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Kyoto University
Derived Categories in Representation Theory

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We survey recent methods of derived categories in the representation theory of algebras.

1 Triangulated Categories and Brown Representability

Definition 1.1 A triangulated category $\mathcal{C}$ is an additive category together with

(1) an autofunctor $T : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$ (i.e. there is $T^{-1}$ such that $T \circ T^{-1} = T^{-1} \circ T = 1_{\mathcal{C}}$) called the translation, and
(2) a collection $T$ of sextuples $(X, Y, Z, u, v, w)$:

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$$

called (distinguished) triangles. These data are subject to the following four axioms:

(TR1) (1) Every sextuple $(X, Y, Z, u, v, w)$ which is isomorphic to a (distinguished) triangle is a (distinguished) triangle.
(2) Every morphism $u : X \to Y$ is embedded in a (distinguished) triangle

$$(1) \quad X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$$

(3) For any $X \in \mathcal{C}$,

$$X \xrightarrow{1} X \to 0 \to T(X)$$

is a (distinguished) triangle
(TR2) A sextuple
\[ X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X) \]
is a (distinguished) triangle if and only if
\[ Y \xrightarrow{u} Z \xrightarrow{w} T(X) \xrightarrow{-T(u)} T(Y) \]
is a (distinguished) triangle.

(TR3) For any (distinguished) triangles \((X, Y, Z, u, v, w), (X', Y', Z', u', v', w')\) and a commutative diagram
\[
\begin{array}{ccc}
X & \xrightarrow{u} & Y \xrightarrow{v} Z \xrightarrow{w} T(X) \\
\downarrow{f} & & \downarrow{g} \\
X' & \xrightarrow{u'} & Y' \xrightarrow{v'} Z' \xrightarrow{w'} T(X')
\end{array}
\]
there exists \(h : Z \to Z'\) which makes a commutative diagram
\[
\begin{array}{ccc}
X & \xrightarrow{u} & Y \xrightarrow{v} Z \xrightarrow{w} T(X) \\
\downarrow{f} & & \downarrow{g} \\
X' & \xrightarrow{u'} & Y' \xrightarrow{v'} Z' \xrightarrow{w'} T(X') \\
\downarrow{h} & & \downarrow{T(f)} \\
\end{array}
\]

(TR4) (Octahedral axiom) For any two consecutive morphisms \(u : X \to Y\) and \(v : Y \to Z\), if we embed \(u, vu\) and \(v\) in (distinguished) triangles \((X, Y, Z, u, i, i')\), \((X, Z, Y', vu, k, k')\) and \((Y, Z, X', v, j, j')\), respectively, then there exist morphisms \(f : Z' \to Y'\), \(g : Y' \to X'\) such that the following diagram commute
\[
\begin{array}{ccc}
X & \xrightarrow{u} & Y \xrightarrow{i} Z' \xrightarrow{i'} T(X) \\
\downarrow{v} & & \downarrow{f} \\
X & \xrightarrow{vu} Z \xrightarrow{k} Y' \xrightarrow{k'} T(X) \\
\downarrow{j} & & \downarrow{T(u)} \\
X' & \xrightarrow{j'} T(Y) \xrightarrow{T(i)} T(Z') \\
\end{array}
\]
and the third column is a triangle.

Sometimes, we write \(X[i]\) for \(T^i(X)\).

**Definition 1.2** (\(\partial\)-functor) Let \(C, C'\) be triangulated categories. An additive functor \(F : C \to C'\) is called a \(\partial\)-functor (sometimes exact functor) provided that there is a functorial isomorphism \(\alpha : FT \sim T_C F\) such that
\[
\begin{array}{ccc}
F(X) & \xrightarrow{F(u)} & F(Y) \xrightarrow{F(v)} F(Z) \\
\xrightarrow{\alpha \times F(w)} & & T_C(F(X))
\end{array}
\]
is a triangle in $C'$ whenever $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T_C(X)$ is a triangle in $C$. Moreover, if a $\partial$-functor $F$ is an equivalence, then $F$ is called a triangulated equivalence. In this case, we denote by $C \cong C'$

For $(F, \alpha), (G, \beta) : C \to C'$ $\partial$-functors, a functorial morphism $\phi : F \to G$ is called a $\partial$-functorial morphism if

$$(T_{\partial} \phi) \circ \alpha = \beta \circ \phi_T$$

We denote by $\partial(C, C')$ the collection of all $\partial$-functors from $C$ to $C'$, and denote by $\partial \text{Mor}(F, G)$ the collection of $\partial$-functorial morphisms from $F$ to $G$.

**Proposition 1.3** Let $F : C \to C'$ be a $\partial$-functor between triangulated categories. If $G : C' \to C$ is a right (or left) adjoint of $F$, then $G$ is also a $\partial$-functor.

**Definition 1.4** A contravariant (resp., covariant) additive functor $H : C \to A$ from a triangulated category $C$ to an abelian category $A$ is called a homological functor (resp., a cohomological functor), if for any triangle $(X, Y, Z, u, v, w)$ in $C$ the sequence

$$H(T(X)) \to H(Z) \to H(Y) \to H(X)$$

(resp., $H(X) \to H(Y) \to H(Z) \to H(T(X))$)

is exact. Taking $H(T^i(X)) = H^i(X)$, we have the long exact sequence:

$$\ldots \to H^{i+1}(X) \to H^i(Z) \to H^i(Y) \to H^i(X) \to \ldots$$

(resp., $\ldots \to H^i(X) \to H^i(Y) \to H^i(Z) \to H^{i+1}(X) \to \ldots$)

**Proposition 1.5** The following hold.

1. If $(X, Y, Z, u, v, w)$ is a triangle, then $vu = 0$, $wv = 0$ and $T(u)w = 0$.

2. For any $X \in C$, $\text{Hom}_C(-, X) : C \to \text{Ab}$ (resp., $\text{Hom}_C(X, -) : C \to \text{Ab}$) is a homological functor (resp., a cohomological functor).

3. For any homomorphism of triangles

$$\begin{array}{ccc}
X & \xrightarrow{u} & Y \\
\downarrow f & & \downarrow g \\
X' & \xrightarrow{u'} & Y'
\end{array}$$

if two of $f$, $g$ and $h$ are isomorphisms, then the rest is also an isomorphism.

**Definition 1.6** (Compact Object) Let $C$ be a triangulated category. An object $C \in C$ is called a compact object in $C$ if the canonical morphism

$$\bigoplus_{i \in I} \text{Hom}_C(C, X_i) \to \text{Hom}_C(C, \bigoplus_{i \in I} X_i)$$

is an isomorphism for any set $\{X_i\}_{i \in I}$ of objects (if $\bigoplus_{i \in I} X_i$ exists in $C$).

For a triangulated category $C$, a set $S$ of compact objects is called a generating set if $\text{Hom}_C(S, X) = 0 \Rightarrow X = 0$, and if $T(S) = S$. A triangulated category $C$ is compactly generated if $C$ contains arbitrary coproducts, and if it has a generating set.
Definition 1.7 (Homotopy Limit) Let $C$ be a triangulated category which contains arbitrary coproducts (resp., products). For a sequence $\{X_i \rightarrow X_{i+1}\}_{i \in \mathbb{N}}$ (resp., $\{X_{i+1} \rightarrow X_i\}_{i \in \mathbb{N}}$) of morphisms in $C$, the homotopy colimit (resp., homotopy limit) of the sequence is the third (resp., second) term of the triangle

$$
\coprod_i X_i \xrightarrow{1-\text{shift}} \coprod_i X_i \rightarrow \hocolim X_i \rightarrow T\left(\coprod_i X_i\right)
$$

(homotopy limit)

where the above shift morphism is the coproduct (resp., product) of $X_i \xrightarrow{f} X_{i+1}$ (resp., $X_{i+1} \xrightarrow{f} X_i$) ($i \in \mathbb{N}$).

Proposition 1.8 Let $C$ be a triangulated category which contains arbitrary coproducts, $\{X_i \rightarrow X_{i+1}\}_{i \in \mathbb{N}}$ a sequence of morphisms in $C$. For a compact object $C$ in $C$, we have

$$\text{Hom}(C, \hocolim X_i) \cong \lim \text{Hom}(C, X_i)$$

Proof. We have an exact sequence

$$0 \rightarrow \coprod_i \text{Hom}(C, X_i) \rightarrow \coprod_i \text{Hom}(C, X_i) \rightarrow \text{Hom}(C, \hocolim X_i) \rightarrow 0 \quad \square$$

Theorem 1.9 (Brown Representability Theorem [Ne]) Let $C$ be a compactly generated triangulated category. If a homological functor $H : C \rightarrow \mathbb{Ab}$ sends coproducts to products, then it is representable, that is, there is an object $X \in C$ such that $H \cong \text{Hom}_C(-, X)$.

Sketch of Proof. Let $S$ be a generating set of $C$. There exist a coproduct $X_1$ of objects of $S$ and a morphism $h_{X_1} \rightarrow H$ such that $\text{Hom}_C(C, X_1) \rightarrow H(C)$ is surjective for any $C \in S$. For a functor $K_1 = \text{Ker}(h_{X_1} \rightarrow H)$ there exists a coproduct $Z_2$ of objects in $S$ and a morphism $h_{Z_2} \rightarrow K_1$ such that $\text{Hom}_C(C, Z_2) \rightarrow K_1(C)$ is surjective for any $C \in S$. Then we have a triangle $Z_2 \xrightarrow{f} X_1 \xrightarrow{g} X_2 \xrightarrow{h} Z_2[1]$. Since $H$ is a homological functor, we have a commutative diagram

$$
\begin{array}{ccc}
H(X_2) & \xrightarrow{f} & H(X_1) & \xrightarrow{g} & H(Z_2) \\
\downarrow \iota & & \downarrow \iota & & \downarrow \iota \\
\text{Mor}(h_{X_2}, H) & \xrightarrow{\text{Mor}(h_{X_1}, f)} & \text{Mor}(h_{X_1}, H) & \xrightarrow{\text{Mor}(h_{Z_2}, g)} & \text{Mor}(h_{Z_2}, H)
\end{array}
$$

Then there is a morphism $X_1 \rightarrow X_2$ satisfying a commutative diagram

$$
\begin{array}{ccc}
0 & \xrightarrow{\text{id}} & K_1 & \xrightarrow{\text{id}} & \text{Hom}_C(-, X_1) & \xrightarrow{\text{Hom}(\text{id}, f)} & H \\
\downarrow & & \downarrow & & \downarrow \iota & & \downarrow \iota \\
0 & \xrightarrow{\text{id}} & K_2 & \xrightarrow{\text{id}} & \text{Hom}_C(-, X_2) & \xrightarrow{\text{Hom}(\text{id}, g)} & H
\end{array}
$$
and we have a morphism of exact sequence

\[
\begin{array}{cccccc}
0 & \longrightarrow & K_1(C) & \longrightarrow & \text{Hom}_C(C, X_1) & \longrightarrow & H(C) & \longrightarrow & 0 \\
& & \downarrow{=} & & \downarrow{=} & & \| & & \\
0 & \longrightarrow & K_2(C) & \longrightarrow & \text{Hom}_C(C, X_2) & \longrightarrow & H(C) & \longrightarrow & 0
\end{array}
\]

for any \( C \in S \). By inductive step, we have a triangle

\[
\bigsqcup_i X_i \xrightarrow{1-\text{shift}} \bigsqcup_i X_i \rightarrow \text{hocolim} X_i \rightarrow T \left( \bigsqcup_i X_i \right)
\]

and we have an exact sequence

\[
\begin{array}{cccccc}
H(\text{hocolim} X_i) & \longrightarrow & \prod_i H(X_i) & \longrightarrow & \prod_i H(X_i) \\
\downarrow{=} & & \downarrow{=} & & \downarrow{=} \\
\text{Mor}(\text{hocolim} X_i, H) & \longrightarrow & \prod_i \text{Mor}(h_{X_i}, H) & \longrightarrow & \prod_i \text{Mor}(h_{X_i}, H)
\end{array}
\]

Therefore there is a morphism \( \text{Hom}_C(-, \text{hocolim} X_i) \rightarrow H \) such that

\[
\text{Hom}_C(C, \text{hocolim} X_i) \cong H(C)
\]

for any \( C \in S \). Hence we have \( \text{Hom}_C(-, \text{hocolim} X_i) \cong H \).

Corollary 1.10 (Adjoint Functor Theorem [Ne]) Let \( C \) be a compactly generated triangulated category. If a \( \partial \)-functor \( F : C \rightarrow D \) commutes with arbitrary coproducts, then there exists a \( \partial \)-functor \( G : D \rightarrow C \) which is a right adjoint of \( F \).

Proof. For any \( Y \in D \), the functor

\[
\text{Hom}_D(F(-), Y) : C \rightarrow \text{Ab}
\]

is a homological functor. By Brown representability theorem there is an object \( GY \in C \) such that

\[
\text{Hom}_D(F(-), Y) \cong \text{Hom}_C(-, GY)
\]

Definition 1.11 (Multiplicative System) Let \( S \) be a multiplicative system in a triangulated category \( C \) which satisfies the following conditions:

(FR0) For a morphism \( s \) in \( C \), if there exist \( f, g \) such that \( sf, gs \in S \), then \( s \in S \).

(FR1) (1) \( 1_X \in S \) for every \( X \in C \).
    (2) For \( s, t \in S \), if \( st \) is defined, then \( st \in S \).
(FR2) Every diagram in $C$

\[
\begin{array}{ccc}
X & \xrightarrow{s} & Y \\
\downarrow f & & \downarrow g \\
X' & \xrightarrow{t} & Y'
\end{array}
\]

with $s \in S$, can be completed to a commutative square

\[
\begin{array}{ccc}
X & \xrightarrow{s} & Y \\
\downarrow f & & \downarrow g \\
X' & \xrightarrow{t} & Y'
\end{array}
\]

with $s, t \in S$. Ditto for the statement with all arrows reversed.

(FR3) For $f, g \in \text{Hom}_C(X,Y)$ the following are equivalent.

1. There exists $s \in S$ such that $sf = sg$.
2. There exists $t \in S$ such that $ft = gt$.

(FR4) For a morphism $u$ in $C$, $u \in S$ if and only if $T(u) \in S$.

(FR5) For triangles $(X, Y, Z, u, v, w), (X', Y', Z', u', v', w')$ and morphisms $f : X \to X'$, $g : Y \to Y'$ in $S$ with $gu = v'u'$, there exists $h : Z \to Z'$ in $S$ such that $(f, g, h)$ is a homomorphism of triangles.

Definition 1.12 (Quotient Category) We define the quotient category $S^{-1}C$ of $C$, as follows:

1. $\text{Ob}(S^{-1}C) = \text{Ob}(C)$.
2. For $X, Y \in \text{Ob}(C)$, let $V(X,Y) = \{(s, Y', f) | s : Y \to Y' \in S, f : X \to Y\}$. In $V(X,Y)$, we define $(s, Y', f) \sim (s', Y'', f')$ if there is $(s'', Y'''', f')$ such that all triangles are commutative in the following diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{f''} & Y'' \\
\downarrow f' & & \downarrow f' \\
X' & \xrightarrow{s} & Y'
\end{array}
\]

Then we define a morphism from $X$ to $Y$ by an equivalence class $s^{-1}f$ of $(s, Y', f)$.

3. For $s^{-1}f : X \to Y, t^{-1}g : Y \to Z$, by (FR2) there are $s' : Z' \to Z'' \in S$ and
$g' : Y' \to Z''$ such that $s' \circ g = g' \circ s$. Then we define $(t^{-1} g \circ s^{-1} f) = (s' \circ t)^{-1} g \circ f$.

Moreover, we define the quotient functor $Q : \mathcal{C} \to S^{-1} \mathcal{C}$ by

(Q1) $Q(X) = X$ for $X \in \mathcal{C}$.

(Q2) $Q(f) = 1^{-1} f$ for a morphism $f : X \to Y$ in $\mathcal{C}$.

**Remark 1.13** Can we define (2) in the above?

**Definition 1.14 (Épaise Subcategory)** Let $\mathcal{C}$ be a triangulated category. A full subcategory $\mathcal{U}$ of $\mathcal{C}$ is called a full triangulated subcategory if $X \to Y$ is a morphism in $\mathcal{U}$, then there is a triangle $X \to Y \to Z \to TX$ with $Z \in \mathcal{U}$.

A full triangulated subcategory $\mathcal{U}$ is called an épaise subcategory if it is closed under direct summands. In this case, let $S(\mathcal{U})$ be the collection of morphisms $s$ such that $X \to Y \to Z \to X[1]$ is a triangle with $Z \in \mathcal{U}$. Then $S(\mathcal{U})$ is a multiplicative system satisfying (FR0) - (FR5). We write $\mathcal{C}/\mathcal{U} = S(\mathcal{U})^{-1} \mathcal{C}$.

In the case that $\mathcal{C}$ contains arbitrary coproducts, a full triangulated subcategory $\mathcal{U}$ is called a localizing subcategory if it is closed under coproducts.

**Remark 1.15** The above definition of an épaise subcategory $\mathcal{U}$ is the same as the original definition [Ve], that is, a full triangulated category satisfying that if $X \to Y$ factors through some object in $\mathcal{U}$ and if there is a triangle $X \to Y \to Z \to TX$ with $Z \in \mathcal{U}$, then $X, Y \in \mathcal{U}$.

**Proposition 1.16 ([BN])** Let $\mathcal{C}$ be a triangulated category which contains arbitrary coproducts. Then any localizing subcategory is an épaise subcategory.

*Sketch of Proof.* Let $\mathcal{U}$ be a localizing subcategory, and $X \in \mathcal{U}$ with $X = Y \oplus Z$ in $\mathcal{C}$. We take a morphism $e : X \to Y \leftarrow X$, and consider the sequence of morphisms

$$X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} \cdots$$

(*)

Then it is easy to see that $Y \cong \text{hocolim}(\cdot) \in \mathcal{U}$. \hfill ∎

**Proposition 1.17** Let $\mathcal{C}$ be a triangulated category. For a multiplicative system $S$ satisfying the conditions (FR0) - (FR5), let $\mathcal{U}(S)$ be the full triangulated subcategory consisting of objects $Z$ which is in a triangle $X \to Y \to Z \to X[1]$ with $s \in S$. Then the following hold.
1. $S(U)$ and $U(S)$ induce an 1 - 1 correspondence between the collection of multiplicative systems $S$ satisfying the conditions (FR0) - (FR5) and the collection of épaisse subcategories $U$.

2. For an épaisse subcategory $U$, $C/U$ is a triangulated category whose (distinguished) triangles are defined to be isomorphic to (distinguished) triangles of $C$.

3. Assume $C$ contains arbitrary coproducts. For a localizing subcategory $U$, $C/U$ also contains arbitrary coproducts.

**Definition 1.18 (stable t-structure)** For full subcategories $U$ and $V$ of a triangulated category $C$, $(U, V)$ is called a stable t-structure in $C$ provided that

1. $U$ and $V$ are stable for translations.

2. $\text{Hom}_C(U, V) = 0$.

3. For every $X \in C$, there exists a triangle $U \rightarrow X \rightarrow V \rightarrow TU$ with $U \in U$ and $V \in V$.

**Proposition 1.19 ([BBD], c.f. [Mi])** Let $C$ be a triangulated category, $(U, V)$ a stable t-structure in $C$, and $i_* : U \rightarrow C, j_* : V \rightarrow C$ the canonical embeddings. Then the following hold.

1. $U$ and $V$ is épaisse subcategories of $C$.

2. $i_*$ (resp., $j_*$) has a right adjoint $i'$ (resp., a left adjoint $j^*$).

3. The adjunction arrows induce a triangle

$$i_*i'X \xrightarrow{\alpha_X} X \xrightarrow{\beta_X} j_*j^*X \xrightarrow{} i_*i'X[1]$$

for any $X \in C$.

4. $C/U$ (resp., $C/V$) exists, and it is triangulated equivalent to $V$ (resp., $U$).

![Diagram](https://via.placeholder.com/150)

**Corollary 1.20** Let $C$ be a compactly generated triangulated category, and $U$ a localizing subcategory of $C$. Then $C/U$ can be defined if and only if there is a full triangulated subcategory $V$ such that $(U, V)$ a stable t-structure in $C$.

**Proof.** If $C/U$ can be defined, then the quotient functor $Q : C \rightarrow C/U$ commutes with coproducts. By Adjoint Functor Theorem, $Q$ has a right adjoint $F : C/U \rightarrow C$. By Proposition 1.19, it is easy to see that $(U, \text{Im} F)$ is a stable t-structure in $C$. 

\[\square\]
2 Derived Categories

Throughout this section, $\mathcal{A}$ is an abelian category and $\mathcal{B}$ is an additive subcategory of $\mathcal{A}$ which is closed under isomorphisms.

Definition 2.1 (Complex) A (cochain) complex is a collection $X^\cdot = (X^n, d^n_X : X^n \to X^{n+1})_{n \in \mathbb{Z}}$ of objects and morphisms of $\mathcal{B}$ such that $d^{n+1}_X d^n_X = 0$. A complex $X^\cdot = (X^n, d^n_X : X^n \to X^{n+1})_{n \in \mathbb{Z}}$ is called bounded below (resp., bounded above, bounded) if $X^n = 0$ for $n \leq 0$ (resp., $n \geq 0$ and $n > 0$).

A complex $X^\cdot = (X^n, d^n_X)$ is called a stalk complex if there exists an integer $n_0$ such that $X^i = 0$ if $i \neq n_0$. We identify objects of $\mathcal{B}$ with a stalk complexes of degree 0.

A morphism $f : X^\cdot \to Y^\cdot$ of complexes is a collection of morphisms $f^n : X^n \to Y^n$ which makes a commutative diagram

\[
\begin{array}{ccc}
\cdots & \longrightarrow & X^n \xrightarrow{d^n_X} X^{n+1} \longrightarrow \cdots \\
\downarrow f^n & & \downarrow f^{n+1} \\
\cdots & \longrightarrow & Y^n \xrightarrow{d^n_Y} Y^{n+1} \longrightarrow \cdots
\end{array}
\]

We denote by $\mathcal{C}(\mathcal{B})$ (resp., $\mathcal{C}^+(\mathcal{B})$, $\mathcal{C}^-(\mathcal{B})$, $\mathcal{C}^b(\mathcal{B})$) the category of complexes (resp., bounded below complexes, bounded above complexes, bounded complexes) of $\mathcal{B}$. An autofunctor $T : \mathcal{C}(\mathcal{B}) \to \mathcal{C}(\mathcal{B})$ is called translation if $(TX^\cdot)^n = X^{n+1}$ and $(Td_X)^n = -d^{n+1}_X$ for any complex $X^\cdot = (X^n, d^n_X)$.

In $\mathcal{C}(\mathcal{A})$, a morphism $u : X^\cdot \to Y^\cdot$ is called a quasi-isomorphism if $H_n(u)$ is an isomorphism for any $n$.

In this section, "*" means "nothing", "+", "-" or "b".

Definition 2.2 For $u \in \text{Hom}_{\mathcal{C}(\mathcal{B})}(X^\cdot, Y^\cdot)$, the mapping cone of $u$ is a complex $M^\cdot(u)$ with

\[
M^n(u) = X^{n+1} \oplus Y^n,
\]

\[
d^n_{M(u)} = \begin{bmatrix} -d^{n+1}_X & 0 \\ u_{n+1} & d^n_Y \end{bmatrix} : X^{n+1} \oplus Y^n \to X^{n+2} \oplus Y^{n+1}.
\]

Definition 2.3 (Homotopy Relation) Two morphisms $f, g \in \text{Hom}_{\mathcal{C}(\mathcal{B})}(X^\cdot, Y^\cdot)$ are said to be homotopic (denote by $f \simeq g$) if there is a collection of morphisms $h = (h^n)$, $h^n : X^n \to Y^{n+1}$ such that $f^n - g^n = d^{n-1}_Y h^n + h^{n+1} d^n_X$ for all $n \in \mathbb{Z}$.

Definition 2.4 (Homotopy Category) The homotopy category $K^*(\mathcal{B})$ of $\mathcal{B}$ is defined by

1. $\text{Ob}(K^*(\mathcal{B})) = \text{Ob}(\mathcal{C}^*(\mathcal{B}))$,

2. $\text{Hom}_{K^*(\mathcal{B})}(X^\cdot, Y^\cdot) = \text{Hom}_{\mathcal{C}^*(\mathcal{B})}(X^\cdot, Y^\cdot)/ \simeq_h$ for $X^\cdot, Y^\cdot \in \text{Ob}(K^*(\mathcal{B}))$. 

Proposition 2.5 A category $K^\bullet(B)$ is a triangulated category whose (distinguished) triangles are defined to be isomorphic to

$$X^\cdot \xrightarrow{u} Y^\cdot \xrightarrow{v} M(u) \xrightarrow{} T(X^\cdot)$$

for any $u : X^\cdot \rightarrow Y^\cdot$ in $K^\bullet(B)$.

Definition 2.6 (Derived Category) The derived category $D^\bullet(A)$ of an abelian category $A$ is $K^\bullet(A)/K^{\bullet\phi}(A)$, where $K^{\bullet\phi}(A)$ is the full subcategory of $K^\bullet(A)$ consisting of null complexes, that is, complexes whose all cohomologies are 0.

Proposition 2.7 The following hold.

1. $D^\bullet(A)$ is a triangulated category, and the canonical functor $Q : K^\bullet(A) \rightarrow D^\bullet(A)$ is a $\delta$-functor.

2. The $i$-th cohomology of complexes is a cohomological functor in the sense of Definition 1.4.

Proposition 2.8 If $0 \rightarrow X^\cdot \xrightarrow{\alpha} Y^\cdot \xrightarrow{\beta} Z^\cdot \rightarrow 0$ is a exact sequence in $C(A)$, then it can be embedded in a triangle in $D(A)$

$$Q(X^\cdot) \xrightarrow{Q(\alpha)} Q(Y^\cdot) \xrightarrow{Q(\beta)} Q(Z^\cdot) \xrightarrow{} TQ(X^\cdot).$$

Definition 2.9 (K-injective Complex) A complex $X^\cdot$ of $K(B)$ is called K-injective (resp., K-projective) if

$$\text{Hom}_{K(B)}(N^\cdot, X^\cdot) = 0 \quad (\text{resp., } \text{Hom}_{K(B)}(X^\cdot, N^\cdot) = 0)$$

for any null complex $N^\cdot$.

Example 2.10 Let $A$ be a ring, $\text{Mod} A$ the category of right $A$-modules, and $\text{Inj} A$ (resp., $\text{Proj} A$) the category of injective (resp., projective) right $A$-modules. Then any complex $I^\cdot \in K^\bullet(\text{Inj} A)$ (resp., $P^\cdot \in K^\bullet(\text{Proj} A)$) is a K-injective (resp., K-projective) complex in $K(\text{Mod} A)$.

Example 2.11 Let $k$ be a field, $A = k[x]/(x^2)$, and

$$X^\cdot : \ldots \xrightarrow{z} A \xrightarrow{z} A \xrightarrow{z} \ldots$$

Then $X^\cdot$ is a null complex of finitely generated projective-injective $A$-modules. But it is neither K-projective nor K-injective, because $\text{Hom}_{K(\text{Mod} A)}(X^\cdot, X^\cdot) \neq 0$.

Theorem 2.12 ([Sp], [Ne], [LAM], [Fr]) Let $K^{\text{inj}}(\text{Mod} A)$ (resp., $K^{\text{proj}}(\text{Mod} A)$) be the category of K-injective (resp., K-projective) complexes, then the following hold.

1. $(K^{\text{proj}}(\text{Mod} A), K^\bullet(\text{Mod} A))$ is a stable t-structure in $K(\text{Mod} A)$, and hence $D(\text{Mod} A)$ exists and is triangulated equivalent to $K^{\text{proj}}(\text{Mod} A)$. 
2. \((K^0(\text{Mod} \ A), K^{i_{ij}}(\text{Mod} \ A))\) is a stable \(t\)-structure in \(K(\text{Mod} \ A)\), and hence \(D(\text{Mod} \ A)\) is triangulated equivalent to \(K^{i_{ij}}(\text{Mod} \ A)\).

3. For a Grothendieck category \(\mathcal{A}\), \((K^0(\mathcal{A}), K^{i_{ij}}(\mathcal{A}))\) is a stable \(t\)-structure in \(K(\mathcal{A})\), and hence \(D(\mathcal{A})\) exists and is triangulated equivalent to \(K^{i_{ij}}(\mathcal{A})\).

Proof. For a complex \(X^\cdot = (X^i, d^i)\), we define the following truncations:

\[
\sigma_{\leq n}X^\cdot : \cdots \to X^{n-2} \to X^{n-1} \to \text{Ker} \, d^n \to 0 \to \cdots
\]

\[
\sigma_{\geq n}X^\cdot : \cdots \to 0 \to \text{Cok} \, d^{n-1} \to X^{n+1} \to X^{n+2} \to \cdots
\]

(1) For any \(n\), there is a complex \(P_n^\cdot \in K^- (\text{Proj} \ A)\) which has a quasi-isomorphism \(P_n^\cdot \to \sigma_{\leq n}X^\cdot\). Then we have the following quasi-isomorphisms (qis)

\[
X^\cdot \cong \varinjlim \sigma_{\leq n}X^\cdot \xleftarrow{\text{qis}} \text{hocolim} \sigma_{\leq n}X^\cdot \xrightarrow{\text{qis}} \text{hocolim} P_n^\cdot
\]

Since \(\text{Hom}_C(\prod_n P_n^\cdot, -) \cong \prod_n \text{Hom}_C(P_n^\cdot, -)\), \(\prod_n P_n^\cdot\) is K-projective. Here \(h^M = \text{Hom}_C(M, -)\) for any object \(M\). It is easy to see that \(\text{hocolim} \, P_n^\cdot\) is K-projective by the following exact sequence

\[
h_{\prod_n P_n^\cdot} \to h_{\prod_n P_n^\cdot} \to h_{\text{hocolim} P_n^\cdot} \to h^T(\prod_n P_n^\cdot) \to h^T(\prod_n P_n^\cdot)
\]

(2) For any \(n\), there is a complex \(I_n^\cdot \in K^+ (\text{Inj} \ A)\) which has a quasi-isomorphism \(\sigma_{\geq -n}X^\cdot \to I_n^\cdot\). Then we have the following quasi-isomorphisms (qis)

\[
X^\cdot \cong \varprojlim \sigma_{\geq -n}X^\cdot \xleftarrow{\text{qis}} \text{holim} \sigma_{\geq -n}X^\cdot \xrightarrow{\text{qis}} \text{holim} I_n^\cdot
\]

by the same reason of (1), we have the statement.

(3) Because there is a ring \(A\) such that \(\mathcal{A}\) is a localization of \(\text{Mod} \ A\) (Gabriel-Popescu Theorem). See [LAM] or [Fr] for details. \(\square\)

Remark 2.13 If \(P^\cdot\) is a K-projective complex (e.g. a bounded above complex of projective \(A\)-modules), then we have

\[
\text{Hom}_{K(\text{Mod} \ A)}(P^\cdot, X^\cdot) \cong \text{Hom}_{D(\text{Mod} \ A)}(P^\cdot, X^\cdot)
\]

for any complex \(X^\cdot\). Similarly, for a K-injective complex \(I^\cdot\) (e.g. bounded below complex of injective \(A\)-modules), then we have

\[
\text{Hom}_{K(\text{Mod} \ A)}(X^\cdot, I^\cdot) \cong \text{Hom}_{D(\text{Mod} \ A)}(X^\cdot, I^\cdot)
\]

for any complex \(X^\cdot\). In particular, given \(A\)-modules \(M, N\), we have

\[
\text{Ext}^1_A(M, N) \cong \text{Hom}_{D(\text{Mod} \ A)}(M, N[i])
\]
Definition 2.14 (Double Complex, Total complex) A double complex $C^*$ is a bigraded object $(C^{p,q})_{p,q \in \mathbb{Z}}$ of $A$ together with $d_{1}^{p,q}: C^{p,q} \to C^{p+1,q}$ and $d_{II}^{p,q}: C^{p,q} \to C^{p,q+1}$ such that

$$C^{q} = (C^{p,q}, d_{1}^{p,q}: C^{p,q} \to C^{p+1,q}), \ C^{p} = (C^{p,q}, d_{II}^{p,q}: C^{p,q} \to C^{p,q+1})$$

are complexes satisfying $d_{1}^{p,q+1}d_{II}^{p,q} - d_{II}^{p+1,q}d_{1}^{p,q} = 0$. For a double complex $C^*$, we define the total complexes

$$\text{Tot}^{1}C^* = (X^n, d^n), \text{where } X^n = \prod_{p+q=n} C^{p,q}, d^n = \prod_{p+q=n} (d_{1}^{p,q} + (-1)^p d_{II}^{p,q})$$

$$\text{Tot}^{II}C^* = (Y^n, d^n), \text{where } Y^n = \prod_{p+q=n} C^{p,q}, d^n = \prod_{p+q=n} (d_{II}^{p,q} + (-1)^p d_{1}^{p,q}).$$

Definition 2.15 (Cartan-Eilenberg Resolution) For a complex $X' \in D(\text{Mod} A)$, let

$$\ldots \to P^{-1} \to P^{0} \to X' \to 0 \ (\text{resp., } 0 \to X' \to I^{0} \to I^{1} \to \ldots)$$

be an exact sequence with $P^n$ (resp., $I^n$) being a complex of projective (resp., injective) $A$-modules. We call $\ldots \to P^{-1} \to P^{0} \to \ldots$ (resp., $I^{0} \to I^{1} \to \ldots$) a Cartan-Eilenberg projective (resp., injective) resolution of $X'$ if the induced complexes $\ldots \to B^n(P^{-1}) \to B^n(P^{0})$ and $\ldots \to H^n(P^{-1}) \to H^n(P^{0})$ (resp., $B^n(I^{0}) \to B^n(I^{1})$ \to \ldots and $H^n(I^{0}) \to H^n(I^{1}) \to \ldots$) are also projective (resp., injective) resolutions of $B^n(X'), H^n(X)$, respectively.

Proposition 2.16 Under the setting of Definition 2.15, the following hold.

1. $\text{Tot}^{1}P^*$ is $K$-projective, and the induced morphism of complexes $\text{Tot}^{1}P^* \to X'$ is a quasi-isomorphism.

2. $\text{Tot}^{II}I^*$ is $K$-injective, and the induced morphism of complexes $X' \to \text{Tot}^{II}I^*$ is a quasi-isomorphism.

Sketch of Proof. We consider the following truncations

$$\sigma_{\leq n}^{ii}P^* : \ldots \to \sigma_{\leq n}P^{-1} \to \sigma_{\leq n}P^{0} \to \sigma_{\geq n}I^0 \to \sigma_{\geq n}I^1 \to \ldots$$

Then it is easy to see $\text{Tot}^{1}\sigma_{\leq n}^{ii}P^*$ (resp., $\text{Tot}^{II}\sigma_{\geq n}^{ii}I^*$) is $K$-projective (resp., $K$-injective), and that the induced morphism of complexes $\text{Tot}^{1}\sigma_{\leq n}^{ii}P^* \to \sigma_{\leq n}X'$ (resp., $\sigma_{\geq n}X' \to \text{Tot}^{II}\sigma_{\geq n}^{ii}I^*$) is a quasi-isomorphism. Therefore we have the following quasi-isomorphisms (qis)

$$X' \leftarrow \text{hocolim } \sigma_{\leq n}X' \leftarrow \text{hocolim } \text{Tot}^{1}\sigma_{\leq n}^{ii}P^* \text{ qis } \text{Tot}^{1}P^*$$

$$(\text{resp., } X' \leftarrow \text{holim } \sigma_{\geq n}X' \leftarrow \text{holim } \text{Tot}^{II}\sigma_{\geq n}^{ii}I^* \text{ qis } \text{Tot}^{II}I^*)$$

and $\text{Tot}^{1}P^*$ (resp., $\text{Tot}^{II}I^*$) is $K$-projective (resp., $K$-injective). \(\square\)
Definition 2.17 (Right Derived Functor) For a \( \partial \)-functor \( F : K^*(A) \rightarrow K(A') \), the right derived functor of \( F \) is a \( \partial \)-functor

\[
R^*F : D^*(A) \rightarrow D(A')
\]

together with a functorial morphism of \( \partial \)-functors

\[
\xi \in \partial \text{Mor}(Q_A \circ F, R^*F \circ Q_A^*)
\]

with the following property:

For \( G \in \partial(D^*(A), D(A')) \) and \( \zeta \in \partial \text{Mor}(Q_A \circ F, G \circ Q_A^*) \), there exists a unique morphism \( \eta \in \partial \text{Mor}(R^*F, G) \) such that

\[
\zeta = (\eta Q_A^*) \xi.
\]

In other words, we can simply write the above using functor categories. For triangulated categories \( C, C' \), the \( \partial \)-functor category \( \partial(C, C') \) is the category \( C_{\partial}(C, C') \) consisting of \( \partial \)-functors from \( C \) to \( C' \) as objects and \( \partial \)-functorial morphisms as morphisms. Then we have

\[
\partial \text{Mor}(Q_A \circ F, -Q_A^*) \cong \partial \text{Mor}(R^*F, -)
\]

as functors from \( \partial(D^*(A), D(A')) \) to \( \text{Set} \).

Proposition 2.18 Let \( A, A' \) be abelian categories, \( F : K(A) \rightarrow K(A') \) a \( \partial \)-functor. If \( A \) is a Grothendieck category, then we have the right derived functor \( RF : D(A) \rightarrow D(A') \) such that \( F(X') \cong RF(X') \) for any \( K \)-injective complex \( X' \).

Remark 2.19 In the setting of Definition 2.17, the left derived functor \( L^*F : D^*(A) \rightarrow D(A') \) can be also defined by reversing arrows of \( \partial \)-functorial morphisms. Let \( R^n F(X') = H^n(RF(X')) \), \( L^n F(X') = H^n(LF(X')) \), then \( R^n F \) (resp., \( L^n F \)) coincides with the ordinary definition of the n-th right (resp., left) derived functor. According to Proposition 2.16, if \( F \) commutes with products (resp., coproducts), then the n-th hypercohomology \( R^n F \) (resp., hyperhomology \( L^n F \)) coincides with \( R^n F \) (resp., \( L^n F \)) (c.f. [CE], [Mc], [We]).

Definition 2.20 (\( \text{Hom}_A^* \otimes_A \)) Let \( X', Y' \) be complexes in \( C(\text{Mod} A) \), \( Z' \) a complex in \( C(\text{Mod} A^{op}) \). We define the complex \( \text{Hom}_A^*(X', Y') \) in \( C(\mathfrak{A} \mathfrak{b}) \) by

\[
\text{Hom}_A^*(X', Y') = \prod_{j-i=n} \text{Hom}_A(X^i, Y^j), \quad d_{\text{Hom}_A^*(X', Y')}(f) = d_X \circ f - (-1)^n f \circ d_Y
\]

for \( f \in \text{Hom}_A^*(X', Y') \). And we define the complex \( X' \otimes_A Z' \) in \( C(\mathfrak{A} \mathfrak{b}) \) by

\[
X' \otimes_A Z' = \prod_{i+j=n} X^i \otimes_A Z^j, \quad d_{X' \otimes Y} = d_X \otimes 1 + (-1)^n 1 \otimes d_Z
\]
Proposition 2.21 Let $A$ be a ring. Then we have a right derived functor

$$R\text{Hom}_A: D(\text{Mod } A)^{op} \times D(\text{Mod } A) \rightarrow D(\mathfrak{A}b)$$

and a left derived functor

$$\hat{\otimes}_A: D(\text{Mod } A) \times D(\text{Mod } A^{op}) \rightarrow D(\mathfrak{A}b)$$

Proposition 2.22 Let $A$ be a ring. For complexes $X'$, $Y'$, we have isomorphisms

$$H^n(\text{Hom}_A(X', Y')) \cong \text{Hom}_{\text{K}(\text{Mod } A)}(X', Y'[n])$$

$$H^n(R\text{Hom}_A(X', Y')) \cong \text{Hom}_{\text{D}(\text{Mod } A)}(X', Y'[n])$$

Definition 2.23 (Perfect Complex) Let $A$ be a ring. A complex $X' \in D(\text{Mod } A)$ is called a perfect complex if $X'$ is quasi-isomorphic to a bounded complex of finitely generated projective $A$-modules.

Let $X$ be a scheme, $D(X)$ the derived category of sheaves of $\mathcal{O}_X$-modules. We denote by $D_{qc}(X)$ the full subcategory of $D(X)$ consisting of complexes whose cohomologies are quasi-coherent sheaves. A complex $X' \in D_{qc}(X)$ is called a perfect complex if $X'$ is locally quasi-isomorphic to a bounded complex of vector bundles (See [TT]).

We denote by $D_{pf}(A)$ the full triangulated subcategory of $D(A)$ consisting of perfect complexes.

Proposition 2.24 ([Rd1], [Ne]) For a ring $A$, the following hold.

1. A complex $X' \in D(\text{Mod } A)$ is perfect if and only if it is a compact object in $D(\text{Mod } A)$.

2. $D(\text{Mod } A)$ is compactly generated.

Theorem 2.25 ([BV]) Let $X$ be a quasi-compact quasi-separated scheme, then the following hold.

1. A complex $X' \in D_{qc}(X)$ is perfect if and only if it is a compact object in $D_{qc}(X)$.

2. $D_{qc}(X)$ is compactly generated.

Theorem 2.26 ([BN]) Let $X$ be a quasi-compact separated scheme, then the canonical functor $D(\text{Qcoh } X) \rightarrow D_{qc}(X)$ is a triangulated equivalence, where $\text{Qcoh } X$ is the category of quasi-coherent sheaves of $\mathcal{O}_X$-modules.

Corollary 2.27 ([BV]) Let $X$ be smooth over a field, then we have

$$D^b(\text{coh } X) \cong D_{pf}(X).$$

where $\text{coh } X$ is the category of coherent sheaves of $\mathcal{O}_X$-modules.
For a ring $A$, we denote by $\text{proj} A$ the category of finitely generated projective $A$-modules.

**Theorem 2.28 ([Rd1], [Rd2])** Let $A, B$ be algebras over a field $k$. The following are equivalent.

1. $\text{D}(\text{Mod} A) \cong \text{D}(\text{Mod} B)$.

2. $\text{K}^b(\text{proj} A) \cong \text{K}^b(\text{proj} B)$.

3. There is a perfect complex $T^\cdot \in \text{D}(\text{Mod} A)$ such that
   (a) $B \cong \text{End}_{\text{D}(\text{Mod} A)}(T^\cdot)$,
   (b) $\text{Hom}_{\text{D}(\text{Mod} A)}(T^\cdot, T^\cdot[i]) = 0$ for $i \neq 0$,
   (c) $\{T^\cdot[i] | i \in \mathbb{Z}\}$ is a generating set in $\text{D}(\text{Mod} A)$.

4. There is a complex $V^\cdot$ of $B$-$A$-bimodules such that
   $$\text{R} \text{Hom}^\cdot_A(V^\cdot, -) : \text{D}(\text{Mod} A) \to \text{D}(\text{Mod} B)$$
   is an equivalence.

In this case, $T^\cdot$ is called a tilting complex for $A$, $V^\cdot$ is called a two-sided tilting complex, and $\text{R} \text{Hom}^\cdot_A(V^\cdot, -)$ is called a standard equivalence.

**Theorem 2.29 ([BO])** Let $X$ be a smooth irreducible projective variety with ample canonical or anticanonical sheaf. If $X'$ is a smooth algebraic variety such that
$$\text{D}^b(\text{coh} X) \cong \Delta \text{D}^b(\text{coh} X'),$$
then $X'$ is isomorphic to $X$.

**Theorem 2.30 ([Be])** Let $\mathbb{P}_k^n$ be the $n$-dimensional projective space over a field $k$, and let $T_1 = \bigoplus_{i=0}^n \mathcal{O}(i)$, $T_2 = \bigoplus_{i=0}^n \mathcal{O}(-i)$, and $B_1 = \text{End}_\mathbb{P}(T_1)$, $B_2 = \text{End}_\mathbb{P}(T_2)$. Then $B_i$ is a finite dimensional $k$-algebra of finite global dimension, and
$$\text{D}^b(\text{coh} \mathbb{P}) \cong \text{D}^b(\text{mod} B_i)$$
where mod $B_i$ is the category of finitely generated $B_i$-modules ($i = 1, 2$).

**Definition 2.31** Let $A$ be an algebra over a field $k$. The derived Picard group of $A$ (relative to $k$) is
$$\text{DPic}_k(A) := \{\text{two-sided tilting complexes } T \in \text{D}^b(\text{Mod} A^e) \} \over \text{isomorphism}$$
with identity element $A$, product $(T_1, T_2) \mapsto T_1 \otimes_A^L T_2$ and inverse $T \mapsto T^\vee := \text{R} \text{Hom}_A(T, A)$.

Given any $k$-linear triangulated category $C$ we let
$$\text{Out}^A_k(C) := \{k\text{-linear triangulated self-equivalences of } C \} \over \text{isomorphism}.$$
Theorem 2.32 ([MY]) Let $k$ be an algebraically closed field, and $A$ a finite dimensional hereditary $k$-algebra. Then we have
\[ \text{DPic}_k(A) = \text{Out}_k^\Delta(D^b(\text{Mod } A)) = \text{Out}_k^\Delta(D^b(\text{mod } A)) \]

M. Kontsevich and A. Rosenberg introduced the notion of non-commutative projective spaces $\text{NP}^n$ [KR], and showed that
\[ D^b(Q\text{coh } \text{NP}^n) \cong D^b(\text{Mod } kQ_n) \]
\[ D^b(\text{coh } \text{NP}^n) \cong D^b(\text{mod } kQ_n) \]
where $Q_n$ is the quiver
\[
\begin{array}{c}
\circ \rightarrow \\
\alpha_n
\end{array}
\]

Corollary 2.33 ([MY]) For non-commutative projective spaces $\text{NP}^n$, we have
\[ \text{Out}_k^\Delta(D^b(Q\text{coh } \text{NP}^n)) \cong \text{Out}_k^\Delta(D^b(\text{coh } \text{NP}^n)) \]
\[ \cong \mathbb{Z} \times (\mathbb{Z} \ltimes \text{PGL}_{n+1}(k)) \]

Theorem 2.34 ([BO]) Let $X$ be a smooth irreducible projective variety with ample canonical or anticanonical sheaf. Then $\text{Out}_k^\Delta(D^b(\text{coh } X))$ is generated by the automorphisms of variety, the twists by invertible sheaves and the translations, and hence $\text{Out}_k^\Delta(D^b(\text{coh } X)) \cong (\text{Aut}_k X \times \text{Pic } X) \times \mathbb{Z}$.

References


