State Extension from Subsystems to the Joint System

(Mathematical Study of Quantum Dynamical Systems and Its Application to Quantum Computer)

Araki, Huzihiro; Moriya, Hajime

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Huzihiro Araki * and Hajime Moriya

1 Introduction

In algebraic approach to quantum systems, a system is described by a C*-algebra $A$ and its state is a normalized positive linear functional $\varphi$, its value $\varphi(A)$ for $A \in A$ being the expectation value of $A$ in that state. Subsystems are described by C*-subalgebras $A_i$ of $A$, $i = 1, 2, \cdots$. Their joint system is the total system described by $A$ if the subalgebras $A_i$ generate $A$ as a C*-algebra. Restrictions $\varphi_i$ of a state $\varphi$ of $A$ to subalgebras $A_i$ are states of $A_i$, $i = 1, 2, \cdots$. Conversely, suppose that states $\varphi_i$ of $A_i$, $i = 1, 2, \cdots$, are first given. Then a state $\varphi$ of $A$ is called a joint extension of states $\varphi_i$ of $A_i$, $i = 1, 2, \cdots$, if the restriction of $\varphi$ to $A_i$ is the given state $\varphi_i$ for each $i$.

For spin or Boson lattice systems, algebras $A_i$ of subsystems with mutually disjoint localization mutually commute and form a tensor product system. If $A$ is the tensor product of $A_i$, $i = 1, 2, \cdots$ and

$$\varphi(\prod_i A_i) = \prod_i \varphi(A_i), \quad A_i \in A_i$$

holds, then $\varphi$ is called the tensor product of $\varphi_i$, $i = 1, 2, \cdots$. Otherwise $\varphi$ is said to be entangled if $\varphi$ is pure. Entanglement in tensor product systems is widely studied.

For Fermion lattice systems, algebras of subsystems with mutually disjoint localization do not mutually commute due to the anticommutativity of Fermion creation and annihilation operators. As electrons are Fermions, a study of Fermion systems seems to have a practical significance. Entanglement for Fermion systems is studied by one of the present authors recently [2].

* Mailing address: Research Institute for Mathematical Sciences, Kyoto University. Kitashirakawa-Oiwakecho, Sakyo, Kyoto 606-8502, Japan
The present work studies the problem of joint extension of states from subsystems to the joint system for (discrete) Fermion systems and generalizes some results in [2].

2 The Fermion Algebra

We consider a C*-algebra $\mathcal{A}$, called a CAR algebra or a Fermion algebra, which is generated by its elements $a_i$ and $a_i^*$, $i \in \mathbb{N}$ ($\mathbb{N} = \{1, 2, \ldots\}$) satisfying the following canonical anticommutation relations (CAR).

\[
\{a_i^*, a_j\} = \delta_{i,j}1 \\
\{a_i^*, a_j^*\} = \{a_i, a_j\} = 0,
\]

($i, j \in \mathbb{N}$), where $\{A, B\} = AB + BA$ (anticommutator) and $\delta_{i,j} = 1$ for $i = j$ and $\delta_{i,j} = 0$ otherwise. For finite subset $I$ of $\mathbb{N}$, $\mathcal{A}(I)$ denotes the C*-subalgebra generated by $a_i$ and $a_i^*$, $i \in I$. A crucial role is played by the unique automorphism $\Theta$ of $\mathcal{A}$ characterized by

\[
\Theta(a_i) = -a_i, \quad \Theta(a_i^*) = -a_i^*
\]

for all $i \in \mathbb{N}$. The even and odd parts of $\mathcal{A}$ and $\mathcal{A}(I)$ are defined by

\[
\mathcal{A}_\pm \equiv \{A \in \mathcal{A} | \Theta(A) = \pm A\},
\]

For any $A \in \mathcal{A}$ (or $\mathcal{A}(I)$), we have the following decomposition

\[
A_\pm = A_+ + A_-, \quad A_\pm = \frac{1}{2}(A \pm \Theta(A)) \in \mathcal{A}_\pm \text{ (or } \mathcal{A}(I)_\pm\text{)}.
\]

A state $\varphi$ of $\mathcal{A}$ or $\mathcal{A}(I)$ is called even if it is $\Theta$-invariant:

\[
\varphi(\Theta(A)) = \varphi(A)
\]

for all $A \in \mathcal{A}$ (or $A \in \mathcal{A}(I)$).

For a state $\varphi$ of a C*-algebra $\mathcal{A}$ ($\mathcal{A}(I)$), $\{\mathcal{H}_\varphi, \pi_\varphi, \Omega_\varphi\}$ denotes the GNS triplet of a Hilbert space $\mathcal{H}_\varphi$, a representation $\pi_\varphi$ of $\mathcal{A}$ (of $\mathcal{A}(I)$), and a vector $\Omega_\varphi \in \mathcal{H}_\varphi$, which is cyclic for $\pi_\varphi(A)$ ($\pi_\varphi(\mathcal{A}(I))$) and satisfies

\[
\varphi(A) = (\Omega_\varphi, \pi_\varphi(A)\Omega_\varphi)
\]

for all $A \in \mathcal{A}$ ($\mathcal{A}(I)$). For any $x \in B(\mathcal{H}_\varphi)$, we write

\[
\overline{\varphi}(x) = (\Omega_\varphi, x\Omega_\varphi).
\]
3 Product State Extension

As subsystems, we consider $\mathcal{A}(I)$ with mutually disjoint subsets $I$'s. For a pair of disjoint subsets $I_1$ and $I_2$ of $\mathbb{N}$, let $\varphi_1$ and $\varphi_2$ be given states of $\mathcal{A}(I_1)$ and $\mathcal{A}(I_2)$, respectively. If a state $\varphi$ of the joint system $\mathcal{A}(I_1 \cup I_2)$ (which is the same as the $C^*$-subalgebra of $\mathcal{A}$ generated by $\mathcal{A}(I_1)$ and $\mathcal{A}(I_2)$) coincides with $\varphi_1$ on $\mathcal{A}(I_1)$ and $\varphi_2$ on $\mathcal{A}(I_2)$, i.e.,

$$\varphi(A_1) = \varphi_1(A_1), \quad A_1 \in \mathcal{A}(I_1),$$

$$\varphi(A_2) = \varphi_2(A_2), \quad A_2 \in \mathcal{A}(I_2),$$

then $\varphi$ is called a joint extension of $\varphi_1$ and $\varphi_2$. As a special case, if

$$\varphi(A_1A_2) = \varphi_1(A_1)\varphi_2(A_2)$$

holds for all $A_1 \in \mathcal{A}(I_1)$ and all $A_2 \in \mathcal{A}(I_2)$, then $\varphi$ is called a product state extension of $\varphi_1$ and $\varphi_2$. For an arbitrary (finite or infinite) number of subsystems, $\mathcal{A}(I_1), \mathcal{A}(I_2), \cdots$ with mutually disjoint $I$'s and a set of given states $\varphi_i$ of $\mathcal{A}(I_i)$, a state $\varphi$ of $\mathcal{A}(\cup_i I_i)$ is called a product state extension if it satisfies (1).

**Theorem 1.** Let $I_1, I_2, \cdots$ be an arbitrary (finite or infinite) number of mutually disjoint subsets of $\mathbb{N}$ and $\varphi_i$ be a given state of $\mathcal{A}(I_i)$ for each $i$.

(1) A product state extension of $\varphi_i$, $i = 1, 2, \cdots$, exists if and only if all states $\varphi_i$ except at most one are even. It is unique if it exists. It is even if and only if all $\varphi_i$ are even.

(2) Suppose that all $\varphi_i$ are pure. If there exists a joint extension of $\varphi_i$, $i = 1, 2, \cdots$, then all states $\varphi_i$ except at most one have to be even. If this is the case, the joint extension is uniquely given by the product state extension and is a pure state.

**Remark.** In Theorem 1 (2), the product state property (1) is not assumed but it is derived from the purity assumption for all $\varphi_i$.

The purity of all $\varphi_i$ does not follow from that of their joint extension $\varphi$ in general. For a product state extension $\varphi$, however, we have the following two theorems about consequences of purity of $\varphi$.

**Theorem 2.** Let $\varphi$ be the product state extension of states $\varphi_i$ with disjoint $I_i$. Assume that all $\varphi_i$ except $\varphi_1$ are even.

(1) $\varphi_1$ is pure if $\varphi$ is pure.

(2) Assume that $\pi_{\varphi_i}$ and $\pi_{\varphi_1}$ are not disjoint. Then $\varphi$ is pure if and only if all $\varphi_i$ are pure. In particular, this is the case if $\varphi$ is even.
Remark. If \( I_1 \) is finite, the assumption of Theorem 2 (2) holds and hence the conclusion follows automatically.

In the case not covered by Theorem 2, the following result gives a complete analysis if we take \( \cup_{i \geq 2} I_i \) in Theorem 2 as one subset of \( \mathbb{N} \).

**Theorem 3.** Let \( \varphi \) be the product state extension of states \( \varphi_1 \) and \( \varphi_2 \) of \( A(I_1) \) and \( A(I_2) \) with disjoint \( I_1 \) and \( I_2 \) where \( \varphi_2 \) is even and \( \varphi_1 \) is such that \( \pi_{\varphi_1} \) and \( \pi_{\varphi_1 \Theta} \) are disjoint.

1. \( \varphi \) is pure if and only if \( \varphi_1 \) and the restriction \( \varphi_2^+ \) of \( \varphi_2 \) to \( A(I_2)_+ \) are both pure.
2. Assume that \( \varphi \) is pure. \( \varphi_2 \) is not pure if and only if

\[
\varphi_2 = \frac{1}{2}(\varphi_2 + \varphi_2 \Theta)
\]

where \( \varphi_2 \) is pure and \( \pi_{\varphi_2} \) and \( \pi_{\varphi_2 \Theta} \) are disjoint.

Remark. The first two theorems are some generalization of results in [3] with the following overlap. The first part of Theorem 1 (1) is given in [3] as Theorem 5.4 (the if part and uniqueness) and a discussion after Definition 5.1 (the only if part). Theorem 1 (2) and Theorem 2 are given in Theorem 5.5 of [3] under the assumption that all \( \varphi_i \) are even.

### 4 Other State Extensions

The rest of our results concerns a joint extension of states of two subsystems, not satisfying the product state property (1). We need a few more notation. For two states \( \varphi \) and \( \psi \) of a \( \mathcal{C}^* \)-algebra \( A(I_1) \), consider any representation \( \pi \) of \( A(I_1) \) on a Hilbert space \( \mathcal{H} \) containing vectors \( \Phi \) and \( \Psi \) such that

\[
\varphi(A) = (\Phi, \pi(A)\Phi), \quad \psi(A) = (\Psi, \pi(A)\Psi).
\]

The transition probability between \( \varphi \) and \( \psi \) is defined ([4]) by

\[
P(\varphi, \psi) \equiv \sup |(\Phi, \Psi)|^2
\]

where the supremum is taken over all \( \mathcal{H}, \pi, \Phi \) and \( \Psi \) as described above. For a state \( \varphi_1 \) of \( A(I_1) \), we need the following quantity

\[
p(\varphi_1) \equiv P(\varphi_1, \varphi_1 \Theta)^{1/2}
\]

where \( \varphi_1 \Theta \) denotes the state \( \varphi_1 \Theta(A) = \varphi_1(\Theta(A)) \), \( A \in A(I_1) \).
If \( \varphi_1 \) is pure, then \( \varphi_1 \Theta \) is also pure and the representations \( \pi_{\varphi_1} \) and \( \pi_{\varphi_1 \Theta} \) are both irreducible. There are two alternatives.

(\( \alpha \)) They are mutually disjoint. In this case \( p(\varphi_1) = 0 \).

(\( \beta \)) They are unitarily equivalent.

In the case (\( \beta \)), there exists a self-adjoint unitary \( u_1 \) on \( \mathcal{H}_{\varphi_1} \) such that
\[
u_1 \pi_{\varphi_1}(A) u_1 = \pi_{\varphi_1}(\Theta(A)), \quad A \in A(I_1),
(\Omega_{\varphi_1}, u_1 \Omega_{\varphi_1}) \geq 0.
\]

For two states \( \varphi \) and \( \psi \), we introduce
\[
\lambda(\varphi, \psi) \equiv \sup \{ \lambda \in \mathbb{R}; \varphi - \lambda \psi \geq 0 \}
\]
Since \( \varphi - \lambda_n \psi \geq 0 \) and \( \lim \lambda_n = \lambda \) imply \( \varphi - \lambda \psi \geq 0 \), we have
\[
\varphi \geq \lambda(\varphi, \psi) \psi.
\]

We need
\[
\lambda(\varphi_2) \equiv \lambda(\varphi_2, \varphi_2 \Theta).
\]

The next Theorem provides a complete answer for a joint extension \( \varphi \) of states \( \varphi_1 \) and \( \varphi_2 \) of \( A(I_1) \) and \( A(I_2) \), when one of them is pure.

**Theorem 4.** Let \( \varphi_1 \) and \( \varphi_2 \) be states of \( A(I_1) \) and \( A(I_2) \) for disjoint subsets \( I_1 \) and \( I_2 \). Assume that \( \varphi_1 \) is pure.

(1) A joint extension \( \varphi \) of \( \varphi_1 \) and \( \varphi_2 \) exists if and only if
\[
\lambda(\varphi_2) \geq \frac{1 - p(\varphi_1)}{1 + p(\varphi_1)}.
\]

(2) If eq. (2) holds and if \( p(\varphi_1) \neq 0 \), then a joint extension \( \varphi \) is unique and satisfies
\[
\varphi(A_1 A_2) = \varphi_1(A_1) \varphi_2(A_2+) + \frac{1}{p(\varphi_1)} f(A_1) \varphi_2(A_2-),

f(A_1) \equiv \overline{\varphi_1}(\pi_{\varphi_1}(A_1) u_1)
\]
for \( A_1 \in A(I_1) \) and \( A_2 = A_{2+} + A_{2-}, \ A_{2\pm} \in A(I_2)_{\pm} \).

(3) If \( p(\varphi_1) = 0 \), (2) is equivalent to evenness of \( \varphi_2 \). If this is the case, at least a product state extension of Theorem 1 exists.

(4) Assume that \( p(\varphi_1) = 0 \) and \( \varphi_2 \) is even. There exists a joint extension
of $\varphi_1$ and $\varphi_2$ other than the unique product state extension if and only if $\varphi_1$ and $\varphi_2$ satisfy the following pair of conditions:

(4-i) $\pi_{\varphi_1}$ and $\pi_{\varphi_2}$ are unitarily equivalent.
(4-ii) There exists a state $\varphi_2$ of $A(I_2)$ such that $\varphi_2 \neq \varphi_2 \Theta$ and

$$\varphi_2 = \frac{1}{2} (\varphi_2 + \varphi_2 \Theta).$$

(5) If $p(\varphi_1) = 0$, then corresponding to each $\varphi_2$ above, there exists a joint extension $\varphi$ which satisfies

$$\varphi(A_1A_2) = \varphi_1(A_1)\varphi_2(A_2) + \overline{\varphi_1}((\pi_{\varphi_1}(A_1)u_1)\varphi_2(A_2).$$

Such extensions along with the unique product state extension (which satisfies eq. (3) for $\varphi_2 = \varphi_2$) exhaust all joint extensions of $\varphi_1$ and $\varphi_2$ when $p(\varphi_1) = 0$.  

Remark. The eq.(2) is sufficient for the existence of a joint extension also for general states $\varphi_1$ and $\varphi_2$.

We have a necessary and sufficient condition for the existence of a joint extension of states $\varphi_1$ and $\varphi_2$ under a specific condition on $\varphi_1$.

**Theorem 5.** Let $\varphi_1$ and $\varphi_2$ be states of $A(I_1)$ and $A(I_2)$ for disjoint subsets $I_1$ and $I_2$. Assume that $\pi_{\varphi_1}$ and $\pi_{\varphi_2}$ are disjoint. Then a joint extension of $\varphi_1$ and $\varphi_2$ exists if and only if $\varphi_2$ is even.

### 5 Examples

**Example 1**

Let $I_1$ and $I_2$ be mutually disjoint finite subsets of $\mathbb{N}$. Let $\varrho \in A(I_1 \cup I_2)$ be an invertible density matrix, namely $\varrho \geq \lambda I$ for some $\lambda > 0$ and $\text{Tr}(\varrho) = 1$, where $\text{Tr}$ denotes the matrix trace on $A(I_1 \cup I_2)$. Take any $x = x^* \in A(I_1)_-$ and $y = y^* \in A(I_2)_-$ satisfying $\|x\| \|y\| \leq \lambda$. Let $\varphi_1(A_1) \equiv \text{Tr}(\varrho A_1)$ for $A_1 \in A(I_1)$ and $\varphi_2(A_2) \equiv \text{Tr}(\varrho A_2)$ for $A_2 \in A(I_2)$. Then

$$\varphi'(A) \equiv \text{Tr}(\varrho' A), \quad \varrho' \equiv \varrho + ixy.$$ 

for $A \in A(I_1 \cup I_2)$ is a state of $A(I_1 \cup I_2)$ and has $\varphi_1$ and $\varphi_2$ as its restrictions to $A(I_1)$ and $A(I_2)$, irrespective of the choice of $x$ and $y$ satisfying the above conditions.

**Example 2**

Let $I_1$ and $I_2$ be mutually disjoint subsets of $\mathbb{N}$. Let $\varphi$ and $\psi$ be states of $A(I_1)$ and $A(I_2)$ such that

$$\varphi = \sum_i \lambda_i \varphi_i, \quad \psi = \sum_i \lambda_i \psi_i, \quad (0 < \lambda_i, \sum_i \lambda_i = 1),$$

$$\varphi_1(A_1) \equiv \text{Tr}(\varrho_1 A_1), \quad \varrho_1 \equiv \varrho + ixy.$$ 

for $A \in A(I_1 \cup I_2)$ is a state of $A(I_1 \cup I_2)$ and has $\varphi_1$ and $\varphi_2$ as its restrictions to $A(I_1)$ and $A(I_2)$, irrespective of the choice of $x$ and $y$ satisfying the above conditions.
where \( \varphi_i \) and \( \psi_i \) are states of \( A(I_1) \) and \( A(I_2) \) which have a joint extension \( \chi_i \) for each \( i \).

\[
\chi = \sum_i \lambda_i \chi_i
\]
is a joint extension of \( \varphi \) and \( \psi \).

This simple example yields next more elaborate ones.

**Example 3**

Let \( \varphi \) and \( \psi \) be states of \( A(I_1) \) and \( A(I_2) \) for disjoint \( I_1 \) and \( I_2 \) with \((\text{non-trivial})\) decompositions

\[
\varphi = \lambda \varphi_1 + (1 - \lambda) \varphi_2, \quad \psi = \mu \psi_1 + (1 - \mu) \psi_2, \quad (0 < \lambda, \mu < 1)
\]

where \( \varphi_1 \) and \( \varphi_2 \) are even. Product state extensions \( \varphi_j \psi_j \) of \( \varphi_i \) and \( \psi_j \) yield

\[
\chi \equiv (\lambda \mu + \kappa) \varphi_1 \psi_1 + (\lambda(1 - \mu) - \kappa) \varphi_1 \psi_2 \\
((1 - \lambda) \mu - \kappa) \varphi_2 \psi_1 + ((1 - \lambda)(1 - \mu) + \kappa) \varphi_2 \psi_2,
\]

which is a joint extension of \( \varphi \) and \( \psi \) for all \( \kappa \in \mathbb{R} \) satisfying

\[
-\min(\lambda \mu, (1 - \lambda)(1 - \mu)) \leq \kappa \leq \min((1 - \lambda) \mu, \lambda(1 - \mu)).
\]

**Example 4**

Let \( \varphi_k, k = 1, \ldots, m \) and \( \psi_l, l = 1, \ldots, n \) be states of \( A(I_1) \) and \( A(I_2) \) for disjoint \( I_1 \) and \( I_2 \). Let

\[
\varphi = \sum_{k=1}^{m} \lambda_k \varphi_k, \quad \psi = \sum_{l=1}^{n} \mu_l \psi_l
\]

with \( \lambda_k, \mu_l > 0 \), \( \sum \lambda_k = \sum \mu_l = 1 \). Assume that there exists a joint extension \( \chi_{kl} \) of \( \varphi_k \) and \( \psi_l \) for each \( k \) and \( l \). Then

\[
\chi = \sum_{kl} (\lambda_k \mu_l + \kappa_{kl}) \chi_{kl}
\]

is a joint extension if

\[
(\lambda_k \mu_l + \kappa_{kl}) \geq 0, \quad \sum_l \kappa_{kl} = \sum_k \kappa_{kl} = 0.
\]

Since the constraint for \( mn \) parameters \( \{\kappa_{kl}\} \) are effectively \( m + n - 1 \) linear relations (because \( \sum_{kl} \kappa_{kl} = 0 \) is common for \( \sum_k \kappa_{kl} = 0 \) and \( \sum_k \kappa_{kl} = 0 \) ), we have \( mn - (m + n - 1) = (m - 1)(n - 1) \) parameters for the joint extension (4).

References


