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Kyoto University
Derived Categories in Representation Theory

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We survey recent methods of derived categories in the representation theory of algebras.

1 Triangulated Categories and Brown Representability

Definition 1.1 A triangulated category $\mathcal{C}$ is an additive category together with
(1) an autofunctor $T: \mathcal{C} \rightarrow \mathcal{C}$ (i.e. there is $T^{-1}$ such that $T \circ T^{-1} = T^{-1} \circ T = 1_{\mathcal{C}}$) called the translation, and
(2) a collection $\mathcal{T}$ of sextuples $(X, Y, Z, u, v, w)$:

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$$

called (distinguished) triangles. These data are subject to the following four axioms:

(TR1) (1) Every sextuple $(X, Y, Z, u, v, w)$ which is isomorphic to a (distinguished) triangle is a (distinguished) triangle.
(2) Every morphism $u: X \rightarrow Y$ is embedded in a (distinguished) triangle

$$
\begin{array}{c}
X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X) \\
\downarrow^{(1)} \\
X \xrightarrow{u} Y
\end{array}
$$

(3) For any $X \in \mathcal{C}$,

$$X \xrightarrow{1} X \rightarrow 0 \rightarrow T(X)$$

is a (distinguished) triangle
(TR2) A sextuple

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$$

is a (distinguished) triangle if and only if

$$Y \xrightarrow{v} Z \xrightarrow{w} T(X) \xrightarrow{T(u)} T(Y)$$

is a (distinguished) triangle.

(TR3) For any (distinguished) triangles $$(X, Y, Z, u, v, w), (X', Y', Z', u', v', w')$$ and a commutative diagram

$$\begin{array}{ccc}
X & \xrightarrow{u} & Y \\
\downarrow{f} & & \downarrow{g} \\
X' & \xrightarrow{u'} & Y'
\end{array}$$

there exists $h : Z \rightarrow Z'$ which makes a commutative diagram

$$\begin{array}{ccc}
X & \xrightarrow{u} & Y \\
\downarrow{f} & & \downarrow{g} \\
X' & \xrightarrow{u'} & Y'
\end{array} \xrightarrow{h} \begin{array}{c} Z \xrightarrow{w} T(X) \\
\downarrow{h} \end{array} \xrightarrow{T(f)} \begin{array}{c} T(X') \xrightarrow{w'} \\
\downarrow{h} \end{array}$$

(TR4) (Octahedral axiom) For any two consecutive morphisms $u : X \rightarrow Y$ and $v : Y \rightarrow Z$, if we embed $u, vu$ and $v$ in (distinguished) triangles $$(X,Y,Z,u,i,i'), (X,Z,Y',vu,k,k')$$ and $$(Y,Z,X',v,j,j')$$, respectively, then there exist morphisms $f : Z' \rightarrow Y', g : Y' \rightarrow X'$ such that the following diagram commute

$$\begin{array}{ccc}
X & \xrightarrow{u} & Y \\
\downarrow{v} & & \downarrow{f} \\
X' & \xrightarrow{u'} & Y'
\end{array} \xrightarrow{i} \begin{array}{c} Z \xrightarrow{i'} \xrightarrow{k} T(X) \\
\downarrow{k'} & & \downarrow{T(f)} \\
Z' & \xrightarrow{w'} \xrightarrow{k'} T(X')
\end{array} \xrightarrow{j} \begin{array}{c} T(X) \xrightarrow{Y} T(X') \\
\downarrow{g} & & \downarrow{T(i)} \xrightarrow{j'} \\
T(Y) & \xrightarrow{T(i)} & T(Z')
\end{array}$$

and the third column is a triangle.

Sometimes, we write $X[i]$ for $T^i(X)$.

**Definition 1.2** (δ-functor) Let $C, C'$ be triangulated categories. An additive functor $F : C \rightarrow C'$ is called a δ-functor (sometimes exact functor) provided that there is a functorial isomorphism $\alpha : FT_c \cong T_c F$ such that

$$\begin{array}{ccc}
F(X) & \xrightarrow{F(u)} & F(Y) \\
\xrightarrow{F(v)} & & \xrightarrow{F(w)} \\
F(Z) & \xrightarrow{\alpha \circ F(w)} & T_c(F(X))
\end{array}$$
is a triangle in \( C' \) whenever \( X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T_{\mathcal{C}}(X) \) is a triangle in \( C \). Moreover, if a \( \partial \)-functor \( F \) is an equivalence, then \( F \) is called a triangulated equivalence. In this case, we denote by \( \mathcal{C} \cong \mathcal{C}' \).

For \((F, \alpha), (G, \beta) : \mathcal{C} \rightarrow \mathcal{C}' \) \( \partial \)-functors, a functorial morphism \( \phi : F \rightarrow G \) is called a \( \partial \)-functorial morphism if

\[
(T_{\mathcal{C}} \phi) \circ \alpha = \beta \circ \phi_{\mathcal{C}}
\]

We denote by \( \partial(\mathcal{C}, \mathcal{C}') \) the collection of all \( \partial \)-functors from \( \mathcal{C} \) to \( \mathcal{C}' \), and denote by \( \partial \text{Mor}(F, G) \) the collection of \( \partial \)-functorial morphisms from \( F \) to \( G \).

**Proposition 1.3** Let \( F : \mathcal{C} \rightarrow \mathcal{C}' \) be a \( \partial \)-functor between triangulated categories. If \( G : \mathcal{C}' \rightarrow \mathcal{C} \) is a right (or left) adjoint of \( F \), then \( G \) is also a \( \partial \)-functor.

**Definition 1.4** A contravariant (resp., covariant) additive functor \( H : \mathcal{C} \rightarrow \mathcal{A} \) from a triangulated category \( \mathcal{C} \) to an abelian category \( \mathcal{A} \) is called a homological functor (resp., a cohomological functor), if for any triangle \( (X, Y, Z, u, v, w) \) in \( \mathcal{C} \) the sequence

\[
H(T(X)) \rightarrow H(Z) \rightarrow H(Y) \rightarrow H(X)
\]  

(resp., \( H(X) \rightarrow H(Y) \rightarrow H(Z) \rightarrow H(T(X)) \))

is exact. Taking \( H(T^i(X)) = H^i(X) \), we have the long exact sequence:

\[
\cdots \rightarrow H^{i+1}(X) \rightarrow H^i(Z) \rightarrow H^i(Y) \rightarrow H^i(X) \rightarrow \cdots
\]  

(resp., \( \cdots \rightarrow H^i(X) \rightarrow H^i(Y) \rightarrow H^i(Z) \rightarrow H^{i+1}(X) \rightarrow \cdots \))

**Proposition 1.5** The following hold.

1. If \( (X, Y, Z, u, v, w) \) is a triangle, then \( vu = 0, vw = 0 \) and \( T(u)w = 0 \).

2. For any \( X \in \mathcal{C} \), \( \text{Hom}_{\mathcal{C}}(-, X) : \mathcal{C} \rightarrow \mathcal{A}\text{b} \) (resp., \( \text{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \rightarrow \mathcal{A}\text{b} \)) is a homological functor (resp., a cohomological functor).

3. For any homomorphism of triangles

\[
\begin{array}{ccc}
X & \xrightarrow{u} & Y \\
\downarrow f & & \downarrow g \\
X' & \xrightarrow{u'} & Y'
\end{array}
\quad
\begin{array}{ccc}
\downarrow h & & \downarrow h' \\
Z & \xrightarrow{w} & T(X) \\
Z' & \xrightarrow{w'} & T(X')
\end{array}
\]

if two of \( f, g \) and \( h \) are isomorphisms, then the rest is also an isomorphism.

**Definition 1.6** (Compact Object) Let \( \mathcal{C} \) be a triangulated category. An object \( X \in \mathcal{C} \) is called a compact object in \( \mathcal{C} \) if the canonical morphism

\[
\bigsqcup_{i \in I} \text{Hom}_{\mathcal{C}}(C, X_i) \xrightarrow{\sim} \text{Hom}_{\mathcal{C}}(C, \bigsqcup_{i \in I} X_i)
\]

is an isomorphism for any set \( \{X_i\}_{i \in I} \) of objects (if \( \bigsqcup_{i \in I} X_i \) exists in \( \mathcal{C} \)).

For a triangulated category \( \mathcal{C} \), a set \( S \) of compact objects is called a generating set if \( \text{Hom}_{\mathcal{C}}(S, X) = 0 \Rightarrow X = 0 \), and if \( T(S) = S \). A triangulated category \( \mathcal{C} \) is compactly generated if \( \mathcal{C} \) contains arbitrary coproducts, and if it has a generating set.
Definition 1.7 (Homotopy Limit) Let $\mathcal{C}$ be a triangulated category which contains arbitrary coproducts (resp., products). For a sequence $\{X_i \to X_{i+1}\}_{i \in \mathbb{N}}$ (resp., $\{X_{i+1} \to X_i\}_{i \in \mathbb{N}}$) of morphisms in $\mathcal{C}$, the homotopy colimit (resp., homotopy limit) of the sequence is the third (resp., second) term of the triangle

$$
\coprod_i X_i \xrightarrow{1-\text{shift}} \coprod_i X_i \to \hocolim X_i \to T \left( \coprod_i X_i \right)
$$

(resp., $T^{-1} \left( \prod_i X_i \right) \to \holim X_i \to \prod_i X_i \xrightarrow{1-\text{shift}} \prod_i X_i$)

where the above shift morphism is the coproduct (resp., product) of $X_i \xrightarrow{f_i} X_{i+1}$ (resp., $X_{i+1} \xrightarrow{f_i} X_i$) ($i \in \mathbb{N}$).

Proposition 1.8 Let $\mathcal{C}$ be a triangulated category which contains arbitrary coproducts, $\{X_i \to X_{i+1}\}_{i \in \mathbb{N}}$ a sequence of morphisms in $\mathcal{C}$. For a compact object $C$ in $\mathcal{C}$, we have

$$\text{Hom}(C, \hocolim X_i) \cong \lim \text{Hom}(C, X_i)$$

Proof. We have an exact sequence

$$0 \to \coprod_i \text{Hom}(C, X_i) \to \coprod_i \text{Hom}(C, X_i) \to \text{Hom}(C, \hocolim X_i) \to 0$$

Theorem 1.9 (Brown Representability Theorem [Ne]) Let $\mathcal{C}$ be a compactly generated triangulated category. If a homological functor $H : \mathcal{C} \to \mathbb{Ab}$ sends coproducts to products, then it is representable, that is, there is an object $X \in \mathcal{C}$ such that $H \cong \text{Hom}_{\mathcal{C}}(-, X)$.

Sketch of Proof. Let $\mathcal{S}$ be a generating set of $\mathcal{C}$. There exist a coproduct $X_1$ of objects of $\mathcal{S}$ and a morphism $h_{X_1} \to H$ such that $\text{Hom}_{\mathcal{C}}(C, X_1) \to H(C)$ is surjective for any $C \in \mathcal{S}$. For a functor $K_1 = \text{Ker}(h_{X_1} \to H)$ there exists a coproduct $Z_2$ of objects in $\mathcal{S}$ and a morphism $h_{Z_2} \to K_1$ such that $\text{Hom}_{\mathcal{C}}(C, Z_2) \to K_1(C)$ is surjective for any $C \in \mathcal{S}$. Then we have a triangle $Z_2 \to X_1 \to X_2 \to Z_2[1]$. Since $H$ is a homological functor, we have a commutative diagram

$$
\begin{array}{ccc}
H(X_2) & \longrightarrow & H(X_1) \\
\downarrow^i & & \downarrow^i \\
\text{Mor}(h_{X_2}, H) & \longrightarrow & \text{Mor}(h_{X_1}, H)
\end{array}
$$

Then there is a morphism $X_1 \to X_2$ satisfying a commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & K_1 & \longrightarrow & \text{Hom}_{\mathcal{C}}(-, X_1) & \longrightarrow & H \\
\downarrow & & \downarrow & & \downarrow & & \parallel \\
0 & \longrightarrow & K_2 & \longrightarrow & \text{Hom}_{\mathcal{C}}(-, X_2) & \longrightarrow & H
\end{array}
$$
and we have a morphism of exact sequence

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & K_1(C) & \longrightarrow & \text{Hom}_C(C, X_1) & \longrightarrow & H(C) & \longrightarrow & 0 \\
\downarrow^0 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_2(C) & \longrightarrow & \text{Hom}_C(C, X_2) & \longrightarrow & H(C) & \longrightarrow & 0
\end{array}
\]

for any \( C \in S \). By inductive step, we have a triangle

\[
\bigsqcup_i X_i \xrightarrow{1-\text{shift}} \bigsqcup_i X_i \to \text{hocolim} X_i \to T\left(\bigsqcup_i X_i\right)
\]

and we have an exact sequence

\[
\begin{array}{ccccccccc}
H(\text{hocolim} X_i) & \longrightarrow & \prod_i H(X_i) & \longrightarrow & \prod_i H(X_i) \quad \text{by adjunction} \\
\downarrow & & \downarrow & & \downarrow \quad \text{by adjunction} \\
\text{Mor}(\text{hocolim} X_i, H) & \longrightarrow & \prod_i \text{Mor}(h_{X_i}, H) & \longrightarrow & \prod_i \text{Mor}(h_{X_i}, H)
\end{array}
\]

Therefore there is a morphism \( \text{Hom}_C(-, \text{hocolim} X_i) \to H \) such that

\[ \text{Hom}_C(C, \text{hocolim} X_i) \cong H(C) \]

for any \( C \in S \). Hence we have \( \text{Hom}_C(-, \text{hocolim} X_i) \cong H \). \( \square \)

**Corollary 1.10 (Adjoint Functor Theorem [Ne])** Let \( C \) be a compactly generated triangulated category. If a \( \partial \)-functor \( F : C \to D \) commutes with arbitrary coproducts, then there exists a \( \partial \)-functor \( G : D \to C \) which is a right adjoint of \( F \).

**Proof.** For any \( Y \in D \), the functor

\[ \text{Hom}_D(F(-), Y) : C \to \text{Ab} \]

is a homological functor. By Brown representability theorem there is an object \( GY \in C \) such that

\[ \text{Hom}_D(F(-), Y) \cong \text{Hom}_C(-, GY) \quad \square \]

**Definition 1.11 (Multiplicative System)** Let \( S \) be a multiplicative system in a triangulated category \( C \) which satisfies the following conditions:

(FR0) For a morphism \( s \) in \( C \), if there exist \( f, g \) such that \( sf, gs \in S \), then \( s \in S \).

(FR1) \( 1_X \in S \) for every \( X \in C \).

(2) For \( s, t \in S \), if \( st \) is defined, then \( st \in S \).
(FR2) Every diagram in C

\[
\begin{array}{c}
X \\ f \\
\downarrow \\
X'
\end{array} \longrightarrow
\begin{array}{c}
Y \\
\downarrow g \\
Y'
\end{array}
\]

with \( s \in S \), can be completed to a commutative square

\[
\begin{array}{c}
X \\ f \\
\downarrow \\
X'
\end{array} \longrightarrow
\begin{array}{c}
Y \\
\downarrow g \\
Y'
\end{array}
\]

with \( s, t \in S \). Ditto for the statement with all arrows reversed.

(FR3) For \( f, g \in \text{Hom}_C(X, Y) \) the following are equivalent.

1. There exists \( s \in S \) such that \( sf = sg \).
2. There exists \( t \in S \) such that \( ft = gt \).

(FR4) For a morphism \( u \in C \), \( u \in S \) if and only if \( T(u) \in S \).

(FR5) For triangles \((X, Y, Z, u, v, w), (X', Y', Z', u', v', w') \) and morphisms \( f : X \to X' \), \( g : Y \to Y' \) in \( S \) with \( gu = u'f \), there exists \( h : Z \to Z' \) in \( S \) such that \((f, g, h)\) is a homomorphism of triangles.

**Definition 1.12 (Quotient Category)** We define the quotient category \( S^{-1}C \) of \( C \), as follows:

1. \( \text{Ob}(S^{-1}C) = \text{Ob}(C) \).
2. For \( X, Y \in \text{Ob}(C) \), let \( V(X, Y) = \{(s, Y', f)|s : Y \to Y' \in S, f : X \to Y\} \). In \( V(X, Y) \), we define \((s, Y', f) \sim (s', Y'', f')\) if there is \((s'', Y'''', f')\) such that all triangles are commutative in the following diagram:

\[
\begin{array}{c}
X \\
\downarrow f \\
Y'
\end{array} \longrightarrow
\begin{array}{c}
Y'' \\
\downarrow f' \\
Y'''
\end{array} \longrightarrow
\begin{array}{c}
Y \\
\downarrow f'' \\
Y'''
\end{array}
\]

Then we define a morphism from \( X \) to \( Y \) by an equivalence class \( s^{-1}f \) of \((s, Y', f)\).

3. For \( s^{-1}f : X \to Y, t^{-1}g : Y \to Z \), by (FR2) there are \( s' : Z' \to Z'' \in S \) and
$g': Y' \to Z''$ such that $s'o g = g'os$. Then we define $(t^{-1}g)o(s^{-1}f) = (s'ot)^{-1}gof$.

Moreover, we define the quotient functor $Q: \mathcal{C} \to \mathcal{S}^{-1}\mathcal{C}$ by

(Q1) $Q(X) = X$ for $X \in \mathcal{C}$.

(Q2) $Q(f) = 1^{-1}_{-}f$ for a morphism $f: X \to Y$ in $\mathcal{C}$.

Remark 1.13 Can we define (2) in the above?

Definition 1.14 (Épaise Subcategory) Let $\mathcal{C}$ be a triangulated category. A full subcategory $\mathcal{U}$ of $\mathcal{C}$ is called a full triangulated subcategory if $X \to Y$ is a morphism in $\mathcal{U}$, then there is a triangle $X \to Y \to Z \to TX$ with $Z \in \mathcal{U}$.

A full triangulated subcategory $\mathcal{U}$ is called an épaise subcategory if it is closed under direct summands. In this case, let $S(\mathcal{U})$ be the collection of morphisms $s$ such that $X \to Y \to Z \to X[1]$ is a triangle with $Z \in \mathcal{U}$. Then $S(\mathcal{U})$ is a multiplicative system satisfying (FR0) - (FR5). We write $\mathcal{C}/\mathcal{U} = S(\mathcal{U})^{-1}\mathcal{C}$.

In the case that $\mathcal{C}$ contains arbitrary coproducts, a full triangulated subcategory $\mathcal{U}$ is called a localizing subcategory if it is closed under coproducts.

Remark 1.15 The above definition of an épaise subcategory $\mathcal{U}$ is the same as the original definition [Ve], that is, a full triangulated category satisfying that if $X \to Y$ factors through some object in $\mathcal{U}$ and if there is a triangle $X \to Y \to Z \to T(X)$ with $Z \in \mathcal{U}$, then $X, Y \in \mathcal{U}$.

Proposition 1.16 ([BN]) Let $\mathcal{C}$ be a triangulated category which contains arbitrary coproducts. Then any localizing subcategory is an épaise subcategory.

Sketch of Proof. Let $\mathcal{U}$ be a localizing subcategory, and $X \in \mathcal{U}$ with $X = Y \oplus Z$ in $\mathcal{C}$. We take a morphism $e: X \to Y \hookrightarrow X$, and consider the sequence of morphisms

(\ast) \quad X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} \ldots

Then it is easy to see that $Y \cong \text{hocolim}(\ast) \in \mathcal{U}$.

\[\square\]

Proposition 1.17 Let $\mathcal{C}$ be a triangulated category. For a multiplicative system $S$ satisfying the conditions (FR0) - (FR5), let $\mathcal{U}(S)$ be the full triangulated subcategory consisting of objects $Z$ which is in a triangle $X \to Y \to Z \to X[1]$ with $s \in S$. Then the following hold.
1. $S(U)$ and $U(S)$ induce an 1-1 correspondence between the collection of multiplicative systems $S$ satisfying the conditions (FR0) - (FR5) and the collection of épaisse subcategories $U$.

2. For an épaisse subcategory $U$, $C/U$ is a triangulated category whose (distinguished) triangles are defined to be isomorphic to (distinguished) triangles of $C$.

3. Assume $C$ contains arbitrary coproducts. For a localizing subcategory $U$, $C/U$ also contains arbitrary coproducts.

**Definition 1.18 (stable t-structure)** For full subcategories $U$ and $V$ of a triangulated category $C$, $(U, V)$ is called a stable t-structure in $C$ provided that

1. $U$ and $V$ are stable for translations.
2. $\text{Hom}_C(U, V) = 0$.
3. For every $X \in C$, there exists a triangle $U \to X \to V \to TU$ with $U \in U$ and $V \in V$.

**Proposition 1.19 ([BBD], c.f. [Mi])** Let $C$ be a triangulated category, $(U, V)$ a stable t-structure in $C$, and $i_* : U \to C, j_* : V \to C$ the canonical embeddings. Then the following hold.

1. $U$ and $V$ is épaisse subcategories of $C$.
2. $i_*$ (resp., $j_*$) has a right adjoint $i^!$ (resp., a left adjoint $j^*$).
3. The adjunction arrows induce a triangle

$$i_* i^! X \xrightarrow{\alpha_X} X \xrightarrow{\beta_X} j_* j^* X \to i_* i^! X[1]$$

for any $X \in C$.
4. $C/U$ (resp., $C/V$) exists, and it is triangulated equivalent to $V$ (resp., $U$).

**Corollary 1.20** Let $C$ be a compactly generated triangulated category, and $U$ a localizing subcategory of $C$. Then $C/U$ can be defined if and only if there is a full triangulated subcategory $V$ such that $(U, V)$ a stable t-structure in $C$.

**Proof.** If $C/U$ can be defined, then the quotient functor $Q : C \to C/U$ commutes with coproducts. By Adjoint Functor Theorem, $Q$ has a right adjoint $F : C/U \to C$. By Proposition 1.19, it is easy to see that $(U, \text{Im} F)$ is a stable t-structure in $C$. $\square$
2 Derived Categories

Throughout this section, $\mathcal{A}$ is an abelian category and $\mathcal{B}$ is an additive subcategory of $\mathcal{A}$ which is closed under isomorphisms.

**Definition 2.1 (Complex)** A (cochain) complex is a collection $X^\cdot = (X^n, d_X^n : X^n \to X^{n+1})_{n \in \mathbb{Z}}$ of objects and morphisms of $\mathcal{B}$ such that $d_X^{n+1}d_X^n = 0$. A complex $X^\cdot = (X^n, d_X^n : X^n \to X^{n+1})_{n \in \mathbb{Z}}$ is called bounded below (resp., bounded above, bounded) if $X^n = 0$ for $n \ll 0$ (resp., $n \gg 0$ and $n \gg 0$).

A complex $X^\cdot = (X^n, d_X^n)$ is called a stalk complex if there exists an integer $n_0$ such that $X^i = 0$ if $i \neq n_0$. We identify objects of $\mathcal{B}$ with a stalk complexes of degree 0.

A morphism $f : X^\cdot \to Y^\cdot$ of complexes is a collection of morphisms $f^n : X^n \to Y^n$ which makes a commutative diagram

$$
\cdots \to X^n \xrightarrow{d_X^n} X^{n+1} \to \cdots \\
\downarrow f^n \quad \downarrow f^{n+1} \\
\cdots \to Y^n \xrightarrow{d_Y^n} Y^{n+1} \to \cdots
$$

We denote by $\mathcal{C}(\mathcal{B})$ (resp., $\mathcal{C}^+(\mathcal{B})$, $\mathcal{C}^-(\mathcal{B})$, $\mathcal{C}^b(\mathcal{B})$) the category of complexes (resp., bounded below complexes, bounded above complexes, bounded complexes) of $\mathcal{B}$. An autofunctor $T : \mathcal{C}(\mathcal{B}) \to \mathcal{C}(\mathcal{B})$ is called translation if $(TX^\cdot)^n = X^{n+1}$ and $(Td_X^n)^n = -d_X^{n+1}$ for any complex $X^\cdot = (X^n, d_X^n)$.

In $\mathcal{C}(\mathcal{A})$, a morphism $u : X^\cdot \to Y^\cdot$ is called a quasi-isomorphism if $H_n(u)$ is an isomorphism for any $n$.

In this section, "+" means "nothing", "+", "−" or "b".

**Definition 2.2** For $u \in \text{Hom}_{\mathcal{C}(\mathcal{B})}(X^\cdot, Y^\cdot)$, the mapping cone of $u$ is a complex $M^\cdot(u)$ with

$$M^n(u) = X^{n+1} \oplus Y^n,$$

$$d^n_{M(u)} = \begin{bmatrix} -d_X^{n+1} & 0 \\ u_{n+1} & d_Y^n \end{bmatrix} : X^{n+1} \oplus Y^n \to X^{n+2} \oplus Y^{n+1}.$$

**Definition 2.3 (Homotopy Relation)** Two morphisms $f, g \in \text{Hom}_{\mathcal{C}(\mathcal{B})}(X^\cdot, Y^\cdot)$ are said to be homotopic (denote by $f \simeq g$) if there is a collection of morphisms $h = (h^n)$, $h^n : X^n \to Y^{n+1}$ such that $f^n - g^n = d_Y^n - h^n + h^{n+1}d_X^n$ for all $n \in \mathbb{Z}$.

**Definition 2.4 (Homotopy Category)** The homotopy category $K^\cdot(\mathcal{B})$ of $\mathcal{B}$ is defined by

1. $\text{Ob}(K^\cdot(\mathcal{B})) = \text{Ob}(\mathcal{C}(\mathcal{B}))$,

2. $\text{Hom}_{K^\cdot(\mathcal{B})}(X^\cdot, Y^\cdot) = \text{Hom}_{\mathcal{C}(\mathcal{B})}(X^\cdot, Y^\cdot)/ \simeq_h$ for $X^\cdot, Y^\cdot \in \text{Ob}(K^\cdot(\mathcal{B}))$. 

**Proposition 2.5** A category $\mathsf{K}^*(\mathcal{B})$ is a triangulated category whose (distinguished) triangles are defined to be isomorphic to

$$X' \xrightarrow{u} Y' \rightarrow M(u) \rightarrow T(X')$$

for any $u : X' \rightarrow Y'$ in $\mathsf{K}^*(\mathcal{B})$.

**Definition 2.6** (Derived Category) The derived category $\mathsf{D}^*(\mathcal{A})$ of an abelian category $\mathcal{A}$ is $\mathsf{K}^*(\mathcal{A})/\mathsf{K}^*(\mathcal{A})$, where $\mathsf{K}^*(\mathcal{A})$ is the full subcategory of $\mathsf{K}^*(\mathcal{A})$ consisting of null complexes, that is, complexes whose all cohomologies are 0.

**Proposition 2.7** The following hold.

1. $\mathsf{D}^*(\mathcal{A})$ is a triangulated category, and the canonical functor $Q : \mathsf{K}^*(\mathcal{A}) \rightarrow \mathsf{D}^*(\mathcal{A})$ is a $\partial$-functor.
2. The $i$-th cohomology of complexes is a cohomological functor in the sense of Definition 1.4.

**Proposition 2.8** If $0 \rightarrow X' \xrightarrow{u} Y' \xrightarrow{v} Z \rightarrow 0$ is an exact sequence in $\mathsf{C}(\mathcal{A})$, then it can be embedded in a triangle in $\mathsf{D}(\mathcal{A})$

$$Q(X') \xrightarrow{Q(u)} Q(Y') \xrightarrow{Q(v)} Q(Z) \rightarrow TQ(X').$$

**Definition 2.9** ($\mathsf{K}$-injective Complex) A complex $X'$ of $\mathsf{K}(\mathcal{B})$ is called $\mathsf{K}$-injective (resp., $\mathsf{K}$-projective) if

$$\text{Hom}_{\mathsf{K}(\mathcal{B})}(N', X') = 0 \quad (\text{resp., } \text{Hom}_{\mathsf{K}(\mathcal{B})}(X', N') = 0)$$

for any null complex $N'$.

**Example 2.10** Let $A$ be a ring, $\text{Mod} A$ the category of right $A$-modules, and $\text{Inj} A$ (resp., $\text{Proj} A$) the category of injective (resp., projective) right $A$-modules. Then any complex $I' \in \mathsf{K}^+ (\text{Inj} A)$ (resp., $P' \in \mathsf{K}^- (\text{Proj} A)$) is a $\mathsf{K}$-injective (resp., $\mathsf{K}$-projective) complex in $\mathsf{K}(\text{Mod} A)$.

**Example 2.11** Let $k$ be a field, $A = k[x]/(x^2)$, and

$$X' : \cdots \xrightarrow{\varepsilon} A \xrightarrow{\varepsilon} A \xrightarrow{\varepsilon} \cdots.$$ 

Then $X'$ is a null complex of finitely generated projective-injective $A$-modules. But it is neither $\mathsf{K}$-projective nor $\mathsf{K}$-injective, because $\text{Hom}_{\mathsf{K}(\text{Mod} A)}(X', X') \neq 0$.

**Theorem 2.12** ([Sp], [Ne], [LAM], [Fr]) Let $\mathsf{K}^{\text{inj}}(\text{Mod} A)$ (resp., $\mathsf{K}^{\text{proj}}(\text{Mod} A)$) be the category of $\mathsf{K}$-injective (resp., $\mathsf{K}$-projective) complexes, then the following hold.

1. $(\mathsf{K}^{\text{proj}}(\text{Mod} A), \mathsf{K}^*(\text{Mod} A))$ is a stable $t$-structure in $\mathsf{D}(\text{Mod} A)$, and hence $\mathsf{D}(\text{Mod} A)$ exists and is triangulated equivalent to $\mathsf{K}^{\text{proj}}(\text{Mod} A)$. 


2. \((K^\mathcal{A}(\text{Mod } A), K^{i\mathcal{A}}(\text{Mod } A))\) is a stable \(t\)-structure in \(K(\text{Mod } A)\), and hence \(D(\text{Mod } A)\) is triangulated equivalent to \(K^{i\mathcal{A}}(\text{Mod } A)\).

3. For a Grothendieck category \(\mathcal{A}\), \((K^\mathcal{A}(\mathcal{A}), K^{i\mathcal{A}}(\mathcal{A}))\) is a stable \(t\)-structure in \(K(\mathcal{A})\), and hence \(D(\mathcal{A})\) exists and is triangulated equivalent to \(K^{i\mathcal{A}}(\mathcal{A})\).

**Proof.** For a complex \(X^\cdot = (X^n, d^n)\), we define the following truncations:

\[
\sigma_{\leq n} X^\cdot : \cdots \to X^{n-2} \to X^{n-1} \to \text{Ker } d^n \to 0 \to \cdots \\
\sigma_{\geq n} X^\cdot : \cdots \to 0 \to \text{Cok } d^n \to X^{n+1} \to X^{n+2} \to \cdots
\]

(1) For any \(n\), there is a complex \(P^n \in K^- (\text{Proj } A)\) which has a quasi-isomorphism \(P^n \to \sigma_{\leq n} X^\cdot\). Then we have the following quasi-isomorphisms (qis)

\[
X^\cdot \cong \lim_{\sigma_{\leq n} X^\cdot \to \text{holim } \sigma_{\leq n} X^\cdot \to \text{holim } P^n}
\]

Since \(\text{Hom}_c(\prod P^n, -) \cong \prod \text{Hom}_c(P^n, -)\), \(\prod P^n\) is \(K\)-projective. Here \(h^M = \text{Hom}_c(M, -)\) for any object \(M\). It is easy to see that \(\text{holim } P^n\) is \(K\)-projective by the following exact sequence

\[
h\prod P^n \to h\prod P^n \to h\text{holim } P^n \to hT(\prod P^n) \to hT(\prod P^n)
\]

(2) For any \(n\), there is a complex \(I^n \in K^+ (\text{Inj } A)\) which has a quasi-isomorphism \(\sigma_{\geq -n} X^\cdot \to I^n\). Then we have the following quasi-isomorphisms (qis)

\[
X^\cdot \cong \lim_{\sigma_{\geq -n} X^\cdot \to \text{holim } \sigma_{\geq -n} X^\cdot \to \lim I^n}
\]

by the same reason of (1), we have the statement.

(3) Because there is a ring \(A\) such that \(\mathcal{A}\) is a localization of \(\text{Mod } A\) (Gabriel-Popescu Theorem). See [LAM] or [Fr] for details.

\[\square\]

**Remark 2.13** If \(P^\cdot\) is a \(K\)-projective complex (e.g. a bounded above complex of projective \(A\)-modules), then we have

\[
\text{Hom}_{K(\text{Mod } A)}(P^\cdot, X^\cdot) \cong \text{Hom}_{D(\text{Mod } A)}(P^\cdot, X^\cdot)
\]

for any complex \(X^\cdot\). Similarly, for a \(K\)-injective complex \(I^\cdot\) (e.g. bounded below complex of injective \(A\)-modules), then we have

\[
\text{Hom}_{K(\text{Mod } A)}(X^\cdot, I^\cdot) \cong \text{Hom}_{D(\text{Mod } A)}(X^\cdot, I^\cdot)
\]

for any complex \(X^\cdot\). In particular, given \(A\)-modules \(M, N\), we have

\[
\text{Ext}^1_A(M, N) \cong \text{Hom}_{D(\text{Mod } A)}(M, N[i])
\]
Definition 2.14 (Double Complex, Total complex) A double complex $C^{\cdot, \cdot}$ is a bigraded object $(C^{p,q})_{p,q \in \mathbb{Z}}$ of $A$ together with $d_{1}^{p,q} : C^{p,q} \rightarrow C^{p+1,q}$ and $d_{II}^{p,q} : C^{p,q} \rightarrow C^{p,q+1}$ such that

$$C^{q} = (C^{q}, d_{1}^{q} : C^{q} \rightarrow C^{q+1})$$

are complexes satisfying $d_{1}^{p,q+1}d_{II}^{p,q} - d_{II}^{p+1,q}d_{1}^{p,q} = 0$. For a double complex $C^{\cdot, \cdot}$, we define the total complexes

$$\text{Tot}^{I} C^{\cdot, \cdot} = (X^{\cdot}, d^{\cdot})$$

where $X^{\cdot} = \prod_{p+q=n} C^{p,q}$, $d^{\cdot} = \prod_{p+q=n} (d_{1}^{p,q} + (-1)^{p}d_{II}^{p,q})$

be an exact sequence with $P^{n}$ (resp., $I^{n}$) being a complex of projective (resp., injective) $A$-modules. We call $\cdots \rightarrow P^{1} \rightarrow P^{0} \rightarrow X^{\cdot} \rightarrow 0$ (resp., $0 \rightarrow X^{\cdot} \rightarrow I^{0} \rightarrow I^{1} \rightarrow \cdots$) a Cartan-Eilenberg projective (resp., injective) resolution of $X^{\cdot}$ if the induced complexes $\cdots \rightarrow B^{n}(P^{1}) \rightarrow B^{n}(P^{0})$ and $\cdots \rightarrow H^{n}(P^{1}) \rightarrow H^{n}(P^{0})$ (resp., $B^{n}(I^{0}) \rightarrow B^{n}(I^{1})$ and $H^{n}(I^{0}) \rightarrow H^{n}(I^{1})$) are also projective (resp., injective) resolutions of $B^{n}(X^{\cdot}), H^{n}(X^{\cdot})$, respectively.

Proposition 2.16 Under the setting of Definition 2.15, the following hold.

1. $\text{Tot}^{I} P^{\cdot}$ is $K$-projective, and the induced morphism of complexes $\text{Tot}^{I} P^{\cdot} \rightarrow X^{\cdot}$ is a quasi-isomorphism.

2. $\text{Tot}^{I} I^{\cdot}$ is $K$-injective, and the induced morphism of complexes $X^{\cdot} \rightarrow \text{Tot}^{I} I^{\cdot}$ is a quasi-isomorphism.

Sketch of Proof. We consider the following truncations

$$\sigma_{\leq n} P^{\cdot} : \cdots \rightarrow \sigma_{\leq n} P^{-1} \rightarrow \sigma_{\leq n} P^{0} \rightarrow \sigma_{\geq n} \sigma_{\geq n} I^{0} \rightarrow \cdots$$

Then it is easy to see $\text{Tot}^{I} \sigma_{\leq n} P^{\cdot}$ (resp., $\text{Tot}^{I} \sigma_{\geq n} I^{\cdot}$) is $K$-projective (resp., $K$-injective), and that the induced morphism of complexes $\text{Tot}^{I} \sigma_{\leq n} P^{\cdot} \rightarrow \sigma_{\leq n} X^{\cdot}$ (resp., $\sigma_{\geq n} X^{\cdot} \rightarrow \text{Tot}^{I} \sigma_{\geq n} I^{\cdot}$) is a quasi-isomorphism. Therefore we have the following quasi-isomorphisms (qis)

$$X^{\cdot} \xleftarrow{\text{qis}} \text{hocolim} \sigma_{\leq n} \xrightarrow{\text{qis}} \text{holim} \sigma_{\geq n} X^{\cdot} \text{qis} \xrightarrow{\text{qis}} \text{hocolim} \text{Tot}^{I} \sigma_{\leq n} P^{\cdot} \text{qis} \xrightarrow{\text{qis}} \text{Tot}^{I} P^{\cdot}$$

(resp., $X^{\cdot} \xleftarrow{\text{qis}} \text{holim} \sigma_{\geq n} \xrightarrow{\text{qis}} \text{holim} \text{Tot}^{I} \sigma_{\geq n} I^{\cdot} \text{qis} \xrightarrow{\text{qis}} \text{Tot}^{I} I^{\cdot}$)

and $\text{Tot}^{I} P^{\cdot}$ (resp., $\text{Tot}^{I} I^{\cdot}$) is $K$-projective (resp., $K$-injective). \qed
Definition 2.17 (Right Derived Functor) For a $\partial$-functor $F : K^\bullet(A) \to K(A')$, the right derived functor of $F$ is a $\partial$-functor

$$R^*F : D^\bullet(A) \to D(A')$$

together with a functorial morphism of $\partial$-functors

$$\xi \in \partial \text{Mor}(Q_{A'} \circ F, R^*F \circ Q_A^*)$$

with the following property:

For $G \in \partial(D^\bullet(A), D(A'))$ and $\zeta \in \partial \text{Mor}(Q_{A'} \circ F, G \circ Q_A^*)$, there exists a unique morphism $\eta \in \partial \text{Mor}(R^*F, G)$ such that

$$\zeta = (\eta Q_{A'})\xi.$$  

In other words, we can simply write the above using functor categories. For triangulated categories $C, C'$, the $\partial$-functor category $\partial(C, C')$ is the category (??) consisting of $\partial$-functors from $C$ to $C'$ as objects and $\partial$-functorial morphisms as morphisms. Then we have

$$\partial \text{Mor}(Q_{A'} \circ F, -Q_A^*) \cong \partial \text{Mor}(R^*F, -)$$

as functors from $\partial(D^\bullet(A), D(A'))$ to $\mathcal{S}et$.

Proposition 2.18 Let $A, A'$ be abelian categories, $F : K(A) \to K(A')$ a $\partial$-functor. If $A$ is a Grothendieck category, then we have the right derived functor $R^*F : D(A) \to D(A')$ such that $F(X^\cdot) \cong R^*F(X^\cdot)$ for any $K$-injective complex $X^\cdot$.

Remark 2.19 In the setting of Definition 2.17, the left derived functor $L^*F : D^\bullet(A) \to D(A')$ can be also defined by reversing arrows of $\partial$-functorial morphisms. Let $R^n F(X^\cdot) = H^n(R^*F(X^\cdot))$, $L^n F(X^\cdot) = H^n(L^*F(X^\cdot))$, then $R^n F$ (resp., $L^n F$) coincides with the ordinary definition of the $n$-th right (resp., left) derived functor. According to Proposition 2.16, if $F$ commutes with products (resp., coproducts), then the $n$-th hypercohomology $R^n F$ (resp., hyperhomology $L^n F$) coincides with $R^n F$ (resp., $L^n F$) (c.f. [CE], [Mc], [We]).

Definition 2.20 $(\text{Hom}_{A'}^{\bullet}, \otimes_A)$ Let $X^\cdot, Y^\cdot$ be complexes in $C(\text{Mod} A)$, $Z^\cdot$ a complex in $C(\text{Mod} A^{op})$. We define the complex $\text{Hom}_{A'}(X^\cdot, Y^\cdot)$ in $C(\text{Ab})$ by

$$\text{Hom}_{A'}(X^\cdot, Y^\cdot) = \prod_{j-i=n} \text{Hom}_A(X^i, Y^j), \quad d^n_{\text{Hom}(X,Y)}(f) = d_X \circ f - (-1)^n f \circ d_Y$$

for $f \in \text{Hom}_{A'}(X^\cdot, Y^\cdot)$. And we define the complex $X^\cdot \otimes_A Z^\cdot$ in $C(\text{Ab})$ by

$$X^\cdot \otimes_A Z^\cdot = \bigoplus_{i+j=n} X^i \otimes_A Z^j, \quad d^n_{X \otimes Y} = d_X \otimes 1 + (-1)^n 1 \otimes d_Z$$
Proposition 2.21 Let $A$ be a ring. Then we have a right derived functor

$$\mathcal{R} \text{Hom}_A : \text{D}(\text{Mod} A)^{\text{op}} \times \text{D}(\text{Mod} A) \to \text{D}(\mathfrak{A}b)$$

and a left derived functor

$$\hat{\mathcal{L}}_A : \text{D}(\text{Mod} A) \times \text{D}(\text{Mod} A^{\text{op}}) \to \text{D}(\mathfrak{A}b)$$

Proposition 2.22 Let $A$ be a ring. For complexes $X^\cdot, Y^\cdot$, we have isomorphisms

$$H^n(\text{Hom}_A(X^\cdot, Y^\cdot)) \cong \text{Hom}_{\text{D}(\text{Mod} A)}(X^\cdot, Y^n)$$

$$H^n(\mathcal{R} \text{Hom}_A(X^\cdot, Y^\cdot)) \cong \text{Hom}_{\text{D}(\text{Mod} A)}(X^\cdot, Y^n)$$

Definition 2.23 (Perfect Complex) Let $A$ be a ring. A complex $X^\cdot \in \text{D}(\text{Mod} A)$ is called a perfect complex if $X^\cdot$ is quasi-isomorphic to a bounded complex of finitely generated projective $A$-modules.

Let $X$ be a scheme, $D(X)$ the derived category of sheaves of $\mathcal{O}_X$-modules. We denote by $D_{qc}(X)$ the full subcategory of $D(X)$ consisting of complexes whose cohomologies are quasi-coherent sheaves. A complex $X^\cdot \in D_{qc}(X)$ is called a perfect complex if $X^\cdot$ is locally quasi-isomorphic to a bounded complex of vector bundles (See [TT]).

We denote by $D_{pf}(A)$ the full triangulated subcategory of $D(A)$ consisting of perfect complexes.

Proposition 2.24 ([Rd1], [Ne]) For a ring $A$, the following hold.

1. A complex $X^\cdot \in \text{D}(\text{Mod} A)$ is perfect if and only if it is a compact object in $\text{D}(\text{Mod} A)$.

2. $\text{D}(\text{Mod} A)$ is compactly generated.

Theorem 2.25 ([BV]) Let $X$ be a quasi-compact quasi-separated scheme, then the following hold.

1. A complex $X^\cdot \in D_{qc}(X)$ is perfect if and only if it is a compact object in $D_{qc}(X)$.

2. $D_{qc}(X)$ is compactly generated.

Theorem 2.26 ([BN]) Let $X$ be a quasi-compact separated scheme, then the canonical functor $D(\text{Qcoh} X) \to D_{qc}(X)$ is a triangulated equivalence, where $\text{Qcoh} X$ is the category of quasi-coherent sheaves of $\mathcal{O}_X$-modules.

Corollary 2.27 ([BV]) Let $X$ be smooth over a field, then we have

$$D^b(\text{coh} X) \stackrel{\Delta}{\cong} D_{pf}(X).$$

where $\text{coh} X$ is the category of coherent sheaves of $\mathcal{O}_X$-modules.
For a ring $A$, we denote by $\text{proj} \ A$ the category of finitely generated projective $A$-modules.

**Theorem 2.28 ([Rd1], [Rd2])** Let $A$, $B$ be algebras over a field $k$. The following are equivalent.

1. $\text{D}(\text{Mod} \ A) \cong \text{D}(\text{Mod} \ B)$.

2. $\text{K}^b(\text{proj} \ A) \cong \text{K}^b(\text{proj} \ B)$.

3. There is a perfect complex $T' \in \text{D}(\text{Mod} \ A)$ such that
   
   (a) $B \cong \text{End}_{\text{D}(\text{Mod} \ A)}(T')$,
   
   (b) $\text{Hom}_{\text{D}(\text{Mod} \ A)}(T', T'[i]) = 0$ for $i \neq 0$,
   
   (c) $\{T'[i] \mid i \in \mathbb{Z}\}$ is a generating set in $\text{D}(\text{Mod} \ A)$.

4. There is a complex $V'$ of $B$-$A$-bimodules such that
   
   $R \text{Hom}_A(V', -) : \text{D}(\text{Mod} \ A) \to \text{D}(\text{Mod} \ B)$

   is an equivalence.

   In this case, $T'$ is called a tilting complex for $A$, $V'$ is called a two-sided tilting complex, and $R \text{Hom}_A(V', -)$ is called a standard equivalence.

**Theorem 2.29 ([BO])** Let $X$ be a smooth irreducible projective variety with ample canonical or anticanonical sheaf. If $X'$ is a smooth algebraic variety such that

$\text{D}^b(\text{coh} \ X) \cong \Delta \text{D}^b(\text{coh} \ X')$,

then $X'$ is isomorphic to $X$.

**Theorem 2.30 ([Be])** Let $\mathbb{P}^n_k$ be the $n$-dimensional projective space over a field $k$, and let $T_1 = \bigoplus_{i=0}^n \mathcal{O}(i)$, $T_2 = \bigoplus_{i=0}^n \mathcal{O}(-i)$, and $B_1 = \text{End}_\mathbb{P}(T_1)$, $B_2 = \text{End}_\mathbb{P}(T_2)$. Then $B_i$ is a finite dimensional $k$-algebra of finite global dimension, and

$\text{D}^b(\text{coh} \ \mathbb{P}) \cong \text{D}^b(\text{mod} \ B_i)$

where $\text{mod} \ B_i$ is the category of finitely generated $B_i$-modules ($i = 1, 2$).

**Definition 2.31** Let $A$ be an algebra over a field $k$. The derived Picard group of $A$ (relative to $k$) is

$\text{DPic}_k(A) := \{\text{two-sided tilting complexes } T \in \text{D}^b(\text{Mod} \ A^e) \mid \text{isomorphism}\}$

with identity element $A$, product $(T_1, T_2) \mapsto T_1 \otimes^L_A T_2$ and inverse $T \mapsto T' := R \text{Hom}_A(T, A)$. Given any $k$-linear triangulated category $\mathcal{C}$ we let

$\text{Out}^A_\Delta(\mathcal{C}) := \{\text{k-linear triangulated self-equivalences of } \mathcal{C} \mid \text{isomorphism}\}$. 
Theorem 2.32 ([MY]) Let $k$ be an algebraically closed field, and $A$ a finite dimensional hereditary $k$-algebra. Then we have

$$\text{DPic}_k(A) = \text{Out}_k^\Delta(D^b(\text{Mod } A)) = \text{Out}_k^\Delta(D^b(\text{mod } A))$$

M. Kontsevich and A. Rosenberg introduced the notion of non-commutative projective spaces $\mathbb{NP}^n$ [KR], and showed that

$$D^b(\text{Qcoh } \mathbb{NP}^n) \cong D^b(\text{mod } kQ_n)$$
$$D^b(\text{coh } \mathbb{NP}^n) \cong D^b(\text{mod } kQ_n)$$

where $Q_n$ is the quiver

$$\alpha_0 \quad \alpha_1 \cdots$$

Corollary 2.33 ([MY]) For non-commutative projective spaces $\mathbb{NP}^n$, we have

$$\text{Out}_k^\Delta(D^b(\text{Qcoh } \mathbb{NP}^n)) \cong \text{Out}_k^\Delta(D^b(\text{coh } \mathbb{NP}^n))$$
$$\cong \mathbb{Z} \times (\mathbb{Z} \ltimes \text{PGL}_{n+1}(k))$$

Theorem 2.34 ([BO]) Let $X$ be a smooth irreducible projective variety with ample canonical or anticanonical sheaf. Then $\text{Out}_k^\Delta(D^b(\text{coh } X))$ is generated by the automorphisms of variety, the twists by invertible sheaves and the translations, and hence $\text{Out}_k^\Delta(D^b(\text{coh } X)) \cong (\text{Aut}_k X \times \text{Pic } X) \times \mathbb{Z}$.

References


