On discrete Morse semi-flow

東北大学大学院理学研究科 数学専攻 堀畑和弘 (Kazuhiro HORIHATA)
Mathematical Institute Tohoku University

1 Introduction.

Set $d$ and $D$ be positive integers greater than 1. Let $B^d$ and $S^D$ be the unit ball centered at the origin in $\mathbb{R}^d$, the unit sphere $S^D$ in $\mathbb{R}^{D+1}$ and $T$ a positive number. Give $Q$ by $(0, T) \times B^d$. This article studies a certain time-difference space-differential system; We call the solution to it “Discrete Morse Semiflow”, which is abbreviated to “DMS”. This system enables us discuss at least two important problems in Geometric evolution problems: Heat flows for harmonic mappings and mean curvature motion. To explain DMS, we introduce a several notation: Let $h$ be a positive number and $N_T = \lceil T/h \rceil + 1$. We put $t_n := nh$ ($n = 0, \ldots, N_T$) and set $k_0 = (1 - h/(16T)\log(1/h))$. $\chi(t) \in C^\infty$ with

$$\chi(t) := \begin{cases} t & t \leq 2, \\ 3 & t > 4, \end{cases} \quad (1.1)$$

Give a mapping $u_0 \in H^{1,2}(B^d; S^D)$. Then DMS is designated by a sequence of mappings $\{u_n\} (n = 1, \ldots, N_T) \subset \{ u \in H^{1,2}(B^d, S^D) ; u - u_0 \in |H^{1,2}(B^d, \mathbb{R}^{D+1}) \}$ of the solution of the following difference-differential systems:

$$\frac{u_n - u_{n-1}}{h} - \Delta u_n + \frac{k_n}{\sqrt{h}} \chi((|u_n|^2 - 1)^2)(|u_n|^2 - 1)u_n = 0 \quad (1.2)$$

in $B^d,$

$$u_n = u_0 \text{ on } \partial B^d. \quad (1.3)$$
An interpolational convention \( u_h(t, x) \) and \( u_{\overline{h}}(t, x) \) \((t > 0)\) respectively indicates

\[
\begin{align*}
  u_h(t, x) & := \frac{t - t_{n-1}}{h}u_{n}(x) + \frac{t_{n} - t}{h}u_{n-1}(x) & \text{for } t_{n-1} < t \leq t_{n}, \\
  u_{\overline{h}}(t, x) & := u_{n}(x) & \text{for } t_{n-1} < t \leq t_{n}.
\end{align*}
\]

Note \( \partial u_h/\partial t(t, x) = (u_{n}(x) - u_{n-1}(x))/h \) for \( t_{n-1} < t < t_{n} \). When no confusion may arise, we say a pair of functions \( u_h \) and \( u_{\overline{h}} \) to be DMS; \( u_h \) and \( u_{\overline{h}} \) satisfy

\[
\begin{align}
  u_{\overline{h}} & \in L^\infty(0, T; H^{1,2}(B^d; S^D)), \\
  \int_{B^d} \left( \frac{\partial u_{h}}{\partial t}, \phi \right) + \left( \nabla u_{\overline{h}}, \nabla \phi \right) \, dx = -\frac{k_{\overline{h}}}{\sqrt[4]{h}} \int_{B^d} (|u_{\overline{h}}|^2 - 1) (u_{\overline{h}}, \phi) \, dx
\end{align}
\]

for all \( \phi \in C_0^\infty(B^d; \mathbb{R}^{D+1}) \),

\[
\begin{align}
  u_h(t, x) - u_0(x) & \in H^{1,2}(B^d; \mathbb{R}^{D+1}) & \text{for every } t \ (0 \leq t \leq N_T h), \\
  \lim_{h \to 0} ||u_h(t, \circ) - u_0(\circ)||_{L^2(B^d)} & = 0. \tag{1.6}
\end{align}
\]

I addict to DMS: We show that DMS satisfies a maximal principle, a few global energy inequalities, a monotonicity inequality for the scaled energy and finally a reverse Poincaré inequality. By using the inequalities above, we prove that DMS converges to a heat flow for harmonic mappings and discuss a partial regularity result on it. Here for any given mapping \( u_0 \in H^{1,2}(B^d, S^D) \), we call \( u \in L^\infty(0, T; H^{1,2}(B^d; S^D)) \cap H^{1,2}(0, T; L^2(B^d; S^D)) \) a heat flow for harmonic mappings provided

\[
\begin{align}
  \frac{\partial u}{\partial t} & = \Delta u + |\nabla u|^2 u & \text{in } Q, \\
  u(0, x) & = u_0(x) & \text{in } \{0\} \times B^d, \\
  u(t, x) & = u_0(x) & \text{in } (0, T) \times \partial B^d.
\end{align}
\]

The following fact is well-known

**Remark 1** (1.7) is equivalent to

\[
\begin{align}
  \frac{\partial u}{\partial t} \wedge u - \Delta u \wedge u & = 0 & \text{in } (C_0^\infty(Q; \mathbb{R}^{D+1}))^*, \\
  |u| & = 1 & \text{in a.e } z \in Q. \tag{1.8}
\end{align}
\]
The parabolic system holds in the following weak sense:

$$
\int_Q \left( \left( \frac{\partial u}{\partial t}, \phi \right) + \langle \nabla u, \nabla \phi \rangle - \langle u, \phi \rangle |\nabla u|^2 \right) \, dz = 0 \text{ for any } \phi \in C_0^\infty(Q; \mathbb{R}^{D+1}),
$$

(1.10)

$$
u(t, x) - u_0(x) \in H^{1,2}(\mathbb{B}^d; \mathbb{R}^{D+1}) \quad \text{for almost every } t \in (0, T),
$$

(1.11)

$$
\lim_{t \to +0} u(t, 0) = u_0(0) \quad \text{in } L^2(\mathbb{B}^d; \mathbb{R}^{D+1}).
$$

(1.12)

My main result of this article is

**Theorem 1 (Partial Regularity)** There exists a heat flow for harmonic mappings and it is smooth on a relative open set in $Q$ whose compliment has 0 $d$-dimensional Hausdorff measure with respect to the parabolic metric.

The proof of Theorem 1 can be performed by combining Theorem 8 with Theorem 9.

### 2 DMS.

In this chapter, we state a discrete maximal principle and a few global energy-estimates. Thereafter we establish a monotonicity inequality for the scaled energies and a reverse Poincaré inequality, which are the main technical tools of this sort of study. The first is

**Theorem 2 (Discrete maximal Principle)** Each of DMS $\{u_n\}$ $(n = 1, \ldots, N_T)$ implies

$$
|u_n| \leq 1 \text{ for all point } x \in \mathbb{B}^d.
$$

**Theorem 3 (Energy Estimate).** For any given mapping $u_0 \in H^{1,2}(\mathbb{B}^d, \mathbb{S}^D)$, DMS $\{u_n\}$ $(n = 1, 2, \ldots, N_T)$ satisfies

$$
\int_{\mathbb{B}^d} \left( \frac{1}{2} |\nabla u_n|^2 + \frac{k_n}{4 \sqrt{h}} \left( |u_n|^2 - 1 \right)^2 \right) \, dx
$$
\[ \leq \frac{1}{2} \int_{\mathbb{B}^d} |\nabla u_0|^2 \, dx \quad \text{for any integer } n \quad (n = 1, \ldots, N_T), \quad (2.1) \]

\[ \frac{h}{2} \sum_{n=1}^{N_T} \int_{\mathbb{B}^d} \left| \frac{u_n - u_{n-1}}{h} \right|^2 \, dx \]

\[ + \frac{\log(1/h)}{16T} h \sum_{n=1}^{N_T} \frac{k_{n-1}}{4 \sqrt{h}} \int_{\mathbb{B}^d} (|u_{n-1}|^2 - 1)^2 \, dx \leq \frac{1}{2} \int_{\mathbb{B}^d} |\nabla u_0|^2 \, dx. \quad (2.2) \]

**Lemma 1** (Global Pokhojaev Identity). $DMS u_n \ (n = 1, 2, \ldots, N_T)$ have the following property:

\[ \frac{1}{2} \int_{\partial \mathbb{B}^d} |\frac{\partial u_n}{\partial|x|}|^2 \, dH^{d-1}_x \]

\[ + \frac{d-2}{2} \int_{\mathbb{B}^d} |\nabla u_n| \, dx + \frac{dk_n}{4 \sqrt{h}} \int_{\mathbb{B}^d} (|u_n|^2 - 1)^2 \, dx \]

\[ = \frac{1}{2} \int_{\partial \mathbb{B}^d} |\nabla_{\tan} u_0|^2 \, dH^{d-1}_x + \int_{\mathbb{B}^d} \langle \frac{u_n - u_{n-1}}{h}, \langle x, \nabla \rangle u_n \rangle \, dx. \quad (2.3) \]

**Corollary 1** (The first derivatives estimates at $\partial \mathbb{B}^d$).

\[ \frac{1}{2} \int_{h}^{T} dt \int_{\mathbb{B}^d} \left| \frac{\partial u_{\overline{h}}}{\partial|x|} \right|^2 \, dH^{d-1}_x \]

\[ \leq 2T \int_{\mathbb{B}^d} |\nabla_{\tan} u_0|^2 \, dH^{d-1}_x + 2(T + 1) \int_{\mathbb{B}^d} |\nabla u_0|^2 \, dx. \quad (2.4) \]

**Corollary 2** (The rate of the convergence). If $\triangle u_0 \in L^{p_0}(\mathbb{B}^d; \mathbb{R}^{D+1})$ for some $p_0 > 1$,

\[ \int_{\mathbb{B}^d} |\nabla (u_1 - u_0)|^2 \, dx \leq 2^{1-2/p_0} ||\nabla u_0||^{2/p'_0}_{L^{2}(\mathbb{B}^d)} ||\triangle u_0||_{L^{p_0}(\mathbb{B}^d)} \cdot h^{1-1/p_0}, \quad (2.5) \]

holds with $1/p_0 + 1/p'_0 = 1$. 
Remark 2 The typical example of map from $B^d$ to $S^{d-1} \subset \mathbb{R}^d$ may be the equator map given by $x/|x|$. If $u_0(x) = x/|x|$, then $\Delta u_0 \in L^{q_0}(B^d)$ as long as $1 < q_0 < d/2$. We refer to F.Bethuel and X.Zheng [1]. Namely the assumption on $L^{q_0}$-integrability about $\Delta u_0$ is just peril.

Lemma 2 (Higher Order Differential Estimates). $DMS u_n \ (n = 1, 2, \ldots, N_T)$ satisfies

$$h \sum_{n=2}^{N_T} \int_{B^d} |\Delta(u_n - u_{n-1})|^2 \, dx$$

$$\leq Ch^{3/2} \int_{B^d} |\nabla u_0|^2 \, dx + \frac{1}{2} \int_{B^d} |\nabla(u_1 - u_0)|^2 \, dx.$$ (2.6)

Now, we are in the position to state a monotonicity inequality for the scaled energy; For $z_0 = (t_{n_0}, x_0) \in Q$ and a positive number $R$, the scaled energy is denoted by

$$E_h(R; z_0) := \frac{1}{2R^d} \int_{t_{n_0} - \theta_0 R^2}^{t_{n_0} - \theta_0(2R)^2} dt \int_{B^d} \left( |\nabla u_h|^2 + \frac{k_h}{2\sqrt[4]{h}} (|u_h|^2 - 1)^2 \right) \times \exp\left(\frac{|x-x_0|^2}{4(t-t_{n_0})}\right) \, dx.$$ (2.7)

Lemma 3 (Monotonicity for the Scaled Energy). For any point $z_0 = (t_{n_0}, x_0)$ and any positive number $R$,

$$\frac{dE_h}{dR}(R; z_0) \geq -\frac{1}{R^{d-1}} \int_{t_{n_0} - \theta_0(2R)^2}^{t_{n_0} - \theta_0 R^2} dt \int_{B^d} \left| \frac{\partial u_h}{\partial t} + \left( \frac{x - x_0}{2(t-t_{n_0})}, \nabla \right) u_h \right|^2 \exp\left(\frac{|x-x_0|^2}{4(t-t_{n_0})}\right) \, dx$$

$$+ \frac{1}{2R^{d+1}} \int_{t_{n_0} - \theta_0(2R)^2}^{t_{n_0} - \theta_0 R^2} k_h dt \int_{B^d} (|u_h|^2 - 1)^2 \exp\left(\frac{|x-x_0|^2}{4(t-t_{n_0})}\right) \, dx$$

$$- C_M(R, R_0; h).$$ (2.8)
where

\[ C_M(R, R_0; h) := C_{M,1} + C_{M,2} \]

\[ C_{M,1} := \frac{CR}{\text{dist}^{d+1}(x_0, \partial \mathbb{B}^d)} \left( \int_{\mathbb{B}^d} |\nabla u_0|^2 \, dx + \int_{\partial \mathbb{B}^d} |\nabla u_0|^2 \, dH_{d-1} \right), \]

\[ C_{M,2} := -C|| \nabla(u_1 - u_0)||_{L^2(\mathbb{B}^d)}||\nabla u_0||_{L^2(\mathbb{B}^d)} + \frac{C\sqrt{h}}{R^{d+1}}||Vu_0||_{L^2(\mathbb{B}^d)}. \]

Hereafter we state a point-wise estimate and the inequality of "a hybrid type". The latter part of the chapter will be devoted to saying these inequalities. We supposed \( \theta_0, \theta_1 \) and \( R \) be positive numbers with

\[ 0 < \theta_0 < 1, \ 1 < \theta_1, \ R > 0, \ \max\left(\frac{2\theta_1}{3}, 2\right) < \frac{\theta_0 R^2}{h}, \quad (2.9) \]

and we set

\[ N_1 := \left\lceil \frac{\theta_0 R^2}{h} \right\rceil, \quad N_2 := \left\lceil \frac{\theta_0 (2R)^2}{h} \right\rceil. \]

We must remark that all \( N_i \) \((i = 1, 2, 3, 4)\) are positive integers by assumption (2.9).

**Theorem 4 (A Point-wise Estimate)** There exists a positive number \( \epsilon_0 \) depending only on \( d \), such that if \( w_k \) satisfies

\[ \int_{t_{n_1}}^{t_{n_1}} dt \int_{B_2R(x_0)} (1 - \langle u_k, K \rangle) \, dx < \epsilon_0 \quad (2.10) \]

for any cylinder \( Q_{2R, 2N_1 h}(t_{n_1}, x_0) \) \( := (t_{n_1} - 2N_1, t_{n_1}) \times B_{2R}(x_0) \subset Q \), then

\[ |\{ z \in Q_{2R, N_1 h}(t_{n_1}, x_0) ; \langle u_k, K \rangle \leq 1 - \delta_0 \}| \]

\[ \leq C \frac{h \log(1/h)}{\delta_0^3} \int_{\mathbb{B}^d} |\nabla u_0|^2 \, dx \quad (2.11) \]

with \( n_1 = n_0 + N_2 - N_1 \) and \( K \) is any vector in \( \mathbb{R}^{D+1} \).

We must remark that all \( N_i \) \((i = 1, 2, 3, 4)\) are positive integers by assumption (2.9).
Theorem 5 (Discrete Hybrid Inequality) DMS $u_{\overline{h}}$ and $u_{h}$ have the following inequality: There exists positive constant $C_{H}$ depending only on $d$ such that for any numbers $\theta_{0}, \theta_{1}, R$ satisfying the condition (2.9), for any cylinders $(t_{n_{0}}, t_{n_{0}+N_{2}}) \times B_{2R}(x_{0}) \subset Q$,

$$
\int_{t_{n_{0}}+N_{2}}^{t_{n_{0}}+N_{2}-N_{1}} dt \int_{B_{R}(x_{0})} \left( \frac{1}{2} |\nabla u_{h}|^{2} + \frac{k_{h}}{4\sqrt{h}} (|u_{h}|^{2} - 1)^{2} + \frac{\theta_{0}R^{2}}{2} \left| \frac{\partial u_{h}}{\partial t} \right|^{2} \right) dx \\
+ \frac{\theta_{0}R^{2}}{10} \int_{B_{R}(x_{0})} \left( \frac{1}{2} |\nabla u_{h}|^{2} + \frac{k_{h}}{4\sqrt{h}} (|u_{h}|^{2} - 1)^{2} \right) dx \bigg|_{t=t_{n_{0}}+N_{2}} \\
\leq C_{H} \max \left( \left( 1 - \frac{\theta_{1}}{N_{1}} \right)^{N_{1}}, \theta_{0}, \delta(\epsilon_{0}) \right) \\
(2.12)
$$

$$
\times \int_{t_{n_{0}}+N_{2}-3N_{1}-1}^{t_{n_{0}}+N_{2}} dt \int_{B_{R}(x_{0})} \left( \frac{1}{2} |\nabla u_{h}|^{2} + \frac{k_{h}}{4\sqrt{h}} (|u_{h}|^{2} - 1)^{2} + \frac{\theta_{0}R^{2}}{2} \left| \frac{\partial u_{h}}{\partial t} \right|^{2} \right) dx \\
+ \left( 1 + \theta_{0} + \frac{1}{\theta_{0}} \right) \frac{C_{H}}{\log(1/\theta_{1})^{2}R^{2}} \int_{t_{n_{0}}+N_{2}-3N_{1}}^{t_{n_{0}}+N_{2}} dt \int_{B_{3R/2}(x_{0})} |u_{h} - K|^{2} dx \\
+ O(h).
$$

where $R_{0} = \min(\sqrt{t_{n_{0}}}/2\theta_{0}, \text{dist}(x_{0}, \partial B^{d}))$, $\epsilon_{0}$ is a certain positive constant appeared in Theorem 4, respectively and $\delta_{0}(\epsilon_{0}) = \epsilon_{0}^{1/d-1/(1+2/d)(1+4/d)}$.

Remark 3 If one takes $\theta_{1}$ being sufficiently large and $\theta_{0}, \epsilon_{0}$ being sufficiently small, then the coefficient of the first term on the right-hand side above is small.


This chapter establishes the existence and a partial regularity on a heat flow for harmonic mappings that are obtained as the limit of DMS. The existence theorem is a slight modification of Y.Chen [3] and see also L.C.Evans [5, p.48, 5.A.1] and J.Shatah [9]. On the other hand the regularity result will be established by means of a blow-up technique. For
The blow-up technique used here, we refer to R. Hardt, D. Kinderlehrer and F.H. Lin [7] and R. Schöen and K. Uhlenbeck [11]. First of all we mention two convergence theorems directly derived from Theorem 2 and Theorem 3.

**Theorem 6 (Convergence)** There exists a subsequence \( \{u_{\overline{h}_{k}}\}, \{u_{h_{k}}\} \) \((k = 1, 2, \ldots)\) of \( \{u_{\overline{h}}\}, \{u_{h}\}\) \((h > 0)\) respectively and a mapping \( u \in L^{\infty}(0, T; H^{1,2}(\mathbb{B}^{d}; \mathbb{R}^{D+1})) \cap H^{1,2}(0, T; L^{2}(\mathbb{B}^{d}; \mathbb{R}^{D+1}))\) such that \( u_{\overline{h}_{k}} \) and \( u_{h} \) respectively converge weakly-* and weakly to \( u \) in \( L^{\infty}(0, T; H^{1,2}(\mathbb{B}^{d}; \mathbb{R}^{D+1})) \) and \( H^{1,2}(0, T; L^{2}(\mathbb{B}^{d}; \mathbb{R}^{D+1})) \), so does \( u_{\overline{h}_{k}} \) strongly to \( u \) in \( L^{2}(Q) \) and \( u_{h} \) point-wisely to \( u \) as \( h \searrow 0 \) (modulo a subsequence of \( h \)).

Theorem 6 enables us state the following existence theorem:

**Theorem 7 (Existence)** Each of DMS: \( u_{\overline{h}} \) and \( u_{h} \) respectively converges to a heat flow for harmonic mappings \( u \) in \( L^{\infty}(0, T; H^{1,2}(\mathbb{B}^{d}; \mathbb{R}^{D+1})) \) and \( H^{1,2}(0, T; L^{2}(\mathbb{B}^{d}; \mathbb{R}^{D+1})) \) as \( h \searrow 0 \).

**Proof of Theorem 7.** Since \( \nabla u_{\overline{h}} \) and \( \partial_{t}u_{h} \) is uniform bounded in \( L^{\infty}(0, T; L^{2}(\mathbb{B}^{d}; \mathbb{R}^{D+1})) \) and \( L^{2}(0, T; L^{2}(\mathbb{B}^{d}; \mathbb{R}^{D+1})) \) respectively, and a subsequence of \( u_{\overline{h}} \) and \( u_{h} \) also converges weakly-* and weakly to a map \( u \) in \( L^{\infty}(0, T; H^{1,2}(\mathbb{B}^{d}; \mathbb{R}^{D+1})) \) and \( H^{1,2}(0, T; L^{2}(\mathbb{B}^{d}; \mathbb{R}^{D+1})) \) respectively, strongly in \( L^{2}(\mathbb{B}^{d}; \mathbb{R}^{D+1}) \), \( |u_{h}| \leq 1 \), almost everywhere in \( Q \) as \( h \searrow 0 \); We show that the map \( u \) is indeed a heat flow for harmonic mappings. Since \( u_{\overline{h}} \) and \( u_{h} \) satisfy

\[
\frac{\partial u_{h}}{\partial t} - \Delta u_{\overline{h}} = \frac{k_{\overline{h}}}{\sqrt{h}}(1 - |u_{\overline{h}}|^{2})u_{\overline{h}},
\]

by taking a wedge product, we have

\[
\left( \frac{\partial u_{h}}{\partial t} \wedge u_{h} - \Delta u_{h} \right) \wedge u_{h} = 0 \text{ in } (C_{0}^{\infty}(Q; \mathbb{R}^{D+1})^{*}). \quad (3.1)
\]

Thus by virtue of \( u \in L^{\infty}(0, T; H^{1,2}(\mathbb{B}^{d}; \mathbb{S}^{D})) \cap H^{1,2}(0, T; L^{2}(\mathbb{B}^{d}; \mathbb{S}^{D}))\), Remark 1, Theorem 6, we observe that \( u \) satisfies (1.10), (1.11) and (1.12), i.e. \( u \) is a heat flow for harmonic mappings.

**Remark 4** In the following, we fix a subsequences \( \{h_{k}\} \) \((k = 1, 2, 3, \ldots)\) of \( \{h\} \) \((h > 0)\) that makes DMS converge to a heat flow for harmonic mappings \( u \).
Definition 1 Fix a point \( z_0 = (t_0, x_0) \in Q \). We indicate \( \mathcal{M}^+ \) by the following rescaled Radon measure:

\[
\mathcal{M}^+ (Q_R(z_0)) := \liminf_{h_k \searrow 0} \frac{1}{2 \theta_0 R^d} \int_{Q_R(z_0)} \left( |\nabla u_{h_k}|^2 + \frac{k_h}{2 \sqrt{h_k}} (|u_{h_k}|^2 - 1)^2 + \theta_0 R^2 \left| \frac{\partial u_{h_k}}{\partial t} \right|^2 \right) dz,
\]

for any positive number \( \theta_0 \) and any cylinder \( Q_R(z_0) \subset Q \).

Remark 5 (Measured Hybrid Inequality) Assume that \( u_{h_k} \) and \( u_{h_k} \) respectively converges weakly-* and weakly in \( L^\infty(0, T; H^{1,2}(B^d; \mathbb{R}^{D+1})) \) and \( H^{1,2}(0, T; L^2(B^d; \mathbb{R}^{D+1})) \cap H^{1,2}(0, T; L^2(B^d; \mathbb{R}^{D+1})) \) as \( h_k \searrow 0 \). Then take the pass to the limit \( h_k \searrow 0 \) in (2.12) to infer the following: For any positive \( \theta_2 \), there exists a positive constant \( C_{HM} \) depending only on \( d \), \( \theta_2 \) such that

\[
\mathcal{M}^+ (Q_R(z_0)) \leq \theta_2 \mathcal{M}^+ (Q_{2R}(z_0)) + C_{HM} \int_{Q_{2R}(z_0)} |u - K|^2 dz \tag{3.2}
\]

holds for any vector \( K \in \mathbb{R}^{D+1} \) and \( Q_R(z_0) \subset Q_{2R}(z_0) \subset Q \) with \( z_0 = (t_0, x_0) \) and \( Q_{2R}(z_0) = (t_0 - \theta_0 (2R)^2, t_0) \times B_{2R}(x_0) \).

In the similar way as in L.Simon [10, Lemma 2, p31], we can assert the following reverse Poincaré inequality:

Corollary 3 (Reverse Poincaré inequality). The rescaled Radon measure implies the reverse Poincaré inequality: whenever \( Q_{4R} \subset Q \),

\[
R^d \mathcal{M}^+ (Q_R(z_0)) \leq C_{PO} \int_{Q_{4R}(z_0)} |u - K|^2 dz \tag{3.3}
\]

holds, where \( C_{PO} \) is a certain positive constant depending only on \( \theta_2 \) and \( d \).

Proof of Corollary 3. Let

\[
M = \sup_{\{Q_{\sigma}(z) : Q_{\sigma}(z) \subset Q_{2R}(z_0)\}} \sigma^{d+2} \mathcal{M}^+ (Q_{\sigma}(z))
\]

and then take any cylinder \( Q_{\sigma}(z) \) with \( Q_{\sigma}(z) \subset Q_{2R}(z_0) \). Notice that such a cylinder can be covered by cylinders \( Q_{\sigma/4}(z_i) \) (\( i = 1, 2, 3, \ldots, S \))
with \( z_i \in Q_\sigma(z) \) and \( Q_\sigma(z_i) \subset Q_{2R}(z_0) \). We can evidently bound the number \( S \) by a fixed constant depending only on \( d \). Recall (3.2); Then

\[
\sigma^{d+2} \mathcal{M}^* (Q_\sigma(z)) \leq 4^{d+2} \sum_{i=1}^{S} \left( \frac{\sigma}{4} \right)^{d+2} \mathcal{M}^* (Q_{\sigma/4}(z_i))
\]

\[
\leq 4^{d+2} \theta_2 \left( \frac{\sigma}{2} \right)^{d+2} \mathcal{M}^* (Q_{\sigma/2}(z_i)) + 4^{d+2} C_{MH} \sum_{i=1}^{S} \int_{Q_{2\sigma}(z_i)} |u - K|^2 \, dz
\]

\[
\leq 4^{d+2} S \theta_2 M + 4^{d+2} C_{MH} S \int_{Q_{4R}(z_0)} |u - K|^2 \, dz.
\]

Taking "sup" on the right-hand side above, we have

\[
M \leq 4^{d+2} S \theta_2 M + 4^{d+2} S C_{MH} S \int_{Q_{4R}(z_0)} |u - K|^2 \, dz,
\]

whereupon \( \theta_2 = 1/(24^{d+2}S) \), we infer

\[
R^{d+2} \mathcal{M}^* (Q_R(z_0)) \leq 24^{d+2} S C_{MH} \int_{Q_{4R}(z_0)} |u - K|^2 \, dz. \tag{3.4}
\]

We can state one of the main assertions:

**Theorem 8 (Energy Improvement)** For some positive numbers \( \epsilon_0, \theta_0 \) and \( \theta_1 \), the following holds: for any positive number \( R \) and point \( z_0 = (t_0, x_0) \) and any measure \( \mathcal{M} \), for any cylinder \( Q_R(z_0) \) (=: \( (t_0 - \theta_0 R^2, t_0) \times B_R(x_0) \)) \( \subset \subset Q \),

\[
\mathcal{M}^* (Q_R(z_0)) < \epsilon_0^2 \quad \text{implies} \quad \mathcal{M}^* (Q_{\theta_1 R}(z_0)) < \frac{1}{2} \mathcal{M}^* (Q_R(z_0)). \tag{3.5}
\]

**Proof of Theorem 8.** The proof can be proceeded by a contradiction: If the statement would be false, then for any positive number \( \theta_1 \) less than \( 1/2 \), there would exist sequences of positive numbers \( R_i \), of points \( z_i = \)
$(t_i, x_i) \subset Q$, of measures $\mathcal{M}_i$ and of heat flow for harmonic mappings $u_i$ ($i = 1, 2, \ldots$) such that for any $\theta_0$ with $Q_{R_i}(z_i) \subset Q$,

$$\mathcal{M}_i \left( Q_{R_i}(z_i) \right) =: \epsilon_i^2 < \frac{1}{i},$$

(3.6)

but

$$\mathcal{M}_i \left( Q_{\theta_i R_i}(z_i) \right) \geq \frac{\epsilon_i^2}{2}.$$  

(3.7)

By rescaling

$$z = (t, x) \rightarrow \bar{z} = (\bar{t}, \bar{x}) = \left( \frac{t - t_i}{\theta_0 R_i^2}, \frac{x - x_i}{R_i} \right),$$

without a loss of generality, we can rewrite (3.6) and (3.7) as

$$\mathcal{M}_i \left( Q_1(0) \right) = \epsilon_i^2 < \frac{1}{i},$$

(3.8)

but

$$\mathcal{M}_i \left( Q_{\theta_1}(0) \right) > \frac{\epsilon_i^2}{2}.$$  

(3.9)

By using the rescaling $\bar{z} = ((t - t_i)/\theta_0 R_i^2, (x - x_i)/R_i)$ and a positive number $r$, (3.2) becomes

$$\mathcal{M}_i \left( Q_{\theta_1}(0) \right) \leq C \int_{Q_{2\theta_1}(0)} |u_i - u_{i, Q_{2\theta_1}}|^2 \, d\bar{z}. \quad (3.10)$$

Set $v_i(\bar{z}) := \frac{1}{\epsilon_i} (u_i(\bar{z}) - u_{i, Q_1})$ for any $r$ with $\theta_1 \leq r \leq \frac{1}{2}$.

By assumption (3.6), a subsequence of $v_i$ converges weakly to a mapping $v_{\infty} \in L^2(0, T; H^{1,2}(B_1(0); \mathbb{R}^{D+1})) \cap H^{1,2}(0, T; L^2(B_1(0); \mathbb{R}^{D+1}))$ as $i \nearrow \infty$ (modulo a subsequence of $i$). In addition, since $v_i$ satisfies the systems:

$$\frac{1}{\theta_0} \frac{\partial v_i}{\partial \bar{t}} - \Delta v_i = \epsilon_i |\nabla v_i|^2 u_i$$

in the sense of $(C_0^\infty(Q_1(0); \mathbb{R}^{D+1}))^*$, by using L.C. Evans [E1, p.39, Theorem 3] and noting (3.6) again, we find that $v_{\infty}$ is the solution of

$$\frac{1}{\theta_0} \frac{\partial v_{\infty}}{\partial \bar{t}} - \Delta v_{\infty} = 0, \quad (3.11)$$

in the classical sense. From the gradient estimate on the solution of the equation (3.11) by S.Campanato [2] and the Rellich-Kondrachev theorem,
it follows that \( \text{ess} \cdot \sup_{z \in Q_{\theta_1}(0)} (|\nabla v_\infty| + |\partial v_\infty/\partial t|) \leq C||\nabla v_\infty||_{L^2(Q_1(0))} \) and \( v_i \) converges strongly to \( v_\infty \) in \( L^2(Q_1(0)) \) as \( i \nearrow \infty \). Thus
\[
\int_{Q_2\theta_1} |u_i|^2 \, d\overline{z} \leq 2 \int_{Q_2\theta_1} |v_\infty|^2 \, d\overline{z} + 2 \int_{Q_2\theta_1} |v_i - v_\infty|^2 \, d\overline{z} \leq Cr^2,
\]
\[
\int_{Q_2\theta_1} |u_i - u_i, Q_{2\theta_1}|^2 \, d\overline{z} \leq \epsilon_i^2 \int_{Q_{2\theta_1}} |u_i|^2 \, d\overline{z} \leq 2C\theta_1^2 \epsilon_i^2,
\]
holds if \( i \) is sufficiently large possibly depending on \( \theta_1 \). Consequently we infer
\[
\mathcal{A}_+ (Q_{\theta_1}(0,0)) \leq C\theta_1 \epsilon_i^2. \tag{3.12}
\]

If we choose \( C\theta_1 < 1/2 \), which is \( \mathcal{A}_+ (Q_{\theta_1}(0)) < \epsilon_i^2/2 \), then we find that this is a contradiction of our choice.

**Theorem 9 (Singular Set)** Let \( \epsilon_0 \) be the positive number appeared in Theorem 8. Define

\[
\text{sing} := \bigcap_{R>0} \{ z_0 \in Q; \mathcal{A}_+(P_R(z_0)) \geq \epsilon_0 \}, \tag{3.13}
\]

with \( P_R(z_0) = (t_0 - \theta_0 R^2, t_0 + \theta_0 R^2) \times B_R(x_0) \). Then \( \text{sing} \) is a relatively closed set and

\[
\mathcal{H}^{(d)}(\text{sing}) = 0. \tag{3.14}
\]

**Proof of Theorem 9.** \( \text{sing} \) is a relatively closed set. Indeed, if \( z_0 \in \text{sing} \cap Q \), some sequence \( z_\nu = (t_\nu, x_\nu) \in \text{sing} \cap Q \) (\( \nu = 1, 2, \ldots \)) implies \( z_\nu \rightarrow z_0 \) as \( \nu \nearrow \infty \), i.e. for any positive \( \delta \), there exists a positive number \( \nu_\delta \) such that \( \text{dist}(z_\nu, z_0) \leq \delta \) holds for any positive integer \( \nu \geq \nu_\delta \). From definition on \( \text{sing} \), for any \( R > \delta \) and any points \( z_\nu (\nu = \nu_\delta, \nu_\delta + 1, \ldots) \), we obtain

\[
\epsilon_0 \leq \lim \inf_{h_k \searrow 0} \frac{k_h}{2\theta_2(R-\delta)^2} \int_{P_{R-\delta}(t_\nu, x_\nu)} \left( |\nabla u_h|^2 + \frac{k_h}{2\sqrt{h_k}} (|u_h|^2 - 1)^2 + \theta_0 (R-\delta)^2 \left| \frac{\partial u_h}{\partial t} \right|^2 \right) \, dz
\]
\[ \leq \frac{\liminf_{h_{k} \searrow 0}}{2\theta_{2}(R - \delta)^{d}} \int_{P_{R}(z_{0})} \left( |\nabla u_{h_{k}}|^{2} + \frac{k_{h_{k}}}{2\sqrt{h_{k}}} (|u_{h_{k}}|^{2} - 1)^{2} + \theta_{0}R_{k}^{2} \left| \frac{\partial u_{h_{k}}}{\partial t} \right|^{2} \right) dz. \]  

(3.15)

By the arbitrariness of $\delta$, passing to the limit $\delta \searrow 0$, we can say $\text{sing} \cap Q \subset \text{sing} \cap Q$, which provides us with our first assertion. Next we estimate the size of $\text{sing}$ in the $d$-dimensional Hausdorff measure with respect to the parabolic metric. Fix a positive $R < 1$ and set a compact set $\text{comp}$ in $Q$. Let $\{P_{2R_{k}}(z_{k})\}$ $(2R_{k} < R)$, be a cover of $\text{sing}$. Since $\text{sing} \cap \text{comp}$ is compact set in $Q$, we can assume that the cover is finite. Moreover the parabolic version of Vitali covering theorem shows that there is a disjoint finite sub-family $\{P_{R_{k}}(z_{k})\}$, $k \in \mathcal{K}$ with

\[ \text{sing} \cap \text{comp} \subset \bigcup_{k \in \mathcal{K}} P_{10R_{k}}(z_{k}), \]

\[ 2\epsilon_{0}R_{k}^{d} \leq \liminf_{h_{k} \searrow 0} \int_{P_{R_{k}}(z_{k})} \left( |\nabla u_{h_{k}}|^{2} + \frac{k_{h_{k}}}{2\sqrt{h_{k}}} (|u_{h_{k}}|^{2} - 1)^{2} + \theta_{0}R_{k}^{2} \left| \frac{\partial u_{h_{k}}}{\partial t} \right|^{2} \right) dz. \]

(3.16)

From Corollary 3, we have

\[ \epsilon_{0}R_{k}^{d} \leq \liminf_{h_{k} \searrow 0} \int_{P_{R_{k}}(z_{k})} \left( |\nabla u_{h_{k}}|^{2} + \frac{k_{h_{k}}}{2\sqrt{h_{k}}} (|u_{h_{k}}|^{2} - 1)^{2} + \theta_{0}R_{k}^{2} \left| \frac{\partial u_{h_{k}}}{\partial t} \right|^{2} \right) dz \]

\[ \leq \frac{C_{P_{0}}}{R_{k}^{2}} \int_{P_{2R_{k}}(z_{k})} |u - u_{P_{2R_{k}}(z_{k})}|^{2} dz \]

\[ \leq C \int_{P_{2R_{k}}(z_{k})} \left( |\nabla u|^{2} + \theta_{0}R_{k}^{2} \left| \frac{\partial u}{\partial t} \right|^{2} \right) dz. \]  

Thus we obtain

\[ \sum_{k=1}^{K} (10R_{k})^{d} \leq C \sum_{k=1}^{K} \int_{P_{2R_{k}}(z_{k})} \left( |\nabla u|^{2} + \theta_{0}R_{k}^{2} \left| \frac{\partial u}{\partial t} \right|^{2} \right) dz. \]

From

\[ \sum_{k=1}^{K} (10R_{k})^{d+2} \leq CR^{2} \int_{Q} \left( |\nabla u|^{2} + \left| \frac{\partial u}{\partial t} \right|^{2} \right) dz \]
and the absolute continuity of the Lebesgue integral, we conclude

$$\mathcal{H}^d(\text{sing} \cap \text{comp}) \leq C \lim_{R \to 0} \sum_{k=1}^{K} (10R_k)^d = 0. \quad (3.17)$$

If we set $\text{comp}_n := \{ z \in Q; \text{dist}(z, \partial Q) \geq 1/n \} \, (n = 1, 2, \ldots)$, by

$$\lim_{n \to \infty} \mathcal{H}^d(\text{sing} \cap \text{comp}_n) = \mathcal{H}^d(\text{sing}),$$

we can deduce our assertion.

**Theorem 10** (Recursive Inequality) The heat flow for harmonic mapping $u$ is Hölder continuous on $Q \setminus \text{sing}$.

**Proof of Theorem 10.** Fix a point $z_0 = (t_0, x_0) \in Q \setminus \text{sing}$ and choose $R$ so that $\mathcal{M}^+ (P_R(z_0)) < \epsilon_0$ with some $\theta_0$ possibly depending on $z_0$ and $R$. Because $Q / \text{sing}$ is an open set, there exists some $P_{R_0}(z_0)$ so that

$$\mathcal{M}^+ (P_{R}(\overline{z}_0)) < \epsilon_0$$

for all point $\overline{z}_0 \in P_{R_0}(z_0)$. Then by Theorem 8, we obtain

$$\mathcal{M}^+ (P_{r}(\overline{z}_0)) \leq C \left( \frac{r}{R} \right)^{\alpha_0} \mathcal{M}^+ (P_R(\overline{z}_0)) \quad (3.18)$$

for any positive number $r > 0$ with $\alpha_0 = \log 2 / \log (1/\theta_1)$. This leads to our claim. $\square$

We next collect a few properties of the heat flow for harmonic mappings obtained by the perturbation of DMS:

**Corollary 4** From (2.2) in Theorem 3, we obtain

$$\lim_{h_k \searrow 0} \sup_{Q} \int \frac{1}{\sqrt{h_k}} \left( |u_{h_k}|^2 - 1 \right)^2 dz = \lim_{h_k \searrow 0} \frac{1}{\log 1 / h_k} = 0. \quad (3.19)$$

From Lemma 4, we infer that there is a positive number $\epsilon_0$ such that for any positive number $\epsilon$ less than $\epsilon_0$ if the heat flow for harmonic mappings $u$ satisfies

$$\int_{Q_{2R}} |u - u_{Q_{4R}}|^2 dz < \epsilon,$$

for any cylinder $Q_{2R}(z_0) \subset Q$, then we infer

$$\lim_{h_k \searrow 0} \sup_{Q_{R}(z_0)} |u_{h_k} - u_{h_k,Q_{2R}}|^2 < C(\epsilon),$$

where $C(\epsilon)$ is a positive number satisfying $C(\epsilon) \searrow 0$ as $\epsilon_0 \searrow 0$. 
Finally, we close this section by showing the strong convergence of $u_{h_k}$ to a heat flow for harmonic mappings $u$ in $H^{1,2}_{\text{loc}}$-topology as $h_k \downarrow 0$;

**Theorem 11** (Strong Convergencity of Gradients) The gradients of $u_{h_k}$ converges strongly to the gradients of $u$ in $L^2(\Omega)$.

**Proof of Theorem 11.** Fix two compact sets $\text{comp} \subset \text{comp}_1 \subset Q$, which are compactly contained each other. Take the difference between (1.10) and (1.5), for a map $\phi \in C_0^\infty(\text{comp}_1; \mathbb{R}^{D+1})$, then we obtain

$$
\int_{\text{comp}_1} \langle \frac{\partial}{\partial t}(u_{h_k} - u), \phi \rangle \, dz + \int_{\text{comp}_1} \langle \nabla(u_{h_k} - u), \nabla \phi \rangle \, dz
= - \int_{\text{comp}_1} |\nabla u|^2 \langle u, \phi \rangle \, dz + \int_{\text{comp}_1} \frac{k_{h_k}}{\sqrt{h_k}} (1 - |u_{h_k}|^2) \langle u_{h_k}, \phi \rangle \, dz. \tag{3.20}
$$

Substituting $\phi$ for $(u_{h_k} - u)\eta_1$, we obtain

$$
\int_{\text{comp}_1} |\nabla(u_{h_k} - u)|^2 \eta_1 \, dz \leq \int_{\text{comp}_1} \left( |\frac{\partial u_{h_k}}{\partial t}| + |\frac{\partial u}{\partial t}| \right) |u_{h_k} - u| \, dz
+ \int_{\text{comp}_1} (|\nabla u_{h_k}| + |\nabla u|) |u_{h_k} - u| |\nabla \eta| \, dz
+ \int_{\text{comp}_1} |\nabla u|^2 |u_{h_k} - u| \, dz
+ \int_{\text{comp}_1} \frac{k_{h_k}}{\sqrt{h_k}} (1 - |u_{h_k}|^2) |u_{h_k} - u| \, dz, \tag{3.21}
$$

where $\eta_1$ is a smooth function with the support of $\text{comp}_1$ and $\eta_1 = 1$ in $\text{comp}$.

By using Schwarz's inequality and recalling the energy inequality (2.1) and (2.2) in Theorem 3 and the strong convergencity of $u_{h_k}$: Theorem 6, we can easily estimate the 1st, the 2nd and the 3rd terms on the right-hand side in (3.21). We estimate the last term of the right-hand side in (3.21). Since $\mathcal{H}^{(d)}(\text{sing}) = 0$ and $\text{sing} \cap \text{comp}$ is compact, from definition of Hausdorff measure, for any positive number $\epsilon$, there exists a positive number $R_\epsilon$ less than $R_\epsilon < R_0 \ (= \text{dist}(\text{comp}, \partial \text{comp}_1))$ and a finite cover
of sing: \( \{Q_{R_{i}}(z_{i})\} \) \((i = 1, 2, \ldots, K_{1})\) with \( R_{i} < R_{\epsilon} \) such that
\[
\text{sing} \subset \bigcup_{i=1}^{K_{1}} Q_{R_{i}}(z_{i}), \quad \mathcal{H}^{(d)}(\text{sing}) \leq \sum_{i=1}^{K_{1}} R_{i}^{d} + \epsilon,
\]
\[
\mathcal{MT}^{+}(Q_{R_{i}}(z_{i})) \leq CR^{\alpha_{0}}.
\]

We decompose the last term as follows:
\[
\int_{\text{comp}_{1}} \frac{k_{\overline{h}_{k}}}{2\sqrt[4]{h_{k}}}(1 - |u_{\overline{h}_{k}}|^{2})|u_{h_{k}} - u| \, dz
\]
\[
\leq \int_{\bigcup_{i=1}^{K_{1}} Q_{R_{i}}(z_{i})} \frac{k_{\overline{h}_{k}}}{2\sqrt[4]{h_{k}}}(1 - |u_{\overline{h}_{k}}|^{2})|u_{h_{k}} - u| \, dz
\]
\[+ \int_{\text{comp}_{1} \setminus \bigcup_{i=1}^{K_{1}} Q_{R_{i}}(z_{i})} \frac{k_{\overline{h}_{k}}}{2\sqrt[4]{h_{k}}}(1 - |u_{\overline{h}_{k}}|^{2})|u_{h_{k}} - u| \, dz. \quad (3.22)
\]

Moreover there exists a finite cover \( \{Q_{R_{j}}\} \) \((j = 1, 2, \ldots, K_{2})\) with \( z_{i} \in \text{comp}_{1} \setminus \bigcup_{i=1}^{K_{1}} Q_{R_{i}}(z_{i}) \), because it is compact; we can proceed to estimate (3.22) as follows:
\[
\int_{\text{comp}_{1}} \frac{k_{\overline{h}_{k}}}{2\sqrt[4]{h_{k}}}(1 - |u_{\overline{h}_{k}}|^{2})|u_{h_{k}} - u| \, dz
\]
\[
\leq \int_{\bigcup_{i=1}^{K_{1}} Q_{R_{i}}(z_{i})} \frac{k_{\overline{h}_{k}}}{2\sqrt[4]{h_{k}}}(1 - |u_{\overline{h}_{k}}|^{2})|u_{h_{k}} - u| \, dz
\]
\[+ \int_{\bigcup_{i=1}^{K_{2}} Q_{R_{j}}(z_{j})} \frac{k_{\overline{h}_{k}}}{2\sqrt[4]{h_{k}}}(1 - |u_{\overline{h}_{k}}|^{2})|u_{h_{k}} - u_{h_{k}Q_{R_{j}}(z_{j})}| \, dz
\]
\[+ \int_{\bigcup_{i=1}^{K_{2}} Q_{R_{j}}(z_{j})} \frac{k_{\overline{h}_{k}}}{2\sqrt[4]{h_{k}}}(1 - |u_{\overline{h}_{k}}|^{2})|u_{Q_{R_{j}}(z_{j})} - u_{h_{k}Q_{R_{j}}(z_{j})}| \, dz\]
\[ + \int_{\bigcup_{i=1}^{K_2} Q_{R_i}(z_i)} \frac{k_{\overline{h}_k}}{2\sqrt[4]{h_k}} (1 - |u_{\overline{h}_k}|^2) |u - u_{Q_{R_i}(z_i)}| \, dz. \tag{3.23} \]

From now on, we estimate the each term of (3.23). First, we majorize the 1st term as follows: Recall (1.5) as \( h = h_k \) and substitute \( u_{\overline{h}_k}\eta_i \) for \( \phi \) in (1.5) where \( \eta_i \) is smooth function having only \( x \)-variable with the compact support in \( B_{2R_i}(x_i) \) satisfying
\[
\eta_i = \begin{cases} 
1 & \text{in } B_{R_i}(x_i), \\
0 & \text{outside } B_{2R_i}(x_i)
\end{cases}
\]
to obtain
\[
\int_{Q_{R_i}(x_i)} \frac{k_{\overline{h}_k}}{2\sqrt[4]{h_k}} (1 - |u_{\overline{h}_k}|^2) \, dz 
\leq \int_{Q_{2R_i}(x_i)} (|\nabla u_{\overline{h}_k}|^2 + \frac{k_{\overline{h}_k}}{2\sqrt[4]{h_k}} (|u_{\overline{h}_k}|^2 - 1)^2) \, dz 
+ \left| \int_{Q_{2R_i}(x_i)} (\langle \frac{\partial u_{h_k}}{\partial t}, u_{h_k} \rangle + \frac{1}{2} \langle |u_{\overline{h}_k}|^2, \nabla \eta_i \rangle) \right| dz. \tag{3.24} \]

From Lemma 3, for \( R_0 = 1/2 \text{ dist (comp, } \partial \text{comp}_1) \), we obtain
\[
\int_{Q_{R_i}(x_i)} \frac{k_{h_k}}{2\sqrt[4]{h_k}} (1 - |u_{h_k}|^2) \, dz \leq C R_i^d \int_Q (|\nabla u_{\overline{h}_k}|^2 + \frac{k_{\overline{h}_k}}{2\sqrt[4]{h_k}} (|u_{\overline{h}_k}|^2 - 1)^2) \, dz 
+ \left| \int_{Q_{2R_i}(x_i)} (\langle \frac{\partial u_{h_k}}{\partial t}, u_{h_k} \rangle \eta_i - \frac{1}{2} \langle |u_{h_k}|^2, \Delta \eta_i \rangle) \right| dz. \tag{3.25} \]

That is, noting that \( \partial u_{h_k}/\partial t \) and \( u_{h_k} \) converges weakly to \( \partial u/\partial t \) and strongly to \( u \) as \( k \nearrow \infty \), respectively and \( |u| = 1 \), a.e,
\[
\limsup_{h_k \searrow \eta} \int_{\bigcup_{i=1}^{K_1} Q_{R_i}(x_i)} \frac{k_{h_k}}{2\sqrt[4]{h_k}} (1 - |u_{h_k}|^2) \, dz \leq C R_i^d 
+ \sum_{i=1}^{K_1} \limsup_{h_k \searrow \eta} \left| \int_{Q_{2R_i}(x_i)} (\langle \frac{\partial u_{h_k}}{\partial t}, u_{h_k} \rangle \eta_i - \frac{1}{2} \langle |u_{h_k}|^2, \Delta \eta_i \rangle) \right| dz.
\]
\[ \leq C \mathcal{H}(\text{sing}) + \epsilon = \epsilon. \] (3.26)

Next we estimate the 2nd and the 4th term on the right-hand side: First recall that since \( z_i \in Q \setminus \text{sing} \), by using Corollary 4 and Theorem 10, we obtain

\[
\limsup_{h_k \downarrow 0} \int_{\bigcup_{j=1}^{K_2} \hat{Q}_{R_j}(z_j)} \frac{k_{h_k}}{2\sqrt{h_k}} (1 - |u_{h_k}|^2) |u_{h_k} - u| dz \\
\leq \limsup_{h_k \downarrow 0} \int_{\bigcup_{j=1}^{K_2} \hat{Q}_{R_j}(z_j)} \frac{k_{h_k}}{2\sqrt{h_k}} (1 - |u_{h_k}|^2) |u_{h_k} - u_{h_k R_j(z_j)}| dz \\
\leq \sum_{j=1}^{K_2} \limsup_{h_k \downarrow 0} \int_{\hat{Q}_{R_j}(z_j)} \frac{k_{h_k}}{2\sqrt{h_k}} (1 - |u_{h_k}|^2) dz \sup_{z \in \hat{Q}_{R_j}(z_j)} |u_{h_k} - u_{h_k R_j(z_j)}| \\
\leq C(R^{\alpha_0}) \limsup_{h_k \downarrow 0} \int_{\bigcup_{j=1}^{K_2} \hat{Q}_{R_j}(z_j)} \frac{k_{h_k}}{2\sqrt{h_k}} (1 - |u_{h_k}|^2) dz, \quad (3.27)
\]

\[
\limsup_{h_k \downarrow 0} \int_{\bigcup_{j=1}^{K_2} \hat{Q}_{R_j}(z_j)} (1 - |u_{h_k}|^2) |u - u_{Q_{R_j}(z_j)}| dz \\
\leq \sum_{i=1}^{K_2} \limsup_{h_k \downarrow 0} \int_{\hat{Q}_{R_j}(z_j)} \frac{k_{h_k}}{2\sqrt{h_k}} (1 - |u_{h_k}|^2) dz \sup_{z \in \hat{Q}_{R_j}(z_j)} |u - u_{Q_{R_j}(z_j)}| \\
\leq C(R^{\alpha_0}) \limsup_{h_k \downarrow 0} \int_{\bigcup_{j=1}^{K_2} \hat{Q}_{R_j}(z_j)} \frac{k_{h_k}}{2\sqrt{h_k}} (1 - |u_{h_k}|^2) dz, \quad (3.28)
\]

where we set \( \hat{Q}_{R_j}(z_j) = Q_{R_j}(z_j), \) \( \hat{Q}_{R_j}(z_j) = Q_{R_j}(z_j) \setminus \bigcup_{i=1}^{j-1} Q_{R_i}(z_i) \) (\( j = 2, 3, \ldots K_2 \)). If we now estimate \( \int_{\bigcup_{j=1}^{K_2} Q_{R_j}(z_j)} k_{h_k}/2\sqrt{h_k} (1 - |u_{h_k}|^2) dz \) in the same way as before, we find

\[
\limsup_{h_k \downarrow 0} \int_{\bigcup_{j=1}^{K_2} Q_{R_j}(z_j)} \frac{k_{h_k}}{2\sqrt{h_k}} (1 - |u_{h_k}|^2) |u - u_{Q_{R_j}(z_j)}| dz \\
\leq C \int_Q \left( |\nabla u_{h_k}|^2 + \frac{k_{h_k}}{2\sqrt{h_k}} (|u_{h_k}|^2 - 1)^2 \right) dz \leq C. \quad (3.29)
\]
Finally, we estimate the 3rd term on the right-hand side:

$$\limsup_{h_k \downarrow 0} \int_{\bigcup_{j=1}^{K_2} Q_{R_j}(z_j)} \frac{k_{h_k}}{2\sqrt{h_k}} (1 - |u_{h_k}|^2) |u_{h_k} - u_{h_k Q_{R_j}(z_j)}| dz \leq \limsup_{h_k \downarrow 0} \int_{\text{comp}} \frac{k_{h_k}}{2\sqrt{h_k}} (1 - |u_{h_k}|^2) dz \\
\times \max \limsup_{j} \frac{1}{|Q_{R_j}|} \int_{Q_{R_j}(z_j)} |u_{h_k} - u| dz = 0. \tag{3.30}$$

By applying (3.29) into (3.27) and (3.28) and gathering the estimates of (3.26), (3.27), (3.28) and (3.30), we arrive at

$$\limsup_{h_k \downarrow 0} \int_{\text{comp}} \frac{k_{h_k}}{2\sqrt{h_k}} (1 - |u_{h_k}|^2) dz \leq C \epsilon + C(R^{\alpha_0}).$$

Let $\epsilon \downarrow 0$ and recall $C(R^{\alpha_0}) \downarrow 0$ as $\epsilon \downarrow 0$ to deduce our claim:

$$\limsup_{h_k \downarrow 0} \int_{\text{comp}} \frac{k_{h_k}}{2\sqrt{h_k}} (1 - |u_{h_k}|^2) dz = 0,$$

which implies

$$\limsup_{h_k \downarrow 0} \int_{\text{comp}} |\nabla (u_{h_k} - u)|^2 dz = 0.$$

4 Final Remarks.

We will mention a few open problems that we should research from now: The 1st is a so-called Federer's dimension reduction argument (For the Federer's dimension reduction argument, we refer to R.Schoen and K.Uhlenbeck [11])

**Conjecture 1** $\mathcal{H}^{(d-1)}(\text{sing}) < \infty$.

Note that the estimate above is sharp in the sense that for any fixed time $t > 0$, any mapping $u(t, \cdot) : \mathbb{B}^3 \to \mathbb{S}^2$ i.e $d = 3, D = 2$ must have as least
one point singularity and "the time" has two-dimension in the parabolic metric.

Next we are drawn into the regularity on the singular set:

Problem 1 sing is rectifiable set.

Author believes that once we can show the above, we also arrive

Problem 2 Let \((s,a) \in Q\) be a singular point of the heat flow for harmonic mappings; There exists a matrix \(R \in O(3)\) (may be independent of \((t,x)\), but possibly depending on \(a\)) such that the heat flow for harmonic mappings \(u\) behaves \(R(x-a)/|x-a|\) around the singular points \(a\) at each time slice \(s\).

These will be proved by establishing the parabolic analogue of L.Simon [10]. We do emphasis that a monotonicity for scaled energy is crucially made the best of his theory.

参考文献


