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# Laplacians of categories and their spectral zeta functions
## (圏のラプラシアンとスペクトルゼータ関数)

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## 1 Cauchy-Schwarz inequalities

Let $\mathbf{K}$ be a category. Denote by $\mathrm{Ob}(\mathbf{K})$ the set of objects of $\mathbf{K}$ (up to isomorphism), by $\mathrm{Mor}(\mathbf{K})$ the set of morphisms of $\mathbf{K}$, and by $\mathrm{Mor}_{\mathbf{K}}(X,Y)$ the set of morphisms from $X$ to $Y$ for $X,Y\in \mathrm{Ob}(\mathbf{K})$. We also write $p:X\to Y$ instead of $P\in \mathrm{Mor}_{\mathbf{K}}(X,Y)$. For given two objects $X$ and $Y$, we put

$$\langle X,Y\rangle_{\mathbf{K}}:=\#\mathrm{Mor}_{\mathbf{K}}(X,Y), \tag{1.1}$$

where $\#A$ denotes the cardinality of a set $A$. In the sequel, we mainly deal with the subset $\mathrm{Ob}_{o}(\mathbf{K})\subset \mathrm{Ob}(\mathbf{K})$ such that $X,Y\in \mathrm{Ob}_{o}(\mathbf{K})$ implies $\langle X,Y\rangle_{\mathbf{K}}<\infty$. In [KuW, I], the following problem is proposed and studied.

**Problem 1.1.** If $\mathbf{K}$ is a "good" category, then does the **Cauchy-Schwarz inequality**

$$\langle X,Y\rangle_{\mathbf{K}}\langle Y,X\rangle_{\mathbf{K}}\leq\langle X,X\rangle_{\mathbf{K}}\langle Y,Y\rangle_{\mathbf{K}} \tag{1.2}$$

holds for any $X,Y\in \mathrm{Ob}_{o}(\mathbf{K})$ ?

We show several examples of the Cauchy-Schwarz inequality (1.2).

**Example 1.1.** Let $\mathrm{Mod}(\mathbb{F}_{q})$ be the category consisting of $\mathbb{F}_{q}$-modules and $\mathbb{F}_{q}$-linear maps. The category $\mathrm{Mod}(\mathbb{F}_{q})$ satisfies (1.2). In fact, $\mathrm{Ob}_{o}(\mathrm{Mod}(\mathbb{F}_{q}))=\{\mathbb{F}_{q}^{k}\mid k\in \mathbb{Z}_{\geq 0}\}$ and

$$\langle \mathbb{F}_{q}^{m},\mathbb{F}_{q}^{n}\rangle_{\mathrm{Mod}(\mathbb{F}_{q})}\langle \mathbb{F}_{q}^{n},\mathbb{F}_{q}^{m}\rangle_{\mathrm{Mod}(\mathbb{F}_{q})}=q^{2mn}\leq q^{m^{2}+n^{2}}=\langle \mathbb{F}_{q}^{m},\mathbb{F}_{q}^{m}\rangle_{\mathrm{Mod}(\mathbb{F}_{q})}\langle \mathbb{F}_{q}^{n},\mathbb{F}_{q}^{n}\rangle_{\mathrm{Mod}(\mathbb{F}_{q})} \tag{1.3}$$

for all $m,n\in \mathbb{Z}_{\geq 0}$.

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**Example 1.2.** The category **Set** of all sets and all maps satisfies (1.2). In fact, $\mathrm{Ob}_o(\mathbf{Set}) = \{ [k] \mid k \in \mathbb{Z}_{\geq 0} \}$ and

$$\langle [m], [n] \rangle_{\mathbf{Set}} \langle [n], [m] \rangle_{\mathbf{Set}} = m^n n^m \leq m^m n^n = \langle [m], [m] \rangle_{\mathbf{Set}} \langle [n], [n] \rangle_{\mathbf{Set}} \tag{1.4}$$

for all $m, n \in \mathbb{Z}_{\geq 0}$. Here we put $[k] := \{1, 2, \ldots, k\}$.

**Example 1.3.** Denote by **Gr** the category consisting of groups and group homomorphisms. Since it is difficult to enumerate the number of homomorphisms between given two finite groups in general, it seems quite hard to prove the Cauchy-Schwarz inequality (1.2) for **Gr**. However, we can prove the Cauchy-Schwarz inequality for the subcategory **Ab** of abelian groups and group homomorphisms because we can determine the number of homomorphisms explicitly (see [KuW]).

*Remark 1.1.* (1) If we consider the inequality

$$\langle X, Y \rangle_{\mathbf{K}}^2 \leq \langle X, X \rangle_{\mathbf{K}} \langle Y, Y \rangle_{\mathbf{K}}, \tag{1.5}$$

instead of (1.2), then, for instance, this is not true even for the category **Set**. In fact,

$$\langle [8], [10] \rangle_{\mathbf{Set}}^2 = 8^{20} > 8^8 \times 10^{10} = \langle [8], [8] \rangle_{\mathbf{Set}} \langle [10], [10] \rangle_{\mathbf{Set}}. \tag{1.6}$$

(2) It is easy to construct an artificial counterexample to the Cauchy-Schwarz inequality (1.2). Let **K** be a category which has only two objects, say $X$ and $Y$, and let the morphisms of **K** be given by

$$\begin{aligned} \mathrm{Mor}_{\mathbf{K}}(X, X) &= \{0, 1\}, \quad \mathrm{Mor}_{\mathbf{K}}(Y, Y) = \{0, 1\}, \\ \mathrm{Mor}_{\mathbf{K}}(X, Y) &= \{0\}, \quad \mathrm{Mor}_{\mathbf{K}}(Y, X) = \{a_1, a_2, \ldots, a_n\} \end{aligned}$$

with the following composition rules:

$$0 \cdot p = 0, \quad p \cdot 0 = 0, \quad 1 \cdot p = 1, \quad p \cdot 1 = 1 \quad (p \in \mathrm{Mor}(\mathbf{K})).$$

Then we have

$$\begin{aligned} \#\mathrm{Mor}_{\mathbf{K}}(X, Y) \#\mathrm{Mor}_{\mathbf{K}}(Y, X) &= n, \\ \#\mathrm{Mor}_{\mathbf{K}}(X, X) \#\mathrm{Mor}_{\mathbf{K}}(Y, Y) &= 4. \end{aligned}$$

Therefore the inequality does not hold if $n > 4$.

We notice that the Cauchy-Schwarz inequality is equivalent to the positivity of the $2 \times 2$ matrix

$$\Delta_{\mathbf{K}}(X, Y) = \begin{pmatrix} \langle X, X \rangle_{\mathbf{K}} & \langle X, Y \rangle_{\mathbf{K}} \\ \langle Y, X \rangle_{\mathbf{K}} & \langle Y, Y \rangle_{\mathbf{K}} \end{pmatrix}.$$

Actually, (1.2) is equivalent to the inequality $\det \Delta_{\mathbf{K}}(X, Y) \geq 0$. The Cauchy-Schwarz inequality is thus regarded as a special case of the following problem.

**Problem 1.2.** Define the **Laplacian** $\Delta$ of a given category $\mathbf{K}$ by

$$\Delta_{\mathbf{K}} := (\langle X, Y\rangle_{\mathbf{K}})_{X,Y\in \mathrm{Ob}_o(\mathbf{K})}. \tag{1.7}$$

If $\mathbf{K}$ is a "good" category, the can one say that $\Delta_{\mathbf{K}}$ is positive in the sense that every principal minor of $\Delta_{\mathbf{K}}$ is positive?

**Example 1.4.** The Laplacians $\Delta_{\mathbf{Set}}, \Delta_{\mathrm{Mod}(\mathbb{F}_q)}$ are positive [KuST]. The Laplacian $\Delta_{\mathbf{Ab}}$ of the category $\mathbf{Ab}$ is expected to be positive but it is not proved yet.

A study of the Laplacian $\Delta_{\mathbf{K}}$ of a given category $\mathbf{K}$ is originally motivated by the study of the zeta functions of categories introduced by Kurokawa for the sake of unifying various zeta functions [Ku]. Let us recall the definition of the zeta function of a category. Assume that $\mathbf{K}$ is a category with a zero object, that is, an object which is initial and terminal. An object $X$ is called simple if $\mathrm{Mor}_{\mathbf{K}}(X, Y)$ consists of monomorphisms for any object $Y \in \mathrm{Ob}(\mathbf{K})$. Denote by $\mathrm{Prim}(\mathbf{K})$ the set of isomorphism classes of simple finite objects in $\mathbf{K}$. The **zeta function** of the category $\mathbf{K}$ is defined by the Euler product

$$\zeta(s, \mathbf{K}) := \prod_{P\in \mathrm{Prim}(\mathbf{K})} (1 - N(P)^{-s})^{-1}, \tag{1.8}$$

where $N(P)$ is the norm of $P$ defined by $N(P) = \#\mathrm{End}_{\mathbf{K}}(X)$ for $X \in P$.

For example, we see the zeta function $\zeta(s, \mathbf{Ab})$ of the category $\mathbf{Ab}$. A simple object of $\mathbf{Ab}$ is a cyclic group of prime order. Namely, we have $\mathrm{Prim}(\mathbf{Ab}) = \{\mathbb{Z}/p\mathbb{Z} \mid p : \text{prime}\}$ and $N(\mathbb{Z}/p\mathbb{Z}) = p$. Therefore we have

$$\zeta(s, \mathbf{Ab}) = \prod_{P\in \mathrm{Prim}(\mathbf{Ab})} (1 - N(P)^{-s})^{-1} = \prod_{p\,:\,\text{prime}} (1 - p^{-s})^{-1} = \zeta(s),$$

which is nothing but the Riemann zeta function. Thus the Riemann zeta function $\zeta(s)$ allows us an interpretation as a zeta function of the category $\mathbf{Ab}$. Related to this fact, the spectrum of $\Delta_{\mathbf{Ab}}$ is studied experimentally in [KuST].

Our main result is stated as follows (the precise definitions of terms are given later).

**Theorem 1.1.** *If $\mathbf{K}$ is an involutive totally ordered category, then its Laplacian $\Delta_{\mathbf{K}}$ is positive.*

As a corollary, the validity of the Cauchy-Schwarz inequality follows.

**Corollary 1.2.** *If $\mathbf{K}$ is an involutive totally ordered category, then $\mathbf{K}$ satisfies the Cauchy-Schwarz inequality.*

In order to prove Theorem 1.1, we employ the representation theory of ordered categories which we explain below.

*Remark 1.2.* Since $\mathbf{Ab}$ is not an involutive totally ordered category, the Cauchy-Schwarz inequality for $\mathbf{Ab}$ is not obtained as an application of Corollary 1.2.

## 2 Ordered categories

**Definition 2.1.** Let $\mathbf{K}$ be a category. Suppose that there exist a partially ordered set (or poset) $\Sigma$ such that

(1) every finite subset $S \subset \Sigma$ has an upper bound,

(2) $\mathrm{Ob}(\mathbf{K})$ is indexed by $\Sigma$, say $\mathrm{Ob}(\mathbf{K}) = \{X_\sigma \mid \sigma \in \Sigma\}$.

We further suppose that there are distinguished morphisms $\lambda_{\beta\alpha} : X_\alpha \to X_\beta$ and $\mu_{\alpha\beta} : X_\beta \to X_\alpha$ for any comparable pair $\alpha < \beta(\alpha, \beta \in \Sigma)$ such that

$$\lambda_{\gamma\beta}\lambda_{\beta\alpha} = \lambda_{\gamma\alpha} \quad (\alpha \leq \beta \leq \gamma)\,, \tag{2.1}$$

$$\mu_{\alpha\beta}\mu_{\beta\gamma} = \mu_{\alpha\gamma} \quad (\alpha \leq \beta \leq \gamma)\,, \tag{2.2}$$

$$\mu_{\alpha\beta}\lambda_{\beta\alpha} = \mathbf{1}_\alpha \quad (\alpha \leq \beta)\,, \tag{2.3}$$

where $\mathbf{1}_\alpha$ denotes the identity of $X_\alpha$. Then we say that $\mathbf{K}$ is a **purely ordered category**. A category which is equivalent to a purely ordered one is said to be an **ordered category**.

*Remark 2.1.* If the index set $\Sigma$ of an ordered category $\mathbf{K}$ is totally ordered, then we say $\mathbf{K}$ is totally ordered category.

**Definition 2.2.** Let $\mathbf{K}$ be an ordered category. Suppose that there exists an involution $P \mapsto P^*$ on $\mathrm{Mor}_{\mathbf{K}}$, i.e. $P^{**} = P, (PQ)^* = Q^*P^*$ such that $\lambda^*_{\beta\alpha} = \mu_{\alpha\beta}$ $(\alpha \leq \beta)$. Then we say that $\mathbf{K}$ is an **involutive ordered category**.

We present several examples of involutive ordered categories.

**Example 2.1.** The category $\mathbf{A}$ of (isomorphism classes of) Hilbert spaces and linear operators is an involutive (totally) ordered category. In fact, $\Sigma = \{0, 1, 2, \dots\} \sqcup \{\infty\}$ and $\mathrm{Ob}(\mathbf{A}) = \{V_n = \mathbb{C}^n \mid n = 0, 1, 2, \dots\} \sqcup \{V_\infty = l^2\}$. The involution is given by the adjoint action $*$ with respect to the equipped inner product $\langle \cdot\,, \cdot \rangle_{V_n}$.

**Example 2.2.** The category $\mathbf{P}$ of (isomorphism classes of) Hilbert spaces and linear operators up to $\mathbb{C}^\times$ (i.e. $\mathrm{Mor}_{\mathbf{P}}(V, W) = \mathrm{Mor}_{\mathbf{A}}(V, W)/\mathbb{C}^\times$) is an involutive (totally) ordered category.

**Example 2.3.** The category $\mathbf{Ab}^{\mathrm{fin}}$ of finite abelian groups and group homomorphisms is an ordered category. (This is a subcategory of $\mathbf{Ab}$ such that $\mathrm{Ob}(\mathbf{Ab}^{\mathrm{fin}}) = \mathrm{Ob}_o(\mathbf{Ab})$.) In fact, an involution is constructed by using the duality of finite abelian groups.

## 3 Representations of categories

We prepare several conventions which are needed below.

**Definition 3.1.** Let $\mathbf{K}$ be a given category. A **linear representation** of $\mathbf{K}$ is a covariant functor $\rho$ from $\mathbf{K}$ to $\mathbf{A}$. A **projective representation** of $\mathbf{K}$ is a covariant functor $\pi$ from $\mathbf{K}$ to $\mathbf{P}$. In the sequel, we simply write 'representation' to mean either linear representation or projective one. We use the adjective 'linear' or 'projective' only when we want to emphasize which kind of representations are dealt with.

**Example 3.1.** For a given category $\mathbf{K}$,

$$o_{\mathbf{K}}(X) = 0 \quad (X \in \mathrm{Ob}(\mathbf{K}))\,, \quad o_{\mathbf{K}}(P) = 0 \quad (P \in \mathrm{Mor}_{\mathbf{K}})$$

defines a representation of $\mathbf{K}$. We call this $o_{\mathbf{K}}$ the **null representation** of $\mathbf{K}$.

**Definition 3.2.** Let $\mathbf{K}$ be an involutive category and let $\rho$ be its representation. We say that $\rho$ is called a **$*$-representation** if and only if

$$\rho(P^*) = \rho(P)^*$$

for all $P \in \mathrm{Mor}_{\mathbf{K}}$.

**Definition 3.3.** Let $\rho$ be a representation of a given category $\mathbf{K}$. A representation $\tau$ of $\mathbf{K}$ is called a **subrepresentation** of $\rho$ if

$$\begin{aligned} \tau(X) &\subset \rho(X) \quad (X \in \mathrm{Ob}(\mathbf{K}))\,, \\ \tau(P) &= \rho(P)|_{\tau(X)} \quad (P : X \to Y)\,. \end{aligned}$$

We say that $\rho$ is **irreducible** if and only if $\rho$ has just two subrepresentations, say, $\rho$ itself and the null representation $o_{\mathbf{K}}$.

**Definition 3.4.** Let $\rho$ be a representation of a category $\mathbf{K}$, and let $W$ be a subset of $\rho(V)$ for an object $V \in \mathrm{Ob}(\mathbf{K})$. Then we can construct a subrepresentation $A$ of $\rho$ by

$$A(X) := \mathrm{Span}_{\mathbb{C}}\{\, Pw \mid P \in \mathrm{Mor}(V, X), w \in W \,\} \subset \rho(X) \quad (X \in \mathrm{Ob}(\mathbf{K}))\,.$$

We call this representation $A$ the **cyclic span** of $W$.

**Definition 3.5.** Let $\rho$ be a representation of a category $\mathbf{K}$. For each object $X \in \mathrm{Ob}(\mathbf{K})$, the restriction $\rho_X := \rho|_{\rho(X)}$ defines a representation of the semigroup $\mathrm{End}(X)$ and/or the group $\mathrm{Aut}(X)$ on the space $\rho(X)$. These representations are called the **subordinate representations** of $\rho$.

**Lemma 3.1.** *If a representation $\rho$ of a category $\mathbf{K}$ is irreducible, the every subordinate representation of $\rho$ is either an irreducible representation or a null representation.*

*Proof.* In fact, if some subordinate representation $\rho_X$ of $\mathrm{End}(X)$ has a nontrivial subrepresentation $W \subset \rho(X)$, then the cyclic span of $W$ gives a nontrivial subrepresentation of $\rho$. □

**Definition 3.6.** Let $\mathbf{K}$ be a category and suppose that two representations $\rho, \rho'$ of $\mathbf{K}$ are given. A family of linear operators

$$T = \{T_\sigma : \rho(X_\sigma) \to \rho'(X_\sigma) \mid \sigma \in \Sigma\} \subset \mathrm{Mor}(\mathbf{A})$$

is called an **intertwiner** between $\rho$ and $\rho'$ if

$$\rho'(P)T_\alpha = T_\beta \rho(P)$$

for any $\alpha, \beta \in \Sigma$ and any $P : X_\alpha \to X_\beta$. We say that an intertwiner $T$ is **invertible** if $T_\sigma$ is invertible whenever $\rho(X_\sigma) \neq 0$. Two representations $\rho, \rho'$ of the same category $\mathbf{K}$ are said to be **equivalent** if they have an invertible intertwiner. For an involutive category $\mathbf{K}$, we denote by $\widehat{\mathbf{K}}$ the set of all equivalence classes of $*$-irreducible representations of $\mathbf{K}$.

# 4 Representation theory of ordered categories

Here we explain how a representation of an ordered category $\mathbf{K}$ is determined by its subordinate representations of semigroups $\mathrm{End}_{\mathbf{K}}(X)$. For simplicity, we put $\Gamma_\sigma := \mathrm{End}_{\mathbf{K}}(X_\sigma)$.

For a comparable pair $\alpha \leq \beta$ of indices, we define $\vartheta_\beta^{(\alpha)} \in \Gamma_\beta$ by $\vartheta_\beta^{(\alpha)} := \lambda_{\beta\alpha}\mu_{\alpha\beta}$. It is elementary to check that

$$(\vartheta_\beta^{(\alpha)})^2 = \vartheta_\beta^{(\alpha)}, \quad \mu_{\alpha\beta}\vartheta_\beta^{(\alpha)} = \mu_{\alpha\beta}, \quad \vartheta_\beta^{(\alpha)}\lambda_{\beta\alpha} = \lambda_{\beta\alpha}.$$

We also notice that

$$\vartheta_\beta^{(\alpha')}\vartheta_\beta^{(\alpha)} = \vartheta_\beta^{(\alpha)}\vartheta_\beta^{(\alpha')} = \vartheta_\beta^{(\alpha')}$$

for $\alpha' \leq \alpha$.

**Lemma 4.1.** *Suppose that $\alpha' \leq \alpha$ and $\beta' \leq \beta$. For any $P \in \mathrm{Mor}(X_{\alpha'}, X_{\beta'})$, there exists a morphism $Q \in \mathrm{Mor}(X_\alpha, X_\beta)$ such that $P = \mu_{\beta'\beta}Q\lambda_{\alpha\alpha'}$. Namely, $\mathrm{Mor}(X_{\alpha'}, X_{\beta'}) = \mu_{\beta'\beta}\mathrm{Mor}(X_\alpha, X_\beta)\lambda_{\alpha\alpha'}$.*

*Proof.* Actually, the morphism $Q = \lambda_{\beta\beta'}P\mu_{\alpha'\alpha}$ satisfies the required conditions. □

**Lemma 4.2.** *Let $\rho$ be a representation of an ordered category $\mathbf{K}$. If there is some $\beta \in \Sigma$ such that $\rho(X_\beta) = 0$, then $\rho(X_\alpha) = 0$ for any $\alpha \leq \beta$.*

*Proof.* We remark that $\rho(\mathbf{1}_\beta) = 0$ by assumption. Therefore it follows that

$$\rho(\mathbf{1}_\alpha) = \rho(\mu_{\alpha\beta}\mathbf{1}_\beta\lambda_{\beta\alpha}) = c\rho(\mu_{\alpha\beta})\rho(\mathbf{1}_\beta)\rho(\lambda_{\beta\alpha}) = 0 \quad \left(c \in \mathbb{C}^\times\right),$$

which implies $\rho(X_\alpha) = \mathrm{im}\,\rho(\mathbf{1}_\alpha) = 0$ for $\alpha \leq \beta$. □

**Lemma 4.3.** *Let $\rho$ be a representation of an ordered category $\mathbf{K}$. The following two conditions are equivalent.*

(a) *$\rho$ is irreducible.*

(b) *Every subordinate representation* $\rho_\sigma := \rho_{X_\sigma}$ $(\sigma \in \Sigma)$ *of* $\rho$ *is irreducible.*

*Proof.* We have proved the implication (a) $\Rightarrow$ (b) in Lemma 3.1. We show the converse (b) $\Rightarrow$ (a). Take any subrepresentation $M$ of $\rho$, and consider the quotient representation $N := \rho/M$. By the assumption of irreducibility of subordinate representations, it follows either $M(X_\sigma) = 0$ or $N(X_\sigma) = 0$ for each $X_\sigma \in \Sigma$. For any $\alpha, \beta \in \Sigma$, there exists a certain $\sigma \in \Sigma$ such that $\alpha, \beta \leq \sigma$. Hence, by Lemma 4.2, it follows either "$M(X_\alpha) = 0$ and $M(X_\beta) = 0$" or "$N(X_\alpha) = 0$ and $N(X_\beta) = 0$." This implies that $M(X_\sigma) = 0$ for all $X_\sigma$ or $N(X_\sigma) = 0$ for all $X_\sigma$. Anyway, $M$ must be a trivial subrepresentation. This implies that $\rho$ is irreducible. $\square$

We notice that $\mu_{\alpha\beta}\lambda_{\beta\alpha} = \mathbf{1}_\alpha$. If we put $U_{\beta\alpha}(P) := \lambda_{\beta\alpha}P\mu_{\alpha\beta} \in \Gamma_\beta$ for $P \in \Gamma_\alpha$, then $U_{\beta\alpha} : \Gamma_\alpha \longrightarrow \Gamma_\beta$ defines an embedding of semigroups. The following equivalence holds.

**Proposition 4.4.** *Let* $\rho$ *be a representation of an ordered category* $\mathbf{K}$, *and assume* $\alpha \leq \beta$. *Then the space* $\operatorname{im}\rho(\vartheta_\beta^{(\alpha)})$ *is invariant under the actions of operators* $\rho(U_{\beta\alpha}(P))$ $(P \in \Gamma_\alpha)$. *Two representations* $(\rho, \rho(X_\alpha))$ *and* $(\rho \circ U_{\beta\alpha}, \operatorname{im}\rho(\vartheta_\beta^{(\alpha)}))$ *of* $\Gamma_\alpha$ *are equivalent.*

*Proof.* It follows from the relation $U_{\beta\alpha}(P)\vartheta_\beta^{(\alpha)} = \vartheta_\beta^{(\alpha)}U_{\beta\alpha}(P)$ that $\operatorname{im}\rho(\vartheta_\beta^{(\alpha)})$ is $\rho(U_{\beta\alpha}(P))$-invariant for any $P \in \Gamma_\alpha$. The family $\rho(\lambda_{\beta\alpha}) : \rho(X_\alpha) \longrightarrow \operatorname{im}\rho(\vartheta_\beta^{(\alpha)})$ gives an intertwiner between $(\rho, \rho(X_\alpha))$ and $(\rho \circ U_{\beta\alpha}, \operatorname{im}\rho(\vartheta_\beta^{(\alpha)}))$, and it is indeed invertible. Hence the representations $(\rho, \rho(X_\alpha))$ and $(\rho \circ U_{\beta\alpha}, \operatorname{im}\rho(\vartheta_\beta^{(\alpha)}))$ of $\Gamma_\alpha$ are equivalent. $\square$

We prepare several conventions.

**Definition 4.1.** Assume $\alpha \leq \beta$. For a representation $\tau$ of the semigroup $\Gamma_\beta$, we put $\operatorname{low}_\beta^\alpha(\tau) := \tau \circ U_{\beta\alpha}$, which is a representation of $\Gamma_\alpha$. This correspondence $\tau \mapsto \operatorname{low}_\beta^\alpha(\tau)$ is called a lowering functor.

**Definition 4.2.** Let $\mathbf{K}$ be an ordered category. Assume that an irreducible representation $\rho_\sigma$ of the semigroup $\Gamma_\sigma$ is given for every $\sigma \in \Sigma$. If these representations $\{\rho_\sigma\}_{\sigma\in\Sigma}$ satisfy $\operatorname{low}_\beta^\alpha\rho_\beta \cong \rho_\alpha$ for any $\alpha \leq \beta$, then we say that it is a compatible system.

The following proposition is fundamental.

**Proposition 4.5.** *Let* $\mathbf{K}$ *be an ordered category. Suppose that a compatible system* $\{\rho_\sigma\}_{\sigma\in\Sigma}$ *is given. Then, there exist uniquely a representation* $\rho$ *of* $\mathbf{K}$ *such that* $\rho(P) = \rho_\sigma(P)$ *for every* $\sigma \in \Sigma$ *and every* $P \in \Gamma_\sigma$.

*Sketch of proof.* We notice that $\mathrm{Mor}_{\mathbf{K}}$ is generated by the endmorphisms $\Gamma_\sigma$ $(\sigma \in \Sigma)$ and $\lambda_{\beta\alpha}$, $\mu_{\alpha\beta}$ $(\alpha \leq \beta)$. Actually, for any $P \in \mathrm{Mor}(X_\alpha, X_\beta)$, we see that

$$P = \mu_{\beta\sigma}(\lambda_{\sigma\beta}P\mu_{\alpha\sigma})\lambda_{\sigma\alpha}, \quad \lambda_{\sigma\beta}P\mu_{\alpha\sigma} \in \Gamma_\sigma$$

for a certain $\sigma \in \Sigma$ such that $\alpha, \beta \leq \sigma$. Based on this fact, we can concretely construct the desired representation $\rho$ from the family $\{\rho_\sigma\}_{\sigma\in\Sigma}$. $\square$

# 5 Decomposition of regular representations

Let $\mathbf{K}$ be an involutive ordered category and denote by $\Sigma$ its index poset. In the sequel we discuss the case where the number $\langle X, Y\rangle_{\mathbf{K}} = \#\mathrm{Mor}_{\mathbf{K}}(X, Y)$ of morphisms is finite.

For $\alpha, \beta \in \Sigma$, we define a representation $(R, L(\mathrm{Mor}_{\mathbf{K}}(X_\beta, X_\alpha)))$ of the semigroup $\Gamma_\alpha \times \Gamma_\beta$ by

$$\begin{aligned} L(\mathrm{Mor}_{\mathbf{K}}(X_\beta, X_\alpha)) &:= \{f : \mathrm{Mor}_{\mathbf{K}}(X_\beta, X_\alpha) \to \mathbb{C}\}, \\ \{R(a,b)f\}(P) &:= f(a^*Pb) \quad (a \in \Gamma_\alpha, b \in \Gamma_\beta). \end{aligned}$$

We recall that as a $G \times G$-module, the regular representation $L(G)$ of a finite group $G$ decomposes as follows:

$$L(G) \cong \sum_{\pi \in \widehat{G}} \pi^* \boxtimes \pi. \tag{5.1}$$

Analogous to this fact, we give a decomposition of the $\Gamma_\alpha \times \Gamma_\beta$-module $L(\mathrm{Mor}_{\mathbf{K}}(X_\beta, X_\alpha))$ when $\alpha$ and $\beta$ are comparable.

**Proposition 5.1.** *Let $\mathbf{K}$ be an involutive ordered category. Suppose that $\alpha \leq \beta$ and the set $\mathrm{Mor}_{\mathbf{K}}(X_\beta, X_\alpha)$ is finite. Then, as a $\Gamma_\alpha \times \Gamma_\beta$-module, the decomposition formula*

$$L(\mathrm{Mor}_{\mathbf{K}}(X_\beta, X_\alpha)) \cong \sum_{\rho \in \widehat{\mathbf{K}}} \rho_\alpha^* \boxtimes \rho_\beta \tag{5.2}$$

*holds.*

*Proof.* For abbreviation we put $M := \mathrm{Mor}_{\mathbf{K}}(X_\beta, X_\alpha)$. First we decompose the left hand side as a $\Gamma_\beta$-module as follows:

$$L(\mathrm{Mor}_{\mathbf{K}}(X_\beta, X_\alpha)) \cong \sum_{\pi \in \widehat{\Gamma}_\beta} \mathrm{Hom}_{\Gamma_\beta}(W_\pi, L(M)) \otimes W_\pi. \tag{5.3}$$

Here we denote by $W_\pi$ the $\Gamma_\beta$-module corresponding to $\pi$. We discuss each $\pi$-component below.

Fix an irreducible representation $\pi \in \widehat{\Gamma}_\beta$. Then there uniquely exists an irreducible representation $\rho$ such that $\rho_\beta \cong \pi$. Thus it is enough to show the equivalence of $\mathrm{Hom}_{\Gamma_\beta}(W_\pi, L(M))$ and $\rho_\alpha^*$ as a $\Gamma_\alpha$-module. By the definition of the lowering functor and a compatible system, we remark that the subordinate representation $\rho_\alpha$ is equivalent to $\mathrm{low}_\beta^\alpha \pi$ on $\mathrm{im}\, \pi(\theta_\beta^\alpha) \subset W_\pi$.

In order to prove the equivalence between $\mathrm{Hom}_{\Gamma_\beta}(W_\pi, L(M))$ and $(\mathrm{low}_\beta^\alpha \pi)^*$ ($\cong \rho_\alpha^*$), we construct an intertwiner as follows: For $\psi \in \mathrm{Hom}_{\Gamma_\beta}(W_\pi, L(M))$,

$$(T\psi)(x) := (\psi x)(\mu_{\alpha\beta}) \quad \left(x \in \mathrm{im}\, \pi(\theta_\beta^\alpha) \subset W_\pi\right). \tag{5.4}$$

This indeed gives an intertwiner. Actually,

$$
\begin{aligned}
((\mathrm{low}^{\alpha}_{\beta}\pi)^*(a)T\psi)(x) &= (T\psi)(\mathrm{low}^{\alpha}_{\beta}\pi(a^*)x) \\
&= (\psi\pi(\lambda_{\beta\alpha}a^*\mu_{\alpha\beta})x)(\mu_{\alpha\beta}) \\
&= \rho(1,\lambda_{\beta\alpha}a^*\mu_{\alpha\beta})(\psi x)(\mu_{\alpha\beta}) \quad (\because \psi \text{ is a } \Gamma_\beta\text{-intertwiner}) \\
&= (\psi x)(\mu_{\alpha\beta}\cdot\lambda_{\beta\alpha}a^*\mu_{\alpha\beta}) \\
&= (\psi x)(a^*\mu_{\alpha\beta}) \quad (\because \mu_{\alpha\beta}\lambda_{\beta\alpha} = 1_\alpha) \\
&= (R(a,1)\psi x)(\mu_{\alpha\beta}) \\
&= (TR(a,1)\psi)(x).
\end{aligned}
$$

Finally we check that $T$ is injective. For $\psi \in \mathrm{Hom}_{\Gamma_\beta}(W_\pi, L(M))$,

$$
\begin{aligned}
T\psi = 0 &\implies (\psi x)(\mu_{\alpha\beta}) = 0 \quad \left(\forall x \in \operatorname{im}\pi(\theta^{\alpha}_{\beta})\right) \\
&\implies (\psi x)(\mu_{\alpha\beta}b) = 0 \quad \left(\forall x \in \operatorname{im}\pi(\theta^{\alpha}_{\beta}), \forall b \in \Gamma_\beta\right).
\end{aligned}
$$

Since $\mathrm{Mor}(X_\beta, X_\alpha) = \mu_{\alpha\beta}\Gamma_\beta$, it follows that $\psi \equiv 0$. Hence $T$ gives an invertible intertwiner as desired. □

# 6 Positivity of Laplacians

## 6.1 Proof of Theorem 1.1

By calculating the dimensions in the decomposition (5.2) of the regular representation, we have the following equality.

**Theorem 6.1.** *Let* **K** *be an involutive ordered category. Denote by* $\Sigma$ *its index poset. If* $\alpha, \beta \in \Sigma$ *are* ***comparable***,

$$
\#\mathrm{Mor}_{\mathbf{K}}(X_\alpha, X_\beta) = \sum_{\rho\in\widehat{\mathbf{K}}} \dim\rho_\alpha \dim\rho_\beta \tag{6.1}
$$

*holds.* □

The positivity of Laplacians is a corollary of the theorem above.

**Theorem 6.2 (Theorem 1.1).** *If* **K** *is an involutive totally ordered category, then its Laplacian* $\Delta_{\mathbf{K}}$ *is positive.*

*Proof.* Recall the Cauchy-Lagrange identity

$$
\det\begin{pmatrix} \langle \boldsymbol{a}_1, \boldsymbol{a}_1\rangle & \dots & \langle \boldsymbol{a}_1, \boldsymbol{a}_m\rangle \\ \vdots & \ddots & \vdots \\ \langle \boldsymbol{a}_m, \boldsymbol{a}_1\rangle & \dots & \langle \boldsymbol{a}_m, \boldsymbol{a}_m\rangle \end{pmatrix} = \sum_{1\le i_1<\cdots<i_m\le n} \det\begin{pmatrix} a_{i_1 1} & \dots & a_{i_1 m} \\ \vdots & \ddots & \vdots \\ a_{i_m 1} & \dots & a_{i_m m}\end{pmatrix}^2,
$$

where $\boldsymbol{a}_i = (a_{i1}, \dots, a_{in})$, $m \le n$. It is easy to see that every principal minor of the Laplacian $\Delta_{\mathbf{K}}$ is written in this form by Theorem 6.1. Thus the positivity follows. □

**Corollary 6.3.** *If* **K** *is an involutive totally ordered category, then* **K** *satisfies the Cauchy-Schwarz inequality.* □

## 6.2 Example: Laplacian and spectral zeta function for PB

Here we deal with the category **PB** attached to the full symmetric group $\mathfrak{S}_\omega$ as an example in which the equality (6.1) is directly checked.

Let us recall the definition of **PB**. An object in **PB** is a finite set $[n] = \{1, 2, \ldots, n\}$. A morphism from $[m]$ to $[n]$ is given by a partial bijection, that is, the triplet $(\varphi, D_\varphi, R_\varphi)$, where $D_\varphi \subset [m]$ and $R_\varphi \subset [n]$ have the same cardinality and $\varphi : D_\varphi \to R_\varphi$ is a bijection. For given two morphisms $\varphi : [l] \to [m]$ and $\psi : [m] \to [n]$, the composition $\psi\varphi : [l] \to [n]$ of $\varphi$ and $\psi$ is defined to be a partial bijection from $D_{\psi\varphi} := \varphi^{-1}(R_\varphi \cap D_\psi)$ to $R_{\psi\varphi} := \psi(R_\varphi \cap D_\psi)$. The distinguished morphisms $\lambda_{nm} : [m] \to [n]$ and $\mu_{mn} : [n] \to [m]$ are defined by

$$\lambda_{nm} : [m] \ni x \mapsto x \in [m] \subset [n],$$
$$\mu_{mn} : [n] \supset [m] \ni x \mapsto x \in [m],$$

for $n \leq m$. For a given partial bijection $\varphi : D_\varphi \to R_\varphi$, $\varphi^*$ is defined to be the partial bijection $\varphi^* : R_\varphi \ni x \mapsto \varphi^{-1}(x) \in D_\varphi$.

**Proposition 6.4.** *The category* **PB** *is an involutive totally ordered category, and hence, the Laplacian* $\Delta_{\mathbf{PB}}$ *is positive definite.* □

By an elementary combinatorial calculation we see that the number of morphisms are given by

$$\langle [m], [n] \rangle_{\mathbf{PB}} = \#\mathrm{Mor}_{\mathbf{PB}}([m], [n]) = \sum_{k=0}^{\min(m,n)} \binom{m}{k}\binom{n}{k} k!. \tag{6.2}$$

Irreducible representations of **PB** are labeled by Young diagrams. Denote by $\rho^\lambda$ the attached irreducible representation of **PB**, and by $\rho_n^\lambda$ the subordinate representation which is the restriction of $\rho^\lambda$ to $\Gamma_n$.

**Proposition 6.5** **([Ne]).** *We have*

$$\dim \rho_n^\lambda = \binom{n}{|\lambda|} \dim \lambda \tag{6.3}$$

*for any* $\lambda \in \mathbb{Y}$. *Here we denote by* $\dim \lambda$ *the dimension of the irreducible* $\mathfrak{S}_{|\lambda|}$*-module corresponding to* $\lambda$. *We remark that* $\binom{n}{k} = 0$ *if* $k > n$.

By using Theorem 6.1 and the well-known fact

$$\sum_{|\lambda|=k} (\dim \lambda)^2 = k!,$$

we have in fact

$$\begin{aligned} \langle [m], [n] \rangle_{\mathbf{PB}} &= \sum_{\lambda \in \mathbb{Y}} \left\{ \binom{m}{|\lambda|} \dim \lambda \times \binom{n}{|\lambda|} \dim \lambda \right\} \\ &= \sum_{k=0}^{\min(m,n)} \binom{m}{k}\binom{n}{k} k! \end{aligned} \tag{6.4}$$

for $m, n \in \mathbb{Z}_{\geq 0}$, which (of course) coincides with the result (6.2) of a combinatorial calculation.

We put $\Delta = \Delta_{\mathbf{PB}}$. One of our main concern is a study of the spectrum of the Laplacian $\Delta$ and its spectral zeta function $\zeta_\Delta(s)$. We put $\Delta_N = (\langle [i], [j] \rangle_{\mathbf{PB}})_{0 \leq i,j \leq N}$, the principal $N$-minor of the Laplacian $\Delta$. Let us denote by $\lambda_{N,j}$ $(0 \leq j \leq N)$ the $(j+1)$-th eigenvalue of $\Delta_N$, that is,

$$0 < \lambda_{N,0} \leq \lambda_{N,1} \leq \cdots \leq \lambda_{N,N}.$$

**Theorem 6.6** **([Kil]).** *For every $k \geq 0$, there exists the limit $\lambda_k := \lim_{N \to \infty} \lambda_{N,k} > 0$.*

We show the numerical estimation of first 10 eigenvalues up to 10 digits (Table 1). These values are calculated as limits of $\lambda_{N,k}$'s. The special values of the spectral zeta function are

$$\begin{aligned}
\lambda_0 &= 0.08487190949\ldots \\
\lambda_1 &= 0.2919019234\ldots \\
\lambda_2 &= 0.8906738137\ldots \\
\lambda_3 &= 2.607762169\ldots \\
\lambda_4 &= 9.640545861\ldots \\
\lambda_5 &= 46.47152499\ldots \\
\lambda_6 &= 273.9773421\ldots \\
\lambda_7 &= 1899.150590\ldots \\
\lambda_8 &= 15101.52483\ldots \\
\lambda_9 &= 135369.6103\ldots
\end{aligned}$$

Table 1: First 10 eigenvalues of $\Delta_{\mathbf{PB}}$

given as follows.

**Theorem 6.7.** *We have*

$$\zeta_\Delta(m) = \sum_{k_1,\ldots,k_m \geq 0} \frac{1}{k_1! \ldots k_m!} \binom{k_1 + k_2}{k_1} \binom{k_2 + k_3}{k_2} \cdots \binom{k_m + k_1}{k_m}. \tag{6.5}$$

*In particular, we have*

$$\zeta_\Delta(1) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{m=0}^{k} \binom{k}{m}^2 = \sum_{k=0}^{\infty} \frac{1}{k!} \binom{2k}{k}, \tag{6.6}$$

$$\zeta_\Delta(2) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{m=0}^{k} \binom{k}{m}^3. \tag{6.7}$$

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