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Algebraic and geometric representations of the two-qudit entanglement

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Abstract

Quantum entanglement is at the core of quantum mechanics and many tasks in quantum information theory. However, apart from some simple cases (pure states or low dimensional mixed states), the mathematical basis of entanglement is not yet fully understood. In this note we discuss a new concept of entanglement witnesses — a special class of observables — both from algebraic and geometric point of views.

1 Introduction

The characterization and classification of entanglement states, introduced by Erwin Schrödinger nearly 70 years ago [1], is one of the most challenging open problems of modern quantum theory. In particular, the role of entanglement in quantum information theory is fundamental. It has been realized over the last years that quantum entanglement is not only a fundamental resource in quantum communication but can also be considered as a resource in quantum computation [2].

In order to explain what the entanglement problem means we shall consider composite quantum systems such that their states are in general mixed and can be represented by density matrices, i.e. by self-adjoint positive definite linear operators of trace one, acting on the Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, which is a tensor product of Hilbert spaces corresponding to subsystems S_A and S_B of the given system S. An important step forward to understand what does entanglement mean is to discriminate first the states which contain classical correlations only. These states are called separable states and their mathematical characterization has been given by Werner [3]. The explicit form of separable states is discussed in the next section. Then we introduce the concept of entanglement witnesses — a special class of observables introduced for the first time by Horodeckis [4] and Terhal [5], and we analyze their algebraic and geometric properties.

2 Composite systems

All systems considered in information theory can be either classical or quantum, or can be hybrids composed of a classical and a quantum part. Therefore one needs a mathematical framework covering all these cases. A proper approach is to characterize each type of system by its algebra of observables.

In quantum case one usually assumes that pure states of the systems in question are elements of a chosen Hilbert space $\mathcal{H}$ and the algebra $\mathcal{A}$ of observables is identified with the subset of all bounded linear operators on the space $\mathcal{H}$, i.e. $\mathcal{A} = \mathcal{B}(\mathcal{H})$. Although $\mathcal{H}$ and $\mathcal{A}$ can be in general infinite dimensional, we shall consider only finite dimensional Hilbert spaces since most research in quantum information theory is done up to now for finite dimensional systems. Hence one can choose $\mathcal{H} = \mathbb{C}^d$ and $\mathcal{A} = \mathcal{B}(\mathcal{H})$ is just the algebra of complex $d \times d$ matrices. The corresponding systems are called d-level systems (qudits) or qubits if $d = 2$ holds. General states (i.e. mixed states) are described in quantum mechanics by density matrices, i.e. by positive semi-definite and normalized operators acting on the space $\mathcal{H}$. Usually one considers density operators as elements of the space $\mathcal{T}(\mathcal{H})$ — the real Banach space of all self-adjoint operators on $\mathcal{H}$ under the trace norm $\|\rho\| := \mathrm{Tr}\,(\rho^*\rho)^{1/2}$. In other words, states of the system are described by density matrices $\rho \in \mathcal{P}(\mathcal{H})$, where the set $\mathcal{P}(\mathcal{H})$ is defined as

$$\mathcal{P}(\mathcal{H}) := \{\rho \in \mathcal{T}(\mathcal{H})\,; \quad \rho \geq 0, \quad \mathrm{Tr}\,\rho = 1\}. \tag{2.1}$$

The set of all semi-positive definite operators $\rho \in \mathcal{T}(\mathcal{H})$ constitutes a positive cone $V^+(\mathcal{H})$ in $\mathcal{T}(\mathcal{H})$. This cone can be also defined as

$$V^+(\mathcal{H}) := \{\rho \in \mathcal{T}(\mathcal{H}) \;; \quad \|\rho\| = \mathrm{Tr}\,\rho\} \tag{2.2}$$

because $\rho \in V^+(\mathcal{H})$ if and only if the equality $\|\rho\| = \mathrm{Tr}\,\rho$ is fulfilled.

Since the difference between classical and quantum systems is an important issue in quantum information theory let us reformulate classical probability theory according to the similar scheme as in the above quantum case.

In the classical case, the algebra of observables is commutative, and can be considered as a space of complex-valued functions on a given set $\mathcal{X}$,

$$\mathcal{A} = C(\mathcal{X}) := \{f : \mathcal{X} \to \mathbb{C}\}. \tag{2.3}$$

A single classical bit corresponds to the choice $\mathcal{X} = \{0, 1\}$. The set $C(\mathcal{X})$ can be also identified with the set of all diagonal operators from $\mathcal{B}(\mathcal{H})$, where $\dim \mathcal{H} = \mathrm{card}\mathcal{X}$. A state p on the classical algebra $C(\mathcal{X})$ is defined by the numbers $p_x := p(e_x)$, where e_x are the functions on $\mathcal{X}$ such that $e_x(z) = 1$ for $x = z$ and zero otherwise. In other words, a state p on the classical algebra $C(\mathcal{X})$ forms a probability distribution on $\mathcal{X}$, i.e. $p(x) \geq 0$ and $\sum_x p(x) = 0$.

Now, let us return to quantum systems. Composed quantum systems occur in many places and situations in quantum information theory. It is theoretically rewarding to describe the physical world in terms of subsystems. However, this means that it is of fundamental importance to be able to describe the state space of a composite system in terms of the simpler state spaces associated with its parts. In quantum theory, a basic axiom states that the state space associated with a bipartite quantum system S made out of two subsystems S_A and S_B is given by the tensor product of the state spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, that is

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B \,. \tag{2.4}$$

As a consequence we must consider also the spaces of observables $\mathcal{B}(\mathcal{H}_{AB})$ and the set of all density matrices on $\mathcal{H}_{AB}$, namely $\mathcal{P}(\mathcal{H}_{AB})$. The crucial point is that this opens the possibility for quantum correlations and entanglement between subsystems. In particular, entanglement is of great importance because it is a main resource in many applications of quantum information theory like quantum computing or teleportation. To explain entanglement in details we have to discuss some concepts which allow us to construct states and observables of the composite system from its subsystems.

To discuss the composition of two arbitrary, i.e. classical or quantum systems it is very convenient to talk about the two subsystems S_A and S_B in terms of their observable algebras $\mathcal{A}_A = \mathcal{B}(\mathcal{H}_A)$ and $\mathcal{A}_B(\mathcal{H}_B)$. The observable algebra of the composite system S_{AB} is then given by the tensor product of $\mathcal{A}_A$ and $\mathcal{A}_B$, i.e.

$$\mathcal{A}_{AB} := \mathrm{span}\,\{A \otimes B, \quad A \in \mathcal{A}_A, \quad B \in \mathcal{A}_B\} \subseteq \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B) \,. \tag{2.5}$$

Let us now consider the special cases arising from different choices for $\mathcal{A}_A$ and $\mathcal{A}_B$. If both subsystems are quantum, that is $\mathcal{A}_A = \mathcal{B}(\mathcal{H}_A)$ and $\mathcal{A}_B = \mathcal{B}(\mathcal{H}_B)$, then we obtain

$$\mathcal{B}(\mathcal{H}_A) \otimes \mathcal{B}(\mathcal{H}_B) = \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B) \,. \tag{2.6}$$

For two classical systems $\mathcal{A}_A = C(\mathcal{X}_A)$ and $\mathcal{A}_B = C(\mathcal{X}_B)$, elements of $\mathcal{A}_A$ and $\mathcal{A}_B$ are complex-valued functions on $\mathcal{X}_A$ and $\mathcal{X}_B$, respectively. Hence the tensor product $C(\mathcal{X}_A) \otimes C(\mathcal{X}_B)$ consists of complex-valued functions on $\mathcal{X}_A \times \mathcal{X}_B$, i.e.

$$C(\mathcal{X}_A) \otimes C(\mathcal{X}_B) = C(\mathcal{X}_A \times \mathcal{X}_B) \,. \tag{2.7}$$

This means that observables and states of the composite system $C(\mathcal{X}_A) \otimes C(\mathcal{X}_B)$ are, in accordance with classical probability theory, given by random variables and probability distribution on the Cartesian product $\mathcal{X}_A \times \mathcal{X}_B$.

3 Hybrid systems. The concept of separability

If one of the subsystems is classical and the other is quantum we have a hybrid system. Such systems occur frequently in quantum information theory whenever a combination of classical and quantum information is considered. The elements of a hybrid system observable algebra $C(\mathcal{X}_A) \otimes \mathcal{B}(\mathcal{H}_B)$ can be considered as operator-valued functions $\mathcal{X} \ni x \mapsto B(x) \in \mathcal{B}(\mathcal{H}_B)$. An important issue is the comparison of correlations between quantum systems on the one hand and classical on the other. To be able to understand similarities and differences let us consider the state space of a system consisting of at least one classical subsystem. In this case, each state of a composite system $\mathcal{A}_A \otimes \mathcal{A}_B$ consisting of a classical ($\mathcal{A}_A = C(\mathcal{X}_A)$) and an arbitrary system $\mathcal{A}_B$ may be represented in the form ([3])

$$\rho = \sum_{i \in \mathcal{X}_A} p_i \rho_i^A \otimes \rho_i^B , \tag{3.1}$$

where $p_i > 0$ and ρ_i^A and ρ_i^B denote restrictions of ρ to $\mathcal{A}_A$ and $\mathcal{A}_B$, respectively. In the classical case, the probability density for ρ_i^A is obtained by integrating out the S_B variables. In the quantum case, it corresponds to the partial trace of density matrix with respect to $\mathcal{H}_B$. It is of great importance that, in general, it is not possible to reconstruct the state ρ from the restrictions ρ_A and ρ_B, which is another way saying that ρ also describes correlations between the subsystems.

If $\mathcal{A}_A$ and $\mathcal{A}_B$ correspond to two quantum systems it is still possible for them to be correlated in the way described in (3.1). However, the crucial point is that not all correlations of quantum systems are of this type. Now, the pure states are given by unit vectors in the tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$ and they can be linear combination of the product vectors. A non-product pure state is a basic example of an entangled state in the sense of the following definition.

Definition 3.1. A state ρ of a composite quantum system $B(\mathcal{H}_A) \otimes B(\mathcal{H}_B)$ is called separable or "classically correlated" if it can be written in the form

$$\rho = \sum p_i |e_i\rangle\langle e_i| \otimes |f_i\rangle\langle f_i| , \tag{3.2}$$

where $\{|e_i\rangle\}$ and $\{|f_i\rangle\}$ are (not necessarily orthonormal) basis in $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively.

It is not difficult to show that the expressions (3.2) and (3.1) are equivalent. We shall denote the set of all separable states by $\operatorname{Sep}\mathcal{P}(\mathcal{H})$. Other states $\rho \in \mathcal{P}(\mathcal{H})$ which do not belong to $\operatorname{Sep}\mathcal{P}(\mathcal{H})$ are called entangled.

In the next section we shall discuss some methods how one can distinguish between separable and entangled states.

4 Entanglement witnesses — an algebraic approach

Below we discuss briefly the entanglement witnesses and positive map characterizations of separability. Witness operators are defined as follows.

DEFINITION 4.1. An operator acting on the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ is an entanglement witness iff:

- for every separable operator $\sigma \in \operatorname{Sep}\mathcal{P}(\mathcal{H})$ it holds that $\operatorname{Tr}(W\sigma) \geq 0$, and
- there exists a $\rho \in \mathcal{P}(\mathcal{H}) \setminus \operatorname{Sep}\mathcal{P}(\mathcal{H})$ such that $\operatorname{Tr}(W\rho) < 0$.

In such a case we say that W detects ρ.

Let us observe that for every ρ which is inseparable (entangled) there exists a witness operator which detects it. In fact, entanglement witnesses have been discussed in the Horodeckis paper [4] but the term "entanglement witness" was introduced by B. Terhal [5]. The most general procedures of constructing entanglement witnesses were introduced in [6, 7].

It follows immediately that entanglement witness has to have some negative eigenvalue. It is easy to check (cf. [6]) that the eigenvector corresponding to this eigenvalue must be entangled. Let us observe that entanglement witnesses represent — in some sense — a kind of Bell inequality which is violated by the entangled state ρ.

There exists an isomorphism [8] between witness operators and linear positive maps which allows to construct maps that detect states which are not separable. Each entanglement witness W acting on $\mathcal{H}_A \otimes \mathcal{H}_B$ defines a positive map Λ that transforms positive operators from $\mathcal{B}(\mathcal{H}_A)$ into positive operators on $\mathcal{B}(\mathcal{H}_B)$. This isomorphism $\mathcal{J} : \mathcal{L}(\mathcal{A}_A, \mathcal{A}_B) \to \mathcal{A}_A \otimes \mathcal{A}_B$, where $\mathcal{A}_A = \mathcal{B}(\mathcal{H}_A)$ and $\mathcal{A}_B = \mathcal{B}(\mathcal{H}_B)$ is defined by

$$W = \mathcal{J}(\Lambda) = \sum E_i^* \otimes \Lambda(E_i)\,. \tag{4.1}$$

Here $\Lambda : \mathcal{A}_A \to \mathcal{A}_B$ and $\{E_i\}$ stands for any orthogonal basis in $\mathcal{A}_A$. In both algebras $\mathcal{A}_A$ and $\mathcal{A}_B$ the scalar product is defined by $(\tau, \sigma) := \operatorname{Tr}(\sigma^*, \tau)$ (for details cf. [8]). An equivalent definition of the isomorphism $\mathcal{J}$ can be given by the equality

$$W = \mathcal{J}(\Lambda) = (I \otimes \Lambda)P_+\,, \tag{4.2}$$

where

$$P_+ := \frac{1}{\dim \mathcal{H}_A} \left(\sum_{i=1}^{\dim \mathcal{H}_A} |i\rangle \otimes |i\rangle \right) \left(\sum_{j=1}^{\dim \mathcal{H}_A} \langle j| \otimes \langle j| \right) \tag{4.3}$$

is the projector onto the maximally entangled state [9]. It is very important that the maps corresponding to entanglement witnesses are positive, but not completely positive, and in particular their extension to $\mathcal{H}_A \otimes \mathcal{H}_B$ allows one to detect the entanglement of ρ.

5 Entanglement witnesses — a geometric approach

From geometric point of view, the existence of entanglement witnesses is a consequence of the Hahn–Banach theorem [9] which states the following: If $\mathcal{R}$ is a convex compact set in a Hilbert space $\mathcal{H}$, and ρ does not belong to $\mathcal{R}$ then there exists a hyperplane which separates ρ from $\mathcal{R}$. In the context of quantum mechanics the set of all density operators $\mathcal{P}(\mathcal{H})$ is a convex and compact subset of the space $\mathcal{B}(\mathcal{H})$. The set of all separable density operators $\mathrm{Sep}\,\mathcal{P}(\mathcal{H})$ is a convex and compact subset of the set $\mathcal{P}(\mathcal{H})$. The state ρ is entangled and therefore $\rho \in \mathcal{P}(\mathcal{H})$ but $\rho \notin \mathrm{Sep}\,\mathcal{P}(\mathcal{H})$. Now the hyperplane which separates ρ from $\mathrm{Sep}\,\mathcal{P}(\mathcal{H})$ is given by those σ which fulfill

$$\mathrm{Tr}\,(\sigma W) = 0\,. \tag{5.1}$$

The expression $(W, \sigma) = \mathrm{Tr}\,(W\sigma)$ has all properties of the scalar product (scalar product in $\mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$. This fact leads us to the observation that all density operators "on one side" of the hyperplane lead to positive values of the trace with W, the ones on the other side to negative values. This intuitive picture of entanglement witnesses also gives an idea how they can be optimized [6]. Performing a parallel transformation of the hyperplane such that it becomes tangent to the set of separable states means that the corresponding witness operator W_0 detects more entangled states than before. Some operational methods which can be used to investigate properties of entanglement witnesses will be published in [10].

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