Gamma_0| + \sum_{i=1}^N \left(\int_{\mathcal{M}} H_i(0, y) dS_y^0\right) r^i, \quad dS_y^0 := dS_y^\gamma(0, \cdot),
\]

so we have \( |\Gamma(t)| = g(r(t)) \). Moreover, (3.2) together with \( |\Omega^-(t)| + |\Omega^+(t)| \equiv |\Omega| \) implies that the volume of the bulk regions are represented in terms of \( v \) as

\[
|\Omega^-| = \frac{h^+(v) - m_0}{h^+(v) - h^-(v)} |\Omega|, \quad |\Omega^+| = \frac{m_0 - h^-(v)}{h^+(v) - h^-(v)} |\Omega|,
\]

from which the first factor in the right hand side of \((\text{IE}^0\text{-b})\) is rewritten as \( h(v(t)) \) with

\[
h(v) = h(v; v_0) := \frac{1}{|\Omega|} \frac{[h^+(v) - h^-(v)]^2}{h^-(v)[h^+(v) - m_0] + h^+(v)[m_0 - h^-(v)]}.
\]

In particular, if the initial pair \((\Gamma_0, v_0)\) is chosen so that \( m_0 \in (\underline{v}, \overline{v}) \), it follows that \( |\Omega^\pm| > 0 \) in (3.4) and therefore we have \( h(v) < 0 \) for all \( v \in I^0 \) (cf. (A1), (A2)). Thus the interface equation \((\text{IE}^0)\) are equivalent to the following initial value problem:

\[\begin{align*}
\dot{r} &= c(v), \\
\dot{v} &= h(v) c(v) g(r), \\
r(0) &= 0, \quad v(0) = v_0.
\end{align*}\]
By virtue of reformulation above and an equivalent expression of $c(v)$

\[(3.6) \quad c(v) = -\frac{S(v)}{\int_{-\infty}^{\infty} \{|Q_z(z;v)|^2\} \, dz}, \]

the interface dynamics are summerized as follows:

- $v \in (v^*, \overline{v}) \implies \dot{r} > 0, \ \dot{v} < 0; \\
  \text{the interface } \Gamma(t) \text{ evolves in such a way that the bulk region } \Omega^-(t) \text{ grows uniformly.}

- $v \in (\underline{v}, v^*) \implies \dot{r} < 0, \ \dot{v} > 0; \\
  \text{the interface } \Gamma(t) \text{ evolves in such a way that the bulk region } \Omega^-(t) \text{ shrinks uniformly.}

- $v = v^* \implies \dot{r} = 0, \ \dot{v} = 0; \\
  \text{the interface } \Gamma(t) \text{ does not evolve.}

We also obtain the following

**Theorem 3 (Unique existence of solutions).** Let $\Gamma_0$ be a smooth initial interface, and a pair $(\Gamma_0, v_0)$ is assumed to satisfy $v_0 \in I^0$ and $m_0 \in (\underline{u}, \overline{u})$. Then the following statements hold true:

1. There exists a constant $T > 0$ such that (IE$^0$) has a unique smooth solution pair $(\Gamma, v)$ on a time interval $[0, T]$.

2. If in addition $v_0$ is sufficiently close to $v^*$, then the unique solution $(\Gamma, v)$ in (1) exists globally in time.

**Proof.** (2) immediately follows from the existence of a constant $R > 0$ such that $r$-component $r(\cdot)$ of the solution to (ODE$^0$) remains in a neighborhood $(-R, R)$ while the corresponding interface $\Gamma(\cdot) = \{x \in \Omega \mid x = \gamma(0, y) + r(\cdot)\nu(0, y), \ y \in \mathcal{M}\}$ is smooth for all $|r| < R$ when we choose $v_0 \approx v^*$.

**Theorem 4 (Stability of equilibrium solutions).** Suppose that a pair $(\Gamma_0, v_0)$ is as in Theorem 3. Then the following statements hold true:

1. $(\Gamma_0, v_0)$ is an equilibrium solution of (IE$^0$) if and only if $v_0 = v^*$.

2. The equilibrium solution $(\Gamma_0, v^*)$ is asymptotically stable relative to (ODE$^0$).

**Proof.** (2) We linearize (ODE$^0$) around the corresponding equilibrium solution $(0, v^*)$ to obtain the eigenvalues 0 and $h(v^*) c'(v^*) |\Gamma_0| < 0$. 

For each $v \in I^0$, the nonlinear term $f(u) - v$ defines a new double-well potential $\mathcal{W}(u;v)$ with two wells located at $u = h^\pm(v)$. Moreover, the potential difference is related to $S(v)$ and $c(v)$ as follows:

\[
\mathcal{W}(h^+(v);v) - \mathcal{W}(h^-(v);v) = -S(v) = c(v) \int_{-\infty}^{\infty} |Q_z(z;v)|^2 \, dz.
\]

Hence it turns out that the 0-th order nonlocal effect equalizes the potential of two wells no matter how the initial state is.
3.2 Higher order equations.

The $j$-th ($j \geq 1$) order equations are as follows:

(IE$^j$-a) \[ R^j_i(t, y) = c'(v^0(t)) v^j(t) + \rho_j(t, y), \quad t > 0, \ y \in \mathcal{M}, \]

(IE$^j$-b) \[ \dot{v}^j(t) = \int_{\mathcal{M}} a(t, y) R^j_i(t, y) dS_y + b(t) v^j(t) + \sigma_j(t), \quad t > 0, \]

(IE$^j$-c) \[ R^j(0, y) = R^j(y), \quad v^j(0) = v_0^j. \]

Recall that $a$ and $b$ are functions depending only on the solution $(\Gamma, v^0)$ to (IE$^0$) (cf. (2.31), (2.32)), while $\rho_j$ and $\sigma_j$ are some functions which can be calculated by using functions with index $k$ ($0 \leq k < j$) in outer and inner expansions.

Each equation (IE$^j$) can be recast as a system of linear non-homogeneous ordinary differential equations. Indeed, by employing a function $r^j$ given by

\[ r^j(t) := R^j(t, y) - R^j(y) - \int_0^t \rho_j(s, y) ds, \]

(IE$^j$-a) and (IE$^j$-b) are respectively expressed as

(IE$^j$-a) \[ \dot{r}^j(t) = c'(v^0(t)) v^j(t), \]

(IE$^j$-b) \[ \dot{v}^j(t) = \left( \int_{\mathcal{M}} a(t, y) dS_y \right) r^j(t) + b(t) v^j(t) \]

\[ + \int_{\mathcal{M}} a(t, y) \left( R^j(y) + \int_0^t \rho_j(s, y) ds \right) dS_y + \sigma_j(t), \]

from which we obtain an initial value problem of the form

(ODE$^j$) \[
\begin{cases}
\dot{r}^j(t) = B(t) v^j(t), \\
\dot{v}^j(t) = C(t) r^j(t) + D(t) v^j(t) + E_j(t), \\
r^j(0) = 0, \quad v^j(0) = v_0^j.
\end{cases}
\]

Due to this reformulation, we have the following

**Theorem 5** (Unique existence of solutions). *Once the initial pair $(R^j(y), v_0^j)$ is given, the equations (IE$^j$) ($j \geq 1$) are successively solvable on a finite time interval $[0, T]$.***

In particular, we can construct a smooth approximate solution $u_A^j$ of (RD) in the sense that

\[ \left\| \frac{\partial u_A^j}{\partial t} - \epsilon^2 \Delta u_A^j - f(u_A^j) + \frac{1}{|\Omega|} \int_\Omega f(u_A^j(\cdot, x)) \, dx \right\|_{L^\infty([0,T] \times \Omega)} = O(\epsilon^{K+1}), \]

\[ \frac{\partial u_A^j}{\partial n} = 0, \quad (t, x) \in [0, T] \times \partial \Omega, \]

by means of unique solutions $(\Gamma, v^0)$ and $(R^j, v^j)$ of (IE$^j$) for $0 \leq j \leq K$.

As the solution $v^0(t)$ approaches the equilibrium state $v^*$, the 0-th order equation (IE$^0$) becomes powerless to approximate the layer dynamics. In this case, we must move our attention to the equation (IE$^1$) for $(R^1, v^1)$ in order to capture the further dynamics of layers. An investigation in such a direction will be our future work.
References


