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# MONOMIAL RELIZATION OF CRYSTAL BASES FOR SPECIAL LINEAR LIE ALGEBRAS

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ABSTRACT. In this article, we give a new realization of crystal bases for finite dimensional irreducible modules over special linear Lie algebras in terms of the monomials introduced by Nakajima. We also discuss the connection between this monomial realization and the tableau realization.

## INTRODUCTION

The theory of *crystal basis* for integrable modules over quantum groups developed by Kashiwara [6, 7] has played a important role in representation theory or mathematical physics. Roughly speaking, crystal bases are bases at $q = 0$ and have a structure of colored oriented graphs, called the *crystal graphs.* Crystal graphs have many nice combinatorial properties reflecting the internal structure of integrable modules. Moreover, crystal bases have a remarkably nice behavior with respect to taking the tensor product. Therefore, it is important to give the explicit crystal structure of representations.

In [13], Littelmann gave a description of crystal bases for all symmetrizable Kac-Moody algebras using the *path model theory* [14, 15]. In [9], Kashiwara and Nakashima gave an explicit realization of crystal bases for finite dimensional irreducible modules using Young tableaux in $\mathfrak{gl}_n$ and their variants in the classical Lie algebras. In [3, 4], Kang, Kashiwara, Misra, Miwa, Nakashima and Nakayashiki developed the theory of *perfect crystals* for general quantum affine algebras and gave a realization of crystal graphs for irreducible highest weight modules over classical quantum affine algebras with arbitrary higher levels in terms of *paths.* Moreover, the crystal bases for basic representations for quantum affine algebras are characterized as the sets of *reduced proper Young walls* [2] and the crystal bases for the classical Lie algebras were realized as the set of reduced proper Young walls satisfying some conditions which appears as a connected component of the crystal basis of the basic representation over affine Lie algebras when we remove all 0-arrows [5].

In [10], Kashiwara and Saito gave a geometric realization of the crystal graph $B(\infty)$ of $U_q^-(\mathfrak{g})$ as the set of irreducible components of a lagrangian subvariety $\mathcal{L}$ of the quiver variety $\mathfrak{M}$ and in [19], Saito extended their idea to the crystal base $B(\lambda)$ of irreducible highest weight

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modules of $U_q(\mathfrak{g})$. In [17], while studying the structure of quiver varieties, H. Nakajima discovered that one can define a crystal structure on the set of irreducible components of a lagrangian subvariety $\mathfrak{Z}$ of the quiver variety $\mathfrak{M}$. These irreducible components are identified with certain monomials, and the action of Kashiwara operators can be interpreted as multiplication by monomials. Moreover, in [8] and [18], M. Kashiwara and H. Nakajima gave a crystal structure on the set $\mathcal{M}$ of monomials and they showed that the connected component $\mathcal{M}(\lambda)$ of $\mathcal{M}$ containing a highest weight vector $M$ with a dominant integral weight $\lambda$ is isomorphic to the irreducible highest weight crystal $B(\lambda)$. Therefore, a natural question arises: for each dominant integral weight $\lambda$, can we give an explicit characterization of the monomials in $\mathcal{M}(\lambda)$?

In this paper, for any dominant integral weight $\lambda$, we give an explicit description of the crystal $\mathcal{M}(\lambda)$ for special linear Lie algebras. In addition, we discuss the connection between the monomial realization and tableau realization of crystal bases given by Kashiwara and Nakashima. More precisely, let $T(\lambda)$ denote the crystal consisting of semistandard tableaux of shape $\lambda$. Then we show that there exists a canonical crystal isomorphism between $\mathcal{M}(\lambda)$ and $T(\lambda)$, which has a very natural interpretation in the language of insertion scheme.

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## 1. Nakajama's monomials

Let $I$ be a finite index set and let $A = (a_{ij})_{i,j \in I}$ be a *generalized Cartan matrix*. We denote by $U_q(\mathfrak{g})$ the quantum group associated with the Cartan datum $(A, P^\vee, P, \Pi^\vee, \Pi)$, where $\mathfrak{h}$ is the Cartan subalgebra, $P^\vee$ is the *dual weight lattice*, $P = \{\lambda \in \mathfrak{h}^* | \lambda(P^\vee) \subset \mathbf{Z}\}$ is the *weight lattice*, $\Pi^\vee = \{h_i | i \in I\}$ is the set of simple coroots, and $\Pi = \{\alpha_i | i \in I\}$ is the set of simple roots. We also denote by $\Lambda_i \in \mathfrak{h}^*$ $(i \in I)$ the *fundamental weights*. See [1] for further details.

For a $U_q(\mathfrak{g})$-module $M$ in the category $\mathcal{O}_{int}$, there exists a unique *crystal base* $(L, B)$, which has nice combinatorial properties reflecting the internal structure of $M$. See for example [1, 6, 7]. In this section, we recall the crystal structure on the set of monomials discovered by H. Nakajima [18]. Our exposition follows that of M. Kashiwara [8].

Let $\mathcal{M}$ be the set of monomials in the variables $Y_i(n)$ for $i \in I$ and $n \in \mathbf{Z}$. Here, a typical elements $M$ of $\mathcal{M}$ has the form $M = Y_{i_1}(n_1)^{a_1} \cdots Y_{i_r}(n_r)^{a_r}$, where $i_k \in I, n_k, a_k \in \mathbf{Z}$ for $k = 1, \cdots, r$. Since $Y_i(n)$'s are commuting variables, we may assume that $n_1 \leq n_2 \leq \cdots \leq n_r$.

For a monomial $M = Y_{i_1}(n_1)^{a_1} \cdots Y_{i_r}(n_r)^{a_r}$, we define

$$(1.1)\qquad \begin{aligned} \mathrm{wt}(M) &= \sum_{k=1}^{r} a_k \Lambda_{i_k} = a_1\Lambda_{i_1} + \cdots a_r\Lambda_{i_r}, \\ \varphi_i(M) &= \max\big(\{ \sum_{\substack{k=1 \\ i_k=i}}^{s} a_k \mid 1 \le s \le r\} \cup \{0\}\big), \\ \varepsilon_i(M) &= \max\big(\{ - \sum_{\substack{k=s+1 \\ i_k=i}}^{r} a_k \mid 1 \le s \le r-1\} \cup \{0\}\big). \end{aligned}$$

It is easy to verify that $\varphi_i(M) \ge 0, \varepsilon_i(M) \ge 0$, and $\langle h_i, \mathrm{wt}M\rangle = \varphi_i(M) - \varepsilon_i(M)$.

First, we define

$$(1.2)\qquad \begin{aligned} n_f &= \text{smallest } n_s \text{ such that } \varphi_i(M) = \sum_{\substack{k=1 \\ i_k=i}}^{s} a_k, \\ n_e &= \text{largest } n_s \text{ such that } \varepsilon_i(M) = - \sum_{\substack{k=s+1 \\ i_k=i}}^{r} a_k. \end{aligned}$$

In addition, choose a set $C = (c_{ij})_{i \ne j}$ of integers such that $c_{ij} + c_{ji} = 1$, and define

$$A_i(n) = Y_i(n) Y_i(n+1) \prod_{j \ne i} Y_j(n + c_{ji})^{\alpha_i(h_j)}.$$

Now, the *Kashiwara operators* $\tilde{e}_i$, $\tilde{f}_i$ $(i \in I)$ on $\mathcal{M}$ are defined as follows:

$$(1.3)\qquad \tilde{f}_i(M) = \begin{cases} 0 & \text{if } \varphi_i(M) = 0, \\ A_i(n_f)^{-1} M & \text{if } \varphi_i(M) > 0, \end{cases} \qquad \tilde{e}_i(M) = \begin{cases} 0 & \text{if } \varepsilon_i(M) = 0, \\ A_i(n_e) M & \text{if } \varepsilon_i(M) > 0. \end{cases}$$

Then the maps $\mathrm{wt} : \mathcal{M} \to P$, $\varphi_i, \varepsilon_i : \mathcal{M} \to \mathbf{Z} \cup \{-\infty\}$, $\tilde{e}_i, \tilde{f}_i : \mathcal{M} \to \mathcal{M} \cup \{0\}$ define a $U_q(\mathfrak{g})$-crystal structure on $\mathcal{M}$ [8, 18].

Moreover, we have

**Proposition 1.1.** [8] *Let $M$ be a monomial with weight $\lambda$ such that $\tilde{e}_i M = 0$ for all $i \in I$, and let $\mathcal{M}(\lambda)$ be the connected component of $\mathcal{M}$ containing $M$. Then there exists a crystal isomorphism from $\mathcal{M}(\lambda)$ to $B(\lambda)$ sending $M$ to $v_\lambda$.*

## 2. Characterization of $\mathcal{M}(\lambda)$

In this section, we give an explicit characterization of the crystal $\mathcal{M}(\lambda)$ for special linear Lie algebras. Let $I = \{1, \cdots, n\}$ and let

$$A = (a_{ij})_{i,j\in I} = \begin{pmatrix} 2 & -1 & \cdots & 0 & 0 \\ -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & \cdots & -1 & 2 \end{pmatrix}$$

be the generalized Cartan matrix of type $A_n$. We define by $U_q(\mathfrak{g}) = U_q(\mathfrak{sl}_{n+1})$ the corresponding quantum group. For simplicity, we take the set $C = (c_{ij})_{i\neq j}$ to be $c_{ij} = 0$ if $i > j$, 1 if $i < j$, and set $Y_0(m)^{\pm 1} = Y_{n+1}(m)^{\pm 1} = 1$ for all $m \in \mathbf{Z}$. Then for $i \in I$ and $m \in \mathbf{Z}$, we have

$$A_i(m) = Y_i(m)Y_i(m+1)Y_{i-1}(m+1)^{-1}Y_{i+1}(m)^{-1}. \tag{2.1}$$

To characterize $\mathcal{M}(\lambda)$, we first focus on the case when $\lambda = \Lambda_k$. Let $M_0 = Y_k(m)$ for $m \in \mathbf{Z}$. By (1.2), we see that $\tilde{e}_i M_0 = 0$ for all $i \in I$ and the connected component containing $M_0$ is isomorphic to $B(\Lambda_k)$ over $U_q(\mathfrak{g})$. For simplicity, we will take $M_0 = Y_k(0)$, even if that does not make much difference.

**Proposition 2.1.** *For $k = 1, \cdots, n$, let $M_0 = Y_k(0)$ be a highest weight vector of weight $\Lambda_k$. Then the connected component $\mathcal{M}(\Lambda_k)$ of $\mathcal{M}$ containing $M_0$ is characterized as*

$$\mathcal{M}(\Lambda_k) = \left\{ \prod_{j=1}^{r} Y_{a_j}(m_{j-1})^{-1}Y_{b_j}(m_j) \;\middle|\; \begin{array}{l} \text{(i) } 0 \le a_1 < b_1 < a_2 < \cdots < a_r < b_r \le n+1, \\ \text{(ii) } k = m_0 > m_1 > \cdots > m_{r-1} > m_r = 0, \\ \text{(iii) } a_j + m_{j-1} = b_j + m_j \textit{ for all } j = 1, \cdots, r \le k. \end{array} \right\}.$$

*Remark* 2.2. If we take $M_0 = Y_k(N)$, then we have only to modify the condition for $m_j$'s as follows:

$$k + N = m_0 > m_1 > \cdots > m_{r-1} > m_r = N.$$

For $i \in I$ and $m \in \mathbf{Z}$, we introduce new variables

$$X_i(m) = Y_{i-1}(m+1)^{-1}Y_i(m). \tag{2.2}$$

Using this notation, every monomial $M = \prod_{j=1}^{r} Y_{a_j}(m_{j-1})^{-1}Y_{b_j}(m_j) \in \mathcal{M}(\Lambda_k)$ may be written as

$$M = \prod_{j=1}^{r} X_{a_j+1}(m_{j-1}-1)X_{a_j+2}(m_{j-1}-2)\cdots X_{b_j}(m_j).$$

For example, we have $M_0 = Y_k(0) = X_1(k-1)X_2(k-2)\cdots X_k(0)$.

Now, it is straightforward to verify that we have another characterization of the crystal $\mathcal{M}(\Lambda_k)$.

**Corollary 2.3.** *For $k = 1, \cdots, n$, we have*

$$\mathcal{M}(\Lambda_k) = \{X_{i_1}(k-1)X_{i_2}(k-2)\cdots X_{i_k}(0) \mid 1 \le i_1 < i_2 < \cdots < i_k \le n+1\}.$$

*Remark* 2.4. If we take $M_0 = Y_k(N)$, then we need to replace $X_i(m)$ by $X_i(m+N)$. That is,

$$\mathcal{M}(\Lambda_k) = \{X_{i_1}(N+k-1)X_{i_2}(N+k-2)\cdots X_{i_k}(N) \mid 1 \le i_1 < i_2 < \cdots < i_k \le n+1\}.$$

We now consider the general case.

**Definition 2.5.** Set $M = \prod_t Y_{a_t}(m_t)^{-1}Y_{b_t}(n_t)$ with $a_t + m_t = b_t + n_t$.

(a) For each $k = 0, \cdots, n-1$, we define $M(k)^+$ to be the product of $Y_{a_t}(m_t)^{-1}Y_{b_t}(n_t)$'s in $M$ with $n_t = k$; that is,

$$M(k)^+ = \prod_{t:n_t=k} Y_{a_t}(m_t)^{-1}Y_{b_t}(n_t) = \prod_t Y_{a_t}(m_t)^{-1}Y_{b_t}(k).$$

(b) For each $k = 1, \cdots, n$, we define $M(k)^-$ to be the product of $Y_{a_t}(m_t)^{-1}Y_{b_t}(n_t)$'s in $M$ with $m_t = k$; that is,

$$M(k)^- = \prod_{t:m_t=k} Y_{a_t}(m_t)^{-1}Y_{b_t}(n_t) = \prod_t Y_{a_t}(k)^{-1}Y_{b_t}(n_t).$$

Now, for $M(k)^+ = \prod_t Y_{a_t}(m_t)^{-1}Y_{b_t}(k)$, we denote by $\lambda^+(M(k))$ the sequence $(b_{i_1}, b_{i_2}, \cdots, b_{i_r})$ whose terms are arranged in such a way that $n+1 \ge b_{i_1} \ge b_{i_2} \ge \cdots \ge b_{i_r}$. Similarly, for $M(k)^- = \prod_t Y_{a_t}(k)^{-1}Y_{b_t}(n_t)$, we denote by $\lambda^-(M(k))$ the sequence $(a_{j_1}, a_{j_2}, \cdots, a_{j_s})$ whose terms are arranged in such a way that $n+1 > a_{j_1} \ge a_{j_2} \ge \cdots \ge a_{j_s}$.

**Definition 2.6.** Let $(\lambda_1, \cdots, \lambda_r)$ and $(\mu_1, \cdots, \mu_s)$ be the sequences such that

$$\lambda_i \ge \lambda_{i+1} \ (1 \le i \le r-1), \quad \mu_j \ge \mu_{j+1} \ (1 \le j \le s-1).$$

We define $(\lambda_1, \cdots, \lambda_r) \prec (\mu_1, \cdots, \mu_s)$ if

$$r \le s \text{ and } \lambda_i < \mu_i \text{ for all } i = 1, \cdots, r.$$

**Theorem 2.7.** *Let $\lambda = a_1\Lambda_1 + \cdots + a_n\Lambda_n$ be a dominant integral weight and let $M_0 = Y_1(0)^{a_1}\cdots Y_n(0)^{a_n}$ be a highest weight vector of weight $\lambda$ in $\mathcal{M}$. The connected component $\mathcal{M}(\lambda)$ in $\mathcal{M}$ containing $M_0$ is characterized as the set of monomials of the form*

$$\prod_i Y_{a_t}(m_t)^{-1}Y_{b_t}(n_t)$$

*with $a_t + m_t = b_t + n_t$ satisfying the following conditions:*

(i) *$\lambda^+(M(k)) \prec \lambda^-(M(k))$ for $k = 1, \cdots, n-1$.*

(ii) *If $\lambda^+(M(k)) = (b_{i_1}, b_{i_2}, \cdots, b_{i_r})$ and $\lambda^-(M(k)) = (a_{j_1}, a_{j_2}, \cdots, a_{j_s})$, then $s - r = a_k$.*

*Remark* 2.8. The crystal $\mathcal{M}(\lambda)$ is obtained by multiplying $a_k$-many monomials in $\mathcal{M}(\Lambda_k)$ $(k = 1, \cdots, n)$. That is,

$$\mathcal{M}(\lambda) = \{M = M_{1,1}\cdots M_{1,a_1}M_{2,1}\cdots M_{n,a_n} \mid M_{k,l} \in \mathcal{M}(\Lambda_k) \text{ for } 1 \le k \le n,\ 1 \le l \le a_k\}.$$

**Example 2.9.** Let $\lambda$ be a dominant integral weight $\Lambda_1 + 2\Lambda_2 + \Lambda_3$ of $A_4$ and let $M = Y_1(0)Y_1(1)Y_1(2)^{-1}Y_2(1)^{-1}Y_3(0)^3$. Then $M$ can be expressed as

$$M = Y_0(3)^{-1}Y_3(0)Y_1(2)^{-1}Y_3(0)Y_0(2)^{-1}Y_1(1)Y_2(1)^{-1}Y_3(0)Y_0(1)^{-1}Y_1(0).$$

Therefore, we have

$$M(0)^+ = Y_0(3)^{-1}Y_3(0)Y_1(2)^{-1}Y_3(0)Y_2(1)^{-1}Y_3(0)Y_0(1)^{-1}Y_1(0),$$
$$M(1)^+ = Y_0(2)^{-1}Y_1(1), \quad M(2)^+ = M(3)^+ = 1,$$

and

$$\begin{aligned}
M(1)^- &= Y_2(1)^{-1}Y_3(0)Y_0(1)^{-1}Y_1(0),\\
M(2)^- &= Y_1(2)^{-1}Y_3(0)Y_0(2)^{-1}Y_1(1),\\
M(3)^- &= Y_0(3)^{-1}Y_3(0),\\
M(4)^- &= 1.
\end{aligned}$$

It is easy to see that $M$ satisfies the conditions of Theorem 2.7. Therefore, $M \in \mathcal{M}(\lambda)$.

**Definition 2.10.** Set $M = \prod_j X_{b_j}(n_j)$.

(i) For each $k = 1, \cdots, n-1$, we define $M(k)$ by the monomial obtained by multiplying all $X_{b_j}(n_j)$ with $n_j = k$ in $M$, that is,

$$M(k) = \prod_{j:n_j=k} X_{b_j}(n_j) = \prod_j X_{b_j}(k).$$

(ii) For $M(k) = \prod_j X_{b_j}(k)$, we define by $\lambda(M(k))$ the sequence $(b_{j_1}, b_{j_2}, \cdots, b_{j_s})$ whose terms are arranged in such a way that $n+1 \geq b_{j_1} \geq b_{j_2} \geq \cdots \geq b_{j_s}$.

**Corollary 2.11.** *Let $\lambda = a_1\Lambda_1 + \cdots + a_n\Lambda_n$. Then $\mathcal{M}(\lambda)$ is expressed as the set of monomials*

$$M = \prod_{\substack{1\leq i\leq n+1\\ 0\leq j\leq n-1}} X_i(j)^{m_{ij}}$$

*such that*

(i) *for each $j = 0, 1, \cdots, n-1$,*

$$\sum_{i=1}^{n+1} m_{ij} = a_{j+1} + \cdots + a_n,$$

(ii) *for each $j = 1, \cdots, n-1$, $\lambda(M(j)) \prec \lambda(M(j-1))$.*

Now, consider the condition (ii) in Corollary 2.11. For $M = \prod_{\substack{1\leq i\leq n+1\\ 0\leq j\leq n-1}} X_i(j)^{m_{ij}}$, there are $m_{i,j}$ -many $i$ entries in the sequence $\lambda(M(j))$. Therefore, the condition $\lambda(M(j)) \prec \lambda(M(j-1))$ implies that

$$\begin{aligned}
&m_{1,n} = 0, \quad m_{ij} = 0 \quad \text{for } 2 \leq i \leq n+1,\ n-i+2 \leq j \leq n,\\
&\sum_{k=i}^{n+1} m_{k,j} \leq \sum_{k=i+1}^{n+1} m_{k,j-1} \quad \text{for } i = 1, \cdots, n+1,\ j = 1, \cdots, n.
\end{aligned} \tag{2.3}$$

Therefore, Corollary 2.11 is expressed as follows:

**Corollary 2.12.** *Let $\lambda = a_1\Lambda_1 + \cdots + a_n\Lambda_n$. Then $\mathcal{M}(\lambda)$ is expressed as the set of monomials*

$$M = \prod_{\substack{1\leq i\leq n+1\\ 0\leq j\leq n-1}} X_i(j)^{m_{ij}}$$

*such that*

(i) $m_{1,n} = 0,\ \ m_{ij} = 0 \quad for\ 2 \le i \le n+1,\ n-i+2 \le j \le n,$

(ii) $\sum_{i=1}^{n+1} m_{ij} = a_{j+1} + \cdots + a_n \quad for\ each\ j = 0, 1, \cdots, n-1,$

(iii) $\sum_{k=i}^{n+1} m_{k,j} \le \sum_{k=i+1}^{n+1} m_{k,j-1} \quad for\ i = 1, \cdots, n+1,\ j = 1, \cdots, n.$

**Example 2.13.** Let $\lambda$ be a dominant integral weight $\Lambda_1 + 2\Lambda_2 + \Lambda_3$ of $A_4$ and let $M$ be a monomial $Y_1(0)Y_1(1)Y_1(2)^{-1}Y_2(1)^{-1}Y_3(0)^3$ given in Example 2.9. Then $M$ can be expressed as

$$X_1(2)X_2(1)^2X_1(1)X_3(0)^3X_1(0)$$

and so it is easy to see that $M$ satisfies the conditions of Corollary 2.12. Therefore, $M \in \mathcal{M}(\lambda)$.

## 3. The connection with Young tableaux

In this section, we give the correspondence between monomial realization and tableau realization of crystal base for the classical Lie algebra $\mathfrak{g} = A_n$. To prove the results in this section, we will adopt the expression of monomials given in Corollary 2.11.

Before we give the correspondence between monomial realization and tableau realization, we introduce certain tableaux with given shape which is different from Young diagram given by Kashiwara and Nakashima.

**Definition 3.1.** (i) We define a *reverse Young diagram* to be a collection of boxes in right-justified rows with a weakly decreasing number of boxes in each row from bottom to top.

(ii) We define a (reverse) *tableau* by a reverse Young diagram filled with positive integers.

(iii) A (reverse) tableau $S$ is called a (reverse) *semistandard tableau* if the entries in $S$ are weakly increasing from left right in each row and strictly increasing from top to bottom in each column.

Note that a reverse Young diagram is just a diagram obtained by reflecting Young diagram to the origin. For a dominant integral weight $\lambda$, let $S(\lambda)$ (resp. $T(\lambda)$) be the set of all (reverse) semistandard tableaux (resp. semistandard tableaux) of shape $\lambda$ with entries on $\{1, 2, \cdots, n\}$, which is realized as crystal basis of finite dimensional irreducible modules [9, 12]. For the fundamental weight $\Lambda_k$ $(k = 1, \cdots, n)$, we have $T(\Lambda_k) = S(\Lambda_k)$.

Let $\lambda = a_1\Lambda_1 + \cdots + a_n\Lambda_n$ be a dominant integral weight and let $M$ be a monomial in $\mathcal{M}(\lambda)$. Then $M$ is expressed as

$$M = \prod_{\substack{1 \le i \le n+1 \\ 0 \le j \le n-1}} X_i(j)^{m_{ij}}.$$

We associate a semistandard tableau $S_M$ with $m_{ij}$-many $i$ entries in $(j+1)$-st row (from bottom to top) for $i = 1, \cdots, n+1$, $j = 0, 1, \cdots, n-1$. Indeed, by the condition (ii) of Corollary 2.12, the tableau $S_M$ is of shape $\lambda$. Moreover, the condition (i) and (iii) imply that $S_M$ is semistandard.

Conversely, let $S$ be a tableau of $S(\lambda)$ with $m_{i,j}$-many $i$ entries in the $j$-th row (from bottom to top) for $i = 1, \cdots, n+1$ and $j = 1, \cdots, n$. We associate a monomial

$$M_T = \prod_{\substack{1 \leq i \leq n+1 \\ 1 \leq j \leq n}} X_i(j-1)^{m_{i,j}}.$$

Then since $S$ is semistandard, it is easy to see that $M_T$ satisfies the condition (i)-(iii) of Corollary 2.12. Moreover, we have

**Theorem 3.2.** *Let $\lambda = a_1\Lambda_1 + \cdots + a_n\Lambda_n$ be a dominant integral weight. Then there is a crystal isomorphism $\psi : \mathcal{M}(\lambda) \to S(\lambda)$.*

**Example 3.3.** Let $\lambda$ be a dominant integral weight $\Lambda_1 + 2\Lambda_2 + \Lambda_3$ of $A_3$ and let $M$ be a monomial $Y_1(3)^{-1}Y_2(0)^2Y_3(1)^{-1}$, then it is expressed as

$$M = X_2(2) \cdot (X_3(1)X_1(1)^2) \cdot (X_4(0)^2X_2(0)^2).$$

Then we have the semistandard tableau

$$S_M = \begin{array}{|c|c|c|c|} \cline{4-4} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 2 \\ \cline{2-4} \multicolumn{1}{c|}{} & 1 & 1 & 3 \\ \hline 2 & 2 & 4 & 4 \\ \hline \end{array} \in S(\Lambda_1 + 2\Lambda_2 + \Lambda_3).$$

We have the following proposition between $S(\lambda)$ and $T(\lambda)$.

**Proposition 3.4.** [11, 12] *For a dominant integral weight $\lambda = a_1\Lambda_1 + \cdots + a_n\Lambda_n$, there is a crystal isomorphism $\varphi : S(\lambda) \to T(\lambda)$ for $U_q(A_n)$-module given by*

$$\varphi(S) = S_{n,1} \leftarrow S_{n,2} \leftarrow \cdots \leftarrow S_{n,a_n} \leftarrow S_{n-1,1} \leftarrow \cdots \leftarrow S_{1,a_1},$$

*where $S_{i,j} \in S(\Lambda_i)$ is the column of $S$ of length $i$ $(1 \leq i \leq n,\ 1 \leq j \leq a_i)$ from right to left.*

**Corollary 3.5.** *Let $\lambda = a_1\Lambda_1 + \cdots + a_n\Lambda_n$ be a dominant integral weight. There is a crystal isomorphism $\phi : \mathcal{M}(\lambda) \to T(\lambda)$.*

**Example 3.6.** Let $M$ be a monomial $Y_1(3)^{-1}Y_2(0)^2Y_3(1)^{-1}$ of $A_3$ given in Example 3.3. Then we have

$$\begin{aligned} \phi(M) &= S_{3,1} \leftarrow S_{2,1} \leftarrow S_{2,2} \leftarrow S_{1,1} \\ &= \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array} \leftarrow \begin{array}{|c|} \hline 1 \\ \hline 4 \\ \hline \end{array} \leftarrow \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \leftarrow \begin{array}{|c|} \hline 2 \\ \hline \end{array} \\ &= \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 4 \\ \hline 2 & 2 & 3 & \multicolumn{1}{c}{} \\ \cline{1-3} 4 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} . \end{aligned}$$

Conversely, let $T$ be a tableau of $T(\Lambda_1 + 2\Lambda_2 + \Lambda_3)$

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 4 \\ \hline 2 & 2 & 3 & \multicolumn{1}{c}{} \\ \cline{1-3} 4 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \; .$$

By applying the reverse bumping rule to the entries from bottom to top and from right to left, we have the following sequence

$$(2, 3, 4, 1, 4, 1, 2, 2).$$

Therefore, we have

$$S_{3,1} = \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array}, \quad S_{2,1} = \begin{array}{|c|} \hline 1 \\ \hline 4 \\ \hline \end{array}, \quad S_{2,2} = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \text{ and } S_{1,1} = \boxed{2},$$

and since

$$\psi^{-1}(S_{3,1}) = X_2(2)X_3(1)X_4(0),\ \psi^{-1}(S_{2,1}) = X_1(1)X_4(0),$$
$$\psi^{-1}(S_{2,2}) = X_1(1)X_2(0),\ \psi^{-1}(S_{1,1}) = X_2(0),$$

we have

$$\begin{aligned} \varphi^{-1}(T) &= \psi^{-1}(S_{3,1})\psi^{-1}(S_{2,1})\psi^{-1}(S_{2,2})\psi^{-1}(S_{1,1}) \\ &= Y_1(3)^{-1}Y_2(0)^2Y_3(1)^{-1}. \end{aligned}$$

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