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Characterizing multipliers by relative invariance

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1 Introduction

Translation invariant operators bounded on $\mathbf{L}^p(\mathbf{R}^n)$ are natural objects to study. They can also be considered as convolution operators. The Hilbert transform, i.e. the convolution with $\frac{1}{x}$ in the principal-value sense, appears both in complex analysis, taking limits onto boundaries, and in harmonic analysis in connection with the convergence of Fourier series. Other examples of translation invariant operators arise in the theory of differential equations. For instance, second order derivatives of the solution to the Laplace equation, $\Delta u = f$. For a survey of the use of these kind of operators in analysis, see [F2].

It can be shown that a bounded translation invariant operator $T : \mathbf{L}^2(\mathbf{R}^n) \to \mathbf{L}^2(\mathbf{R}^n)$ is, what is called, a multiplier operator $T = T_m$, i.e. on the Fourier transform side the operator corresponds to multiplication with a bounded function, the multiplier. So we have

$$\mathcal{F}(T_m(f))(\lambda) = m(\lambda)\mathcal{F}(f)(\lambda),$$

where $\mathcal{F}(f)(\lambda) = \int_{\mathbf{R}^n} e^{2\pi i x\cdot\lambda} f(x)\, dx$. In one dimension, the most fundamental multiplier operator is the Hilbert transform, H, which is defined by

$$Hf(x) = \lim_{\epsilon\to 0} \frac{1}{\pi} \int_{|y|\geq\epsilon} \frac{f(x-y)}{y}\, dy.$$

The corresponding multiplier is $m(\lambda) = i \operatorname{sgn} \lambda$. This operator has the following invariance properties

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- it is translation invariant
- it commutes with positive dilations and anti-commutes with negative ones.

Conversely, a continuous operator $T : \mathbf{L}^2(\mathbf{R}) \to \mathbf{L}^2(\mathbf{R})$ with these two properties must, up to multiplication with a scalar, be the Hilbert transform. This observation is regarded as a characterization of the Hilbert transform by means of invariance under the affine transformation group $\mathbf{GL}(1,\mathbf{R}) \ltimes \mathbf{R}$.

In higher dimensions the natural generalizations of the Hilbert transform are the Riesz transforms defined by

$$R_j f(x) = \lim_{\epsilon \to 0} c_n \int_{|y| \geq \epsilon} \frac{y_j}{|y|^{n+1}} f(x-y)\, dy,$$

where $c_n = \Gamma(\frac{n+2}{2})/\pi^{(n+2)/2}$. The corresponding multipliers, m_j, again have a simple form

$$m_j(\lambda) = i \frac{\lambda_j}{|\lambda|}.$$

Again there is a characterization

Theorem 1 ([S] sect. 3.1 Proposition 2). *A family of multiplier operators $\bar{T} = (T_1, \ldots, T_n)$ bounded on $\mathbf{L}^2(\mathbf{R}^n)$ and commuting with positive dilations, satisfies the identity $l_{\rho^{-1}} \circ \bar{T} \circ l_\rho = \pi_\rho \circ \bar{T}$, where π_ρ is the standard representation of $\mathrm{O}(n)$ on $\mathbf{R}^n$, if and only if $m_i(\lambda) = C\lambda_i/|\lambda|$ $(1 \leq i \leq n)$. That is, up to a constant, the family of operators is the family of Riesz transforms.*

The natural representation of $\mathrm{O}(n)$ can be identified with the representation on spherical harmonics of degree 1. Stein has also extended the Riesz transforms to higher Riesz transforms by using spherical harmonics of higher degrees, see Stein [S] section III.3.4.

In the characterization of the Riesz transforms in Theorem 1, one can observe that the conformal transformation group $\mathrm{CO}(n) \ltimes \mathbf{R}^n \simeq (\mathbf{R}^\times - \mathrm{O}(n)) \ltimes \mathbf{R}^n$ appears, and conversely this is (in some sense) a maximal group of (relative) invariance of the Riesz transforms. In this paper, we consider two different, but natural, procedures

- One way is to start with a multiplier and then
  - find a (maximal) group of relative invariance.
  - After that, we solve the equation of invariance and ask for uniqueness in the sense that the solution space should be finite dimensional.

  If we get back the original operator, or a finite family containing it, one can regard the multiplier as characterized by this invariance.

- Another way is to begin with some nice group action and then to find all functions that satisfy the invariance conditions. For this, take a subgroup of affine transformations and give an equation of (relative) invariance by this group such that the space of solutions is finite dimensional, or preferably 1-dimensional. Then use this to find new multiplier operators.

We shall give a general formulation of "relative invariant operators" in Theorem 2 which contains Stein's characterization of the Riesz transforms in Theorem 1(in a different but essentially equivalent way) and relative invariant of prehomogeneous vector spaces as special cases. Then we shall give some examples of invariant multipliers when the groups are

$$\mathbf{R}_+^* \times \mathrm{SO}(p,q) \ltimes \mathbf{R}^{p+q} \subset \mathrm{Aff}(\mathbf{R}^{p+q}),$$

$$\mathrm{O}(m) \times \mathrm{GL}(k,\mathbf{R}) \ltimes \mathbf{R}^{km} \subset \mathrm{Aff}(\mathbf{R}^{km}).$$

We also examine $\mathbf{L}^p$-boundedness in some of the cases at the end of the paper.

2 General results

2.1 Affine action

In this section we will generalize the set-up from the introduction to be able to consider other groups acting on $\mathbf{R}^n$. Let H be a subgroup of $\mathrm{GL}(n,\mathbf{R})$ and take a finite dimensional irreducible representation (π, V) of H. H acts on $\mathbf{R}^n$, hence also, by the contragredient action: $\lambda \mapsto (h^t)^{-1}\lambda$, on the character group $\hat{\mathbf{R}}^n$. For every open orbit $\mathcal{O}$ there exists an element λ_0 such that $H/H_{\lambda_0} \cong \mathcal{O}$. We will assume that there exists a finite set of open orbits, $\mathcal{O}_1, \ldots, \mathcal{O}_N$ such that their union is conull in $\hat{\mathbf{R}}^n$. The orbits correspond to quotients $\mathcal{O}_j \cong H/H_j$ as above. Let $\mathcal{C}_{bdd}(\mathcal{O}_j)$ denote the complex vector space consisting of bounded continuous functions on $\mathcal{O}_j$, on which the group H acts by pullback of functions.

Theorem 2. *Let $\mathcal{B}_H(\mathbf{L}^2(\mathbf{R}^n), V \otimes \mathbf{L}^2(\mathbf{R}^n))$ be the vector space of bounded, translation invariant operators $T : \mathbf{L}^2(\mathbf{R}^n) \to V \otimes \mathbf{L}^2(\mathbf{R}^n)$ satisfying*

$$\begin{tikzcd} \mathbf{L}^2(\mathbf{R}^n) \arrow[r, "T"] \arrow[d, "l_g"'] & V \otimes \mathbf{L}^2(\mathbf{R}^n) \arrow[d, "\pi(g)\otimes l_g"] \\ \mathbf{L}^2(\mathbf{R}^n) \arrow[r, "T"] & V \otimes \mathbf{L}^2(\mathbf{R}^n), \end{tikzcd} \tag{1}$$

for all $g \in H$. Then we have an isomorphism

$$\mathcal{B}_H(\mathbf{L}^2(\mathbf{R}^n), V \otimes \mathbf{L}^2(\mathbf{R}^n)) \cong \bigoplus_{j=1}^{N} \mathrm{Hom}_H(V^*, \mathcal{C}_{bdd}(\mathcal{O}_j))$$

as vector spaces. Thus the left-hand-side will be one dimensional if there is only one orbit and

$$\dim \mathrm{Hom}_H(V^*, \mathcal{C}_{bdd}(\mathcal{O}_1)) = 1.$$

Corollary 1. *If* $\dim V = 1$ *then we always have*

$$\dim \mathrm{Hom}_H(V^*, \mathcal{C}_{bdd}(\mathcal{O}_j)) \leq \dim \mathrm{Hom}_H(V^*, \mathcal{C}(\mathcal{O}_j)) \leq 1.$$

Thus in that case the multiplier is unique, up to a scalar, on each orbit if it exists.

Example 1. *Stein's result is the case where* $H = \mathbf{R}_+ \times \mathrm{O}(n)$, $N = 1$, $\mathcal{O}_1 = \mathbf{R}^n \setminus \{0\}$ *and* π *is the tensor product of the trivial representation with a spherical representation. The subgroup leaving the vector* $v = (1, 0, \dots, 0)$ *fixed is* $H_v = \mathrm{O}(n-1)$ *and the quotient* $H/H_v \cong \mathbf{R}^n$ *is a reductive symmetric space.*

Example 2. *In the theory of prehomogeneous vector spaces, a non-trivial function on* $\mathcal{O}_j$ *contained in the image of* $\mathrm{Hom}_H(V^*, \mathcal{C}(\mathcal{O}_j))$, *where* (π, V) *is assumed to be one-dimensional, is called a* **relative invariant**, *and the corresponding one dimensional representation* (π^*, V^*) *defines a function on* H *by* $h \mapsto \pi^*(h)$, *which is called a* b-**function**. *We shall give some examples in sections 3.1 and 3.2.*

Example 3. *If the quotient* H/H_{λ_j} *is a reductive symmetric space then the dimension of the space* $\mathrm{Hom}_H(V, \mathcal{C}(H/H_{\lambda_j}))$ *is* ≤ 1. *Hence, if all the orbits* H/H_{λ_j} *are reductive symmetric spaces, then, by the previous theorem, we obtain* $\dim \mathcal{B}_H(\mathbf{L}^2(\mathbf{R}^n), V \otimes \mathbf{L}^2(\mathbf{R}^n)) \leq N$.

The above three examples treat cases where either $\dim V = 1$ or the orbits $\mathcal{O}_j$ are symmetric spaces. Later we will consider an example where $\mathrm{O}(k) \times \mathrm{GL}(m, \mathbf{R})$ is acting on $\mathbf{R}^{mk}$. In this example $\dim \mathcal{B}_H(\mathbf{L}^2(\mathbf{R}^n), V \otimes \mathbf{L}^2(\mathbf{R}^n)) \leq 1$, even though $\dim V$ can be > 1 and the orbit is not a symmetric space.

We end this section with the following remark for non-unitarizable representations (π, V).

Proposition 1. $\mathcal{B}_H(\mathbf{L}^2(\mathbf{R}^n), V \otimes \mathbf{L}^2(\mathbf{R}^n)) = \{0\}$ *if* (π, V) *is a non-unitarizable representation of a reductive Lie group* H.

For example this is the case if $H = \mathrm{SL}(n, \mathbf{R})$ and π is the natural representation of H on $V = \mathbf{R}^n$, $(n > 1)$.

3 Examples

3.1 $\mathrm{GL}(2, \mathbf{R})$ acting on $\mathbf{R}^3$

We will identify the set of symmetric matrices $S = \mathrm{Symm}(2)$ with $\mathbf{R}^3$ by the map

$$\begin{pmatrix} x & z \\ z & y \end{pmatrix} \mapsto (x, y, z).$$

and let GL(2) act on symmetric matrices by $l_g : X \mapsto gXg^t$. The group GL(2) has two natural families of one-dimensional irreducible unitary representations:

$$\pi_{\epsilon,\alpha} : g \mapsto \operatorname{sgn}(\det g)^{\epsilon} |\det g|^{i\alpha}, \tag{2}$$

where $\epsilon \in \mathbf{Z}_2$ and $\alpha \in \mathbf{R}$. We define three open subsets in $S^* \cong \mathbf{R}^3$ by

$$\begin{aligned}
\mathcal{O}_{++} &= \{\lambda = (\lambda_1, \lambda_2, \lambda_3) : \lambda_1 + \lambda_2 > 0, \lambda_1\lambda_2 - \lambda_3^2 > 0\},\\
\mathcal{O}_{+-} &= \{\lambda = (\lambda_1, \lambda_2, \lambda_3) : \lambda_1\lambda_2 - \lambda_3^2 < 0\},\\
\mathcal{O}_{--} &= \{\lambda = (\lambda_1, \lambda_2, \lambda_3) : \lambda_1 + \lambda_2 < 0, \lambda_1\lambda_2 - \lambda_3^2 > 0\}.
\end{aligned}$$

(Thus $\mathcal{O}_{++}$ corresponds to matrices with trace and determinant > 0, $\mathcal{O}_{--}$ to matrices with trace and determinant < 0 and $\mathcal{O}_{+-}$ to matrices with determinant < 0) Each of them is a single orbit of GL(2), since matrices with the same signature are conjugate, and their union $\mathcal{O}_{++} \cup \mathcal{O}_{+-} \cup \mathcal{O}_{--}$ is open dense. For $\beta \in \mathbf{R}$ and $\delta \in \{++, +-, --\}$ we define a function supported on the orbit corresponding to the sign delta

$$m_\delta^\beta(\lambda) = \begin{cases} |\lambda_1\lambda_2 - \lambda_3^2|^{-\frac{i\beta}{2}} & (\lambda \in \mathcal{O}_\delta),\\ 0 & (\lambda \notin \mathcal{O}_\delta).\end{cases}$$

Theorem 3. *Let $T : L^2(\mathbf{R}^3) \to L^2(\mathbf{R}^3)$ be a bounded, translation invariant operator, which satisfies*

$$T \circ l_g = \pi_{\epsilon,\alpha}(g)\, l_g \circ T \tag{3}$$

for all $g \in GL(2)$. Then, if $\epsilon = 0$ the corresponding multiplier is of the form

$$m(\lambda) = C_1 m_{++}^{\alpha}(\lambda) + C_2 m_{+-}^{\alpha}(\lambda) + C_3 m_{--}^{\alpha}(\lambda),$$

for some $C_1, C_2, C_3 \in \mathbf{C}$, but if $\epsilon = 1$ we get

$$m(\lambda) = 0.$$

Proof. By using the bilinear map

$$\langle\ ,\ \rangle : \mathrm{Symm}(2) \times \mathrm{Symm}(2) \mapsto \mathbf{R}, \langle u, v\rangle \mapsto Trace(uv),$$

We shall identify S^* with Symm(2), and hence also with $\mathbf{R}^3$. The contragredient representation of GL(2) on S^* is given by

$$l_g^*\lambda = (g^{-1})^t \lambda g^{-1},$$

for $\lambda \in \mathrm{Symm}(2)$. We note that

$$\langle l_g u, l_g^*\lambda\rangle = \langle u, \lambda\rangle.$$

For $\delta \in \{++, +-, --\}$ and $\alpha \in \mathbf{C}$ we obtain

$$\begin{aligned} \pi_{0,\alpha}^*(g) m_\delta^\alpha(\lambda) &= |\det g|^{-i\alpha} m_\delta^\alpha(\lambda) \\ &= |\det(g^t \lambda g)|^{-\frac{i\alpha}{2}} \\ &= m_\delta^\alpha(g^t \lambda g) \\ &= m_\delta^\alpha(l_{g^{-1}}^* \lambda), \end{aligned}$$

for $\lambda \in \mathcal{O}_\delta$. We see that they generate $\mathrm{Hom}_{\mathrm{GL}(2)}(\mathbf{C}, \mathcal{C}_{bdd}(\mathcal{O}_\delta))$. Hence the result for $\epsilon = 0$ follows from Corollary 1.

To show that there are no non-trivial multipliers for $\epsilon = 1$ we just note that

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

and that it leaves some element of each orbit invariant. For $\mathcal{O}_{++}$ we have

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which implies that $-m(1,1,0) = m(1,1,0)$, i.e. m has to be equal to zero on $\mathcal{O}_{++}$. For $\mathcal{O}_{+-}$ we take

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which implies that $-m(0,0,1) = m(0,0,1)$, i.e. m has to be equal to zero on $\mathcal{O}_{+-}$. Finally, we look at $\mathcal{O}_{--}$.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

which implies that $-m(-1,-1,0) = m(-1,-1,0)$, i.e. m has to be equal to zero on $\mathcal{O}_{--}$. □

3.2 GL(2) × GL(2) acting on $\mathbf{R}^4$

Let us consider $\mathbf{R}^4$, with GL(4) acting in the usual way. Consider the map $\mathrm{GL}(2) \times \mathrm{GL}(2) \to \mathrm{GL}(4)$ given by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \rho \right) \to \begin{pmatrix} a\rho & b\rho \\ c\rho & d\rho \end{pmatrix}.$$

The kernel is $K = \{(\lambda I_2, \lambda^{-1} I_2)\}$, where $\lambda \in \mathbf{R}$ and I_2 is the 2×2-identity matrix. The induced action of $\mathrm{GL}(2) \times \mathrm{GL}(2)$ on $\mathbf{R}^4$ is the same as the natural action on $\mathbf{R}^2 \otimes \mathbf{R}^2$, which in turn is identified with $\mathbf{R}^4$. Another way of portraying $\mathbf{R}^4$ is as M(2).

Theorem 4. *Let* $T : \mathbf{L}^2(\mathbf{R}^4) \to \mathbf{L}^2(\mathbf{R}^4)$ *be a bounded, translation invariant operator, which satisfies the relation*

$$T \circ l_{(g_1,g_2)} = \pi_{\epsilon,\alpha}(g_1)\pi_{\epsilon,\alpha}(g_2)\, l_{(g_1,g_2)} \circ T.$$

for all $g_1, g_2 \in \mathrm{GL}(2)$*, where* $\pi_{\epsilon,\alpha}$ *is given by (2). Then the corresponding multiplier function has the form*

$$m(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = C \operatorname{sgn}(\lambda_1\lambda_4 - \lambda_2\lambda_3)^\epsilon |\lambda_1\lambda_4 - \lambda_2\lambda_3|^{i\alpha},$$

where C *is a constant.*

Proof. Observe first that the representation $(g_1, g_2) \to \pi_{\epsilon,\alpha}(g_1)\pi_{\epsilon,\alpha}(g_2)$ is also a representation of $\mathrm{GL}(2) \times \mathrm{GL}(2)/K$. Hence, we are in a situation where Theorem 2 applies. Transferring the relation to the Fourier transform side gives us the following identity for the multiplier

$$f((g_1, g_2)\lambda) = \operatorname{sgn}(\det g_1)^\epsilon \operatorname{sgn}(\det g_2)^\epsilon |\det g_1|^{i\alpha} |\det g_2|^{i\alpha} f(\lambda), \tag{4}$$

whenever $g_1, g_2 \in \mathrm{GL}(2)$ and $\lambda \in \mathbf{R}^4$. By Corollary 1 all we need to do is to verify that the function m in the statement of Theorem satisfies this invariance relation. Under the mapping $\mathbf{R}^4 \to \mathrm{M}(2)$ the vector $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ maps to the matrix

$$\Lambda = \begin{pmatrix} \lambda_1 & \lambda_3 \\ \lambda_2 & \lambda_4 \end{pmatrix}.$$

In this notation, an element (g_1, g_2) acts by multiplication on both sides:

$$(g_1, g_2)\Lambda = g_2 \Lambda g_1^t.$$

It is now obvious that the function m satisfies the identity (4). This completes the proof. □

3.3 $\mathrm{SO}(p, q) \times \mathbf{R}_+$ acting on $\mathbf{R}^{p+q}$

In light of local isomorphisms of Lie groups

$$SL(2, \mathbb{R}) \approx SO(2, 1),$$

$$SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \approx SO(2, 2),$$

the previous two examples may be explained in a more general setting as follows.

For $p, q \geq 1$, we let $G_1 := SO_0(p, q)$, the identity component of the indefinite orthogonal group

$$O(p, q) = \{g \in GL(p + q, \mathbb{R}) : Q(gx) = Q(x) \text{ for any } x \in \mathbb{R}^{p+q}\},$$

where Q is the quadratic form given by

$$Q(x) := x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2.$$

We shall consider a direct product group

$$G := G_1 \times \mathbb{R}_+,$$

the group acting conformally on the standard flat pseudo-Riemannian manifold $\mathbb{R}^{p,q}$ equipped with $ds^2 = dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2$. We define a family of one dimensional unitary representations of G by

$$\pi_\alpha : G \to \mathbb{C}^\times, \qquad (h, a) \mapsto a^{i\alpha} \qquad (p+q \geq 3),$$

$$\pi_{\alpha,\beta} : G \to \mathbb{C}^\times, \qquad \left(\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}, a \right) \mapsto a^{i\alpha} e^{it\beta} \qquad (p+q = 2),$$

for $\alpha, \beta \in \mathbb{R}$.

We also define bounded functions by

$$Q_+(\lambda)^{i\alpha} := \begin{cases} Q(\lambda)^{i\alpha} & \text{if } Q(\lambda) > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$Q_-(\lambda)^{i\alpha} := \begin{cases} |Q(\lambda)|^{i\alpha} & \text{if } Q(\lambda) < 0 \\ 0 & \text{otherwise} \end{cases}$$

$$Q_+^{(\pm)}(\lambda)^{i\alpha} := \begin{cases} Q(\lambda)^{i\alpha} & \text{if } Q(\lambda) > 0 \text{ and } \pm\lambda_1 > 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\lambda_\pm^{i\alpha} = \begin{cases} |\lambda|^{i\alpha} & \text{if } \pm\lambda > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 5. *Let $p, q \geq 1$. Let $T : L^2(\mathbb{R}^{p+q}) \to L^2(\mathbb{R}^{p+q})$ be a bounded translation invariant operator, which satisfies the relation*

$$T \circ l_g = \begin{cases} \pi_\alpha(g) l_g \circ T & (p+q \geq 2) \\ \pi_{\alpha,\beta}(g) l_g \circ T & (p+q = 2) \end{cases}$$

for all $g \in G$. Then the corresponding multiplier function has the form:

$$m(\lambda) \tag{5}$$
$$= \begin{cases} c_1 Q_+(\lambda)^{-\frac{1}{2}i\alpha} + c_2 Q_-(\lambda)^{-\frac{1}{2}i\alpha} & (p, q \geq 3) \\ c_1 Q_+^{(+)}(\lambda)^{-\frac{1}{2}i\alpha} + c_2 Q_+^{(-)}(\lambda)^{-\frac{1}{2}i\alpha} + c_3 Q_-(\lambda)^{-\frac{1}{2}i\alpha} & (p = 1, q \geq 2) \\ \displaystyle\sum_{\varepsilon_1 = \pm, \varepsilon_2 = \pm} c_{\varepsilon_1, \varepsilon_2} (\lambda_1 + \lambda_2)_{\varepsilon_1}^{-\frac{1}{2}i(\alpha+\beta)} (\lambda_1 - \lambda_2)_{\varepsilon_2}^{-\frac{1}{2}i(\alpha-\beta)} & (p = q = 1) \end{cases}$$

for some constants $c_1, c_2, c_3, c_{\varepsilon_1,\varepsilon_2} \in \mathbb{C}$. The case $p \geq 2$ and $q = 1$ is similar to the second case.

Remark 1. *Here we have treated the connected group* $\mathrm{SO}_0(p,q)$. *The cases* $\mathrm{SO}(p,q)$ *and* $\mathrm{O}(p,q)$ *can be reduced to this one. However, the number of orbits are different. In particular, for* $\mathrm{O}(p,q)$ *with* $p,q \geq 2$ *we have only one orbit and thus obtain a solution unique up to multiplication with a scalar.*

Proof. Consider the natural action of $G = \mathrm{SO}_0(p,q) \times \mathbf{R}_+$ on $\mathbb{R}^{p+q}$. Then, the following union of open G-orbits

$$\begin{array}{ll} \mathcal{O}_+ \cup \mathcal{O}_- & (p,q \geq 3) \\ \mathcal{O}_+^{(+)} \cup \mathcal{O}_+^{(-)} \cup \mathcal{O}_- & (p=1, q \geq 2) \\ \mathcal{O}_+^{(+)} \cup \mathcal{O}_+^{(-)} \cup \mathcal{O}_-^{(+)} \cup \mathcal{O}_-^{(-)} & (p=q=1) \end{array}$$

is dense in $\mathbb{R}^{p+q}$, respectively, where we put

$$\begin{aligned} \mathcal{O}_\pm &:= \{\lambda \in \mathbb{R}^{p+q} : \pm Q(\lambda) > 0\}, \\ \mathcal{O}_+^{(\pm)} &:= \{\lambda \in \mathcal{O}_+ : \pm\lambda_1 > 0\} \qquad (p=1), \\ \mathcal{O}_-^{(\pm)} &:= \{\lambda \in \mathcal{O}_- : \pm\lambda_{p+1} > 0\} \qquad (q=1). \end{aligned}$$

Owing to Corollary 1, Theorem 5 follows if we are able to show that the functions m in equation (5) satisfies the relation

$$m(g^t\lambda) = \begin{cases} \pi_{-\alpha}(g)m(\lambda) & (p+q \geq 2) \\ \pi_{-\alpha,-\beta}(g)m(\lambda) & (p+q=2) \end{cases}$$

for any $g \in G$ on each orbit. A simple computation shows that this is indeed the case. □

4 $\mathrm{O}(m) \times \mathrm{GL}(k,\mathbf{R})$ acting on $\mathbf{R}^{mk}$

This section provides an example of Theorem 2 where the invariance conditions determine multiplier operators up to scalar, even in the setting that (π, V) is not one dimensional and H-orbits are not symmetric.

Let $n = mk$ $(m \geq k)$, and $H := G_1 \times G_2 = \mathrm{O}(m) \times \mathrm{GL}_+(k,\mathbf{R})$. Then H acts on $\mathbf{R}^n \simeq \mathrm{M}(m,k;\mathbf{R})$ by

$$X \mapsto aXb^{-1}$$

for $(a,b) \in H$. We define a subset of $M(m,k;\mathbf{R})$ by

$$\mathcal{O} = \{X \in \mathrm{M}(m,k;\mathbf{R}) : \operatorname{rank} X = k\}.$$

Then $\mathcal{O}$ is open dense in $\mathbf{R}^n \simeq \mathrm{M}(m,k;\mathbf{R})$. Furthermore, if $X \in \mathcal{O}$, then X^tX is positive definite, and in particular $\det(X^tX) > 0$.

For a subset $I \subset \{1,2,\dots,m\}$ with $|I| = k$, we define a function

$$m_I : \mathcal{O} \to \mathbf{R},\ X \mapsto \frac{\det(X_{ij})_{i\in I, 1\leq j\leq k}}{\det(X^tX)^{\frac{1}{2}}}, \tag{6}$$

where X_{ij} is the (i,j)'th minor. We shall regard m_I as a bounded function on $\mathbf{R}^n$.

Theorem 6. *The set of multipliers $\{m_I\}$ defines a bounded translation invariant operator*

$$T : L^2(\mathbf{R}^n) \to \wedge^k(\mathbf{R}^m) \otimes L^2(\mathbf{R}^n)$$

which is characterized, up to a scalar, by the intertwining property (1). Here, we regard the k-th exterior tensor $\wedge^k(\mathbf{R}^m)$ as an H-module, by extending the natural action of $\mathrm{O}(m)$ on $\wedge^k(\mathbf{R}^m)$ trivially to the second factor $\mathrm{GL}_+(k, \mathbf{R})$.

Remark 2. *If $k = 1$ then $\det(X^tX)^{\frac{1}{2}}$ is nothing but the norm $|X|$ of a vector $X \in \mathbf{R}^n$ and $m_I(X) = \frac{X_i}{|X|}$ for $I = \{i\}$. Thus, Theorem in the case $k = 1$ corresponds to Stein's Theorem characterizing the usual Riesz transforms.*

Proof. We shall apply Theorem 2. It follows from the Gram-Schmidt orthogonalization procedure that H acts transitively on $\mathcal{O}$. Since $\mathcal{O}$ is open dense in $\mathbf{R}^n$, Theorem 6 is a consequence of the following lemma. □

Lemma 1. *For a representation π of $\mathrm{O}(m)$, we shall denote by $\widetilde{\pi}$ the extention of π to H by letting $\mathrm{GL}_+(k, \mathbf{R})$ act trivially. For any irreducible (finite dimensional) representation π of $\mathrm{O}(m)$,*

$$\mathrm{Hom}_H(\widetilde{\pi}, C_{bdd}(\mathcal{O})) \leq 1.$$

If π is the natural representation of $\mathrm{O}(m)$ on the exterior algebra $\wedge^k(\mathbf{R}^m)$, then

$$\mathrm{Hom}_H(\widetilde{\pi}, C_{bdd}(\mathcal{O})) = 1.$$

and the image of $\widetilde{\pi}$ in $C_{bdd}(\mathcal{O})$ is spanned by the basis $\{m_I : |I| = k\}$ as a complex vector space.

Remark 3. *In this case the dimension of the representation space is no longer one dimensional so Corollary 1 does not apply. Also the orbit is not a reductive symmetric space so it does not fit with example 3 either. Nevertheless, Theorem 6 asserts that one can characterize invariant multipliers up to scalar by the invariance condition. The idea of the proof is to show that there is a reductive symmetric space for which the dimension of the space of homomorphisms dominate the dimension of the space of homomorphisms for our space.*

Proof. We write $\mathcal{C}(\mathcal{O})^{G_2}$ for the set of G_2-invariant continuous functions of $\mathcal{O}$. Then, $\mathcal{C}(\mathcal{O})^{G_2}$ is a submodule of $\mathcal{C}(\mathcal{O})$, and we have a natural bijection:

$$\mathrm{Hom}_H(\widetilde{\pi}, \mathcal{C}(\mathcal{O})) \simeq \mathrm{Hom}_{G_1}(\pi, \mathcal{C}(\mathcal{O})^{G_2}).$$

Let us consider the right-hand side. To see $\mathcal{O}$ as a homogeneous space of $H = G_1 \times G_2$, we note that the isotropy subgroup L at $\begin{pmatrix} I_k \\ O \end{pmatrix} \in \mathcal{O}$, is given by

$$\begin{aligned} L &= \left\{ \left(\begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix}, b \right) : b \in \mathrm{SO}(k), c \in \mathrm{O}(m-k) \right\} \\ &\simeq \mathrm{SO}(k) \times \mathrm{O}(m-k). \end{aligned}$$

Then we shall identify $\mathcal{O}$ with the homogeneous space H/L.

Let $\iota : G_1 \to H$, $a \mapsto (a, \mathrm{I}_k)$ be the natural injection. Then, it is not difficult to see that the pull-back ι^* induces an isomorphism of G_1-modules:

$$\mathcal{C}(H/L)^{G_2} \simeq \mathcal{C}(G_1/\iota^{-1}((G_1 \times \mathrm{I}_k) \cap L(\mathrm{I}_n \times G_2))).$$

In our setting, $L(\mathrm{I}_k \times G_2) = (\mathrm{SO}(k) \times \mathrm{O}(m-k)) \times \mathrm{GL}_+(k, \mathbf{R})$, and therefore

$$\mathcal{C}(\mathcal{O})^{G_2} \simeq \mathcal{C}(\mathrm{O}(m)/(\mathrm{SO}(k) \times \mathrm{O}(m-k))).$$

Thus we have shown

$$\mathrm{Hom}_H(\widetilde{\pi}, \mathcal{C}(\mathcal{O})) \simeq \mathrm{Hom}_{\mathrm{O}(m)}(\pi, \mathcal{C}(\mathrm{O}(m)/(\mathrm{SO}(k) \times \mathrm{O}(m-k)))).$$

Since $\mathrm{O}(m)/(\mathrm{SO}(k) \times \mathrm{O}(m-k))$ is a reductive symmetric space, the dimension of the right-hand side is not greater than one. Hence,

$$\dim \mathrm{Hom}_H(\widetilde{\pi}, \mathcal{C}_{bdd}(\mathcal{O})) \leq \dim \mathrm{Hom}_H(\widetilde{\pi}, \mathcal{C}(\mathcal{O})) \leq 1.$$

This shows the first statement. The second statement easily follows from the explicit construction of the base m_I. □

5 $\mathbf{L}^p$-boundedness

In this section we will consider the question of $\mathbf{L}^p$-boundedness for some of the operators that have appeared in the examples. Standard multiplier theory tells us that a multiplier operator bounded on $\mathbf{L}^p(\mathbf{R}^n)$ must also be bounded on $\mathbf{L}^2(\mathbf{R}^n)$, see for example [H] Corollary 1.3. There is no general theory for the converse statement. Hence, we are tempted to ask for which set of p's the multiplier operators we have seen remain bounded.

Theorem 7. *The operators characterized by Theorem 3 in section 3.1 are bounded only on* $\mathbf{L}^2(\mathbf{R}^3)$.

Proof. If the multiplier operator with multiplier m_δ^β in section 3.1 is bounded on $\mathbf{L}^p(\mathbf{R}^3)$ then also the operator corresponding to $m_{\bar{\delta}}^{-\beta}$ is bounded on the same space, because it is obtained by taking the complex conjugate which preserves $\mathbf{L}^p$. Composing the operators shows that the operator, given by the characteristic function of the orbit as multiplier, must also be bounded on $\mathbf{L}^p(\mathbf{R}^3)$. The case $\delta = +-$ can be reduced to the others (and the argument below) by taking the identity operator minus the operator. For $\delta = ++$ or $= --$, it is easy to see that the orbit is a rotated cone. Now, by taking the intersection with a suitable hyperplane we will see that p has to be equal to 2. This follows from deLeeuw's Theorem [T], Theorem 2.4, which says that the restriction of an $\mathbf{L}^p$-multiplier to a hyperplane is also an $\mathbf{L}^p$-multiplier, and Fefferman's result that the characteristic function for the unit ball is a bounded multiplier only for $p = 2$ if the dimension is at least 2, see [F1]. □

In the same way we find that

Theorem 8. *The operators characterized by Theorem 4 in section 3.2 are bounded only on* $\mathbf{L}^2(\mathbf{R}^4)$.

In this case the relevant operator, after a suitable change of variables, is the one corresponding to the characteristic function of the set $\{\lambda; \lambda_1^2 + \lambda_2^2 \geq \lambda_3^2 + \lambda_4^2\}$. Here we do not intersect with a hyperplane to get a contradiction, but a plane of codimension 2.

It also follows in a similar manner that

Theorem 9. 1) *The multiplier operator given by the function* m *defined by equation (5) in Theorem 5 is bounded only on* $\mathbf{L}^2(\mathbf{R}^{p+q})$, *if* $p + q \geq 3$.

2) *If* $p + q = 2$, *the operator is bounded on* $\mathbf{L}^r(\mathbf{R}^2)$, *for all* $1 < r < \infty$.

Proof. When $p + q \geq 3$ the guiding operator is the one given by the characteristic function of the set $\{\lambda; \lambda_1^2 + \ldots + \lambda_p^2 \geq \lambda_{p+1}^2 + \ldots \lambda_{p+q}^2\}$, where we might assume that $p \geq q$. The first result then follows as before.

If $p = q = 1$ we are considering the multiplier

$$\sum_{\varepsilon_1=\pm,\varepsilon_2=\pm} c_{\varepsilon_1,\varepsilon_2}(\lambda_1 + \lambda_2)_{\varepsilon_1}^{-\frac{1}{2}i(\alpha+\beta)}(\lambda_1 - \lambda_2)_{\varepsilon_2}^{\frac{1}{2}i(\alpha-\beta)}.$$

We want to show that the connected multiplier operator is bounded on $\mathbf{L}^r(\mathbf{R}^2)$ for all $1 < r < \infty$. To do this it is enough to consider the factors separately

$$\begin{aligned} m_{1,\epsilon}^{\alpha}(\lambda) &= (\lambda_1 + \lambda_2)_{\epsilon}^{i\alpha}, \\ m_{2,\epsilon}^{\alpha}(\lambda) &= (\lambda_1 - \lambda_2)_{\epsilon}^{i\alpha}. \end{aligned}$$

Clearly, they are all simple rotations of the multiplier

$$m(\lambda) = \begin{cases} |\lambda_1|^{i\alpha} & \text{if } \lambda_1 > 0 \\ 0 & \text{otherwise.} \end{cases}$$

But this multiplier is just the identity in one variable and a one-dimensional multiplier, well-known to be bounded on all $\mathbf{L}^r$ for $1 < r < \infty$, in the second variable, see [S] page 96. Hence, the resulting operator is also bounded on $\mathbf{L}^r$ for $1 < r < \infty$, which proves the second statement of the Theorem. □

It is not known to the authors for which p the operators characterized in Theorem 6 are L^p-bounded except for the special case $k = 1$. We note that if $k = 1$ the transforms are nothing but the Riesz transforms, which are well-known to be bounded on $\mathbf{L}^p$ for $1 < p < \infty$, see [S],page 57 and Theorem 3, or [T], page 269.

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