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Abstract. Let $H_K(E)$ be a reproducing kernel Hilbert space comprising complex-valued functions $\{f\}$ on $E$ and $L_j (j = 1, 2, \ldots)$ be a bounded linear operator on $H_K(E)$ into a Hilbert space $H_j$. Then, for $d_j \in H_j$ we shall consider the simultaneous operator equations $L_j f = d_j (j = 1, 2, \ldots)$ with the best approximation problem, for given $d_j \in H_j$
\[
\inf_{f \in H_K(E)} \sum_j \|L_j f - d_j\|^2_{H_j}.
\]

Furthermore we shall give a general idea and method for approximations of $L_2$ functions by Sobolev Hilbert spaces by using the Tikhonov regularization. We shall illustrate examples by figures for approximations of $L_2$ functions by the first and second order Sobolev Hilbert spaces.

Keywords: Reproducing kernel, operator equations, bounded linear operator, Tikhonov regularization, Sobolev space, best approximation, Green's function, simultaneous linear partial differential equation, generalized inverse.

1 Introduction and Background Theorems

We shall formulate our background theorem which has many concrete applications based on [2-6].

Let $H_K$ be a Hilbert space comprising complex-valued functions $\{f\}$ on a set $E$ admitting a reproducing kernel $K(x, y)$ and let $L$ be a bounded linear
operator on $H_K$ into a Hilbert space $H$. We introduce the inner product in the space $H_K$, for any fixed $\lambda > 0$

$$\lambda(f_1, f_2)_{H_K} + (Lf_1, Lf_2)_H.$$  \hspace{1cm} (1)

Then, it forms a Hilbert space and this Hilbert space $H_K(L; \lambda)$ admits a reproducing kernel $K_L(x, y; \lambda)$ on $E$. Then, we have the relation of $K(x, y)$ and $K_L(x, y; \lambda)$

$$K_L(x, y; \lambda) + \frac{1}{\lambda}(LK_{L}(., y; \lambda), LK(., x))_H = \frac{1}{\lambda}K(x, y).$$ \hspace{1cm} (2)

**Theorem 1** The best approximation $f^*_{\lambda,g,f_0}$ in the sense, for any $f_0 \in H_K$ and for any $g \in H$

$$\inf_{f \in \mathcal{H}_K} \{\lambda\|f - f_0\|_{H_K}^2 + \|Lf - g\|_H^2\}$$

$$= \lambda\|f^*_{\lambda,g,f_0} - f_0\|_{H_K}^2 + \|Lf^*_{\lambda,g,f_0} - g\|_H^2$$ \hspace{1cm} (3)

exists uniquely and it is represented by

$$f^*_{\lambda,g,f_0}(x) = \lambda(f_0(\cdot), K_L(\cdot, x; \lambda))_{H_K} + (g(\cdot), LK_L(\cdot, x; \lambda))_{H}.$$ \hspace{1cm} (4)

As simple and typical reproducing kernel Hilbert spaces, we shall consider the Sobolev Hilbert spaces $H_{K_1}$ and $H_{K_2}$ admitting the reproducing kernels

$$K_1(x, y) = \frac{1}{2}e^{-|x-y|} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi(x-y)} \frac{d\xi}{\xi^2 + 1}$$ \hspace{1cm} (5)

and

$$K_2(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\xi(x-y)}}{\xi^4 + \xi^2 + 1} d\xi.$$ \hspace{1cm} (6)

The norms in $H_{K_1}$ and $H_{K_2}$ are given by

$$\|f\|_{H_{K_1}}^2 = \int_{-\infty}^{\infty} (|f'(x)|^2 + |f(x)|^2) dx$$

and

$$\|f\|_{H_{K_2}}^2 = \int_{-\infty}^{\infty} (|f''(x)|^2 + |f'(x)|^2 + |f(x)|^2) dx,$$

respectively. We shall examine the best approximation in (3) for some typical Hilbert spaces $H$ and bounded linear operators $L$. In general, we are interested in the behaviours of the best approximation functions for $\lambda$ tending to zero from the viewpoint of the Tikhonov regularization. So, we wish to illustrate the behaviours of the best approximations for $\lambda$ tending to zero.
2 Typical Examples

See [3] for many concrete reproducing kernel forms for which Theorem 1 is applied. We can see a general example and a general approach for simultaneous linear partial differential equations in N. Aronszajn [1] who discussed deeply Green's functions in connection with reproducing kernels. We shall give typical examples.

2.1 Let

\[ G(x, y) = \frac{1}{2} e^{-|x-y|}. \]  

(7)

Then \( G(x, y) \) is the reproducing kernel for the Hilbert Sobolev space \( H_G \) comprising all absolutely continuous functions \( f(x) \) on \( \mathbb{R} \) with finite norms

\[ \left\{ \int_{-\infty}^{\infty} \left( |f'(x)|^2 + |f(x)|^2 \right) dx \right\}^{\frac{1}{2}} < \infty. \]  

(8)

Hence, we can examine the best approximation problem as follows: For any given \( F_1, F_2 \in L_2(\mathbb{R}) \),

\[ \inf_{f \in H_G} \int_{-\infty}^{\infty} (|F_1(x) - f'(x)|^2 + |F_2(x) - f(x)|^2) dx. \]  

(9)

2.2 For the first order Sobolev Hilbert space \( H_{K_1} \) we shall consider the two bounded linear operators \( L_1 : H_{K_1} \rightarrow L_1 f = f \in L_2(\mathbb{R}) \) and \( L_2 : H_{K_1} \rightarrow L_2 f = f' \in L_2(\mathbb{R}) \). Then, the associated reproducing kernels \( K_{1,1}(x, y; \lambda) \) and \( K_{1,2}(x, y; \lambda) \) for the RKHSs with the norms

\[ \lambda\|f\|_{H_{K_1}}^2 + \|f\|_{L_2(\mathbb{R})}^2 \]

and

\[ \lambda\|f\|_{H_{K_1}}^2 + \|f'\|_{L_2(\mathbb{R})}^2 \]

are given by

\[ K_{1,1}(x, y; \lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\xi(x-y)}}{\lambda \xi^2 + (\lambda + 1)} d\xi \]

\[ = \frac{1}{2\sqrt{\lambda(\lambda + 1)}} \exp \left\{ -\sqrt{\frac{\lambda + 1}{\lambda}} |x-y| \right\} \]  

(10)

and

\[ K_{1,2}(x, y; \lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\xi(x-y)}}{(\lambda + 1) \xi^2 + \lambda} d\xi \]
\[
\frac{1}{2\sqrt{\lambda(\lambda+1)}} \exp \left\{ -\sqrt{\frac{\lambda}{\lambda+1}} |x - y| \right\},
\]
respectively. Hence, the best approximate functions \( f_{1,1}^*(x; \lambda, g) \) and \( f_{1,2}^*(x; \lambda, g) \) in the senses, for any \( g \in L_2(\mathbb{R}) \)
\[
\inf_{f \in H_{K_1}} \left\{ \lambda \|f\|_{H_{K_1}}^2 + \|f - g\|_{L_2(\mathbb{R})}^2 \right\}
= \lambda \|f_{1,1}^* (\cdot; \lambda, g)\|_{H_{K_1}}^2 + \|f_{1,1}^* (\cdot; \lambda, g) - g\|_{L_2(\mathbb{R})}^2
\]
and
\[
\inf_{f \in H_{K_1}} \left\{ \lambda \|f\|_{H_{K_1}}^2 + \|f' - g\|_{L_2(\mathbb{R})}^2 \right\}
= \lambda \|f_{1,2}^* (\cdot; \lambda, g)\|_{H_{K_1}}^2 + \|f_{1,2}^* (\cdot; \lambda, g) - g\|_{L_2(\mathbb{R})}^2
\]
are given by
\[
f_{1,1}^*(x; \lambda, g) = \int_{-\infty}^{\infty} g(\xi) \frac{1}{2\sqrt{\lambda(\lambda+1)}} \exp \left\{ -\sqrt{\frac{\lambda+1}{\lambda}} |\xi - x| \right\} d\xi
\]
and
\[
f_{1,2}^*(x; \lambda, g) = \int_{-\infty}^{\infty} g(\xi) \frac{1}{2\sqrt{\lambda(\lambda+1)}} \frac{\partial}{\partial \xi} \exp \left\{ -\sqrt{\frac{\lambda}{\lambda+1}} |\xi - x| \right\} d\xi,
\]
respectively. Note that \( f_{1,2}^*(x; \lambda, g) \) can be considered as an approximate and generalized solution of the differential equation
\[
y' = g(x) \quad \text{on} \quad \mathbb{R}
\]
in the first order Sobolev Hilbert space \( H_{K_1} \). See Figure 3.
Figure 1: Examples of approximated functions in (9). (a) $F_1(x) = 0$ and $F_2(x) = \chi_{[-1,1]}$ (top thin curve). (b) $F_1(x) = F_2(x) = \chi_{[-1,1]}$ (middle bold curve). (c) $F_1(x) = \chi_{[-1,1]}$ and $F_2(x) = 0$ (bottom curve).

2.3 For the second order Sobolev Hilbert space $H_{K_2}$ we shall consider the three bounded linear operators into $L_2(\mathbb{R})$ defined by

$L_1 : f \rightarrow f$

$L_2 : f \rightarrow f'$

and

$L_3 : f \rightarrow f''$.

Then, the reproducing kernels $K_{2,1}(x, y; \lambda), K_{2,2}(x, y; \lambda)$ and $K_{2,3}(x, y; \lambda)$ for the Hilbert spaces with the norms

$\lambda\|f\|^2_{H_{K_2}} + \|f\|^2_{L_2(\mathbb{R})},$

$\lambda\|f\|^2_{H_{K_2}} + \|f'\|^2_{L_2(\mathbb{R})},$

and

$\lambda\|f\|^2_{H_{K_2}} + \|f''\|^2_{L_2(\mathbb{R})},$

are given by

$K_{2,1}(x, y; \lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\xi(x-y)}}{\lambda\xi^4 + \lambda\xi^2 + (\lambda + 1)} d\xi,$

(17)

$K_{2,2}(x, y; \lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\xi(x-y)}}{\lambda\xi^4 + (\lambda + 1)\xi^2 + \lambda} d\xi,$

(18)
Figure 2: Graphs of $f_{1,1}^*(x; \lambda, g)$ in (14) (top) and $f_{2,1}^*(x; \lambda, g)$ in (20) (bottom) for $g(x) = \chi_{[-1,1]}$. 
Figure 3: Graphs of $f_{1,2}^*(x; \lambda, g)$ in (15) (top) and $f_{2,2}^*(x; \lambda, g)$ in (21) (bottom) for $g(x) = \chi_{[-1,1]}$. 
Figure 4: Graphs of \( f_{2,3}^*(x; \lambda, g) \) in (22) for \( g(x) = \chi_{[-1,1]} \).

Figure 5: Examples of approximated functions in (25). (a) \( F_1(x) = F_2(x) = F_3(x) = \chi_{[-1,1]} \) (top bold curve). (b) \( F_1(x) = \chi_{[-1,1]} \) and \( F_2(x) = F_3(x) = 0 \) (the second bold curve) (c) \( F_1(x) = F_2(x) = 0 \) and \( F_3(x) = \chi_{[-1,1]} \) (the thin curve). (d) \( F_1(x) = F_3(x) = 0 \) and \( F_2(x) = \chi_{[-1,1]} \) (the rest curve).
and
\[
K_{2,3}(x, y; \lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\xi(x-y)}}{(\lambda+1)\xi^4 + \lambda\xi^2 + \lambda} d\xi.
\] (19)

Then, the corresponding best approximate functions \(f_{2,1}^{*}(x; \lambda, g), f_{2,2}^{*}(x; \lambda, g),\) and \(f_{2,3}^{*}(x; \lambda, g)\) are given by, for any \(g \in L_{2}(\mathbb{R})\)
\[
f_{2,1}^{*}(x; \lambda, g) = \int_{-\infty}^{\infty} g(\xi) d\xi \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\eta(\xi-x)}}{\lambda\eta^4 + \lambda + (\lambda+1)} d\eta,
\] (20)
\[
f_{2,2}^{*}(x; \lambda, g) = \int_{-\infty}^{\infty} g(\xi) d\xi \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{-i\eta \cdot e^{-i\eta(\xi-x)}}{\lambda\eta^4 + (\lambda+1)\eta^2 + \lambda} d\eta,
\] (21)
and
\[
f_{2,3}^{*}(x; \lambda, g) = \int_{-\infty}^{\infty} g(\xi) d\xi \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{-\eta^2 \cdot e^{-i\eta(\xi-x)}}{(\lambda+1)\eta^4 + \lambda\eta^2 + \lambda} d\eta,
\] (22)
respectively. We shall give another type applications of Theorem 1. 
Note that
\[
K(x, y) = \frac{1}{4} e^{-|x-y| \{1 + |x-y|\}}
\] (23)
is the reproducing kernel of the Sobolev space \(H_{K}\) with finite norms
\[
\left\{ \int_{-\infty}^{\infty} (|f''(x)|^2 + 2|f'(x)|^2 + |f(x)|^2) dx \right\}^{\frac{1}{2}} < \infty.
\] (24)
Therefore, we can examine the approximate problem as follows:
\[
\inf_{f \in H_{K}} \int_{-\infty}^{\infty} (|F_{1}(x) - f''(x)|^2 + 2|F_{2}(x) - f'(x)|^2 + |F_{3}(x) - f(x)|^2) dx.
\] (25)
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