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Abstract. Let $H_K(E)$ be a reproducing kernel Hilbert space comprising complex-valued functions $\{f\}$ on $E$ and $L_j$ ($j = 1, 2, \ldots$) be a bounded linear operator on $H_K(E)$ into a Hilbert space $H_j$. Then, for $d_j \in H_j$ we shall consider the simultaneous operator equations $L_j f = d_j$ ($j = 1, 2, \ldots$) with the best approximation problem, for given $d_j \in H_j$

$$\inf_{f \in H_K(E)} \sum_j \|L_j f - d_j\|_{H_j}^2.$$ 

Furthermore we shall give a general idea and method for approximations of $L_2$ functions by Sobolev Hilbert spaces by using the Tikhonov regularization. We shall illustrate examples by figures for approximations of $L_2$ functions by the first and second order Sobolev Hilbert spaces.

Keywords: Reproducing kernel, operator equations, bounded linear operator, Tikhonov regularization, Sobolev space, best approximation, Green's function, simultaneous linear partial differential equation, generalized inverse.

1 Introduction and Background Theorems

We shall formulate our background theorem which has many concrete applications based on [2-6].

Let $H_K$ be a Hilbert space comprising complex-valued functions $\{f\}$ on a set $E$ admitting a reproducing kernel $K(x, y)$ and let $L$ be a bounded linear
operator on $H_K$ into a Hilbert space $H$. We introduce the inner product in the space $H_K$, for any fixed $\lambda > 0$

$$\lambda(f_1, f_2)_{H_K} + (Lf_1, Lf_2)_H.$$  \hfill (1)

Then, it forms a Hilbert space and this Hilbert space $H_K(L; \lambda)$ admits a reproducing kernel $K_L(x, y; \lambda)$ on $E$. Then, we have the relation of $K(x, y)$ and $K_L(x, y; \lambda)$

$$K_L(x, y; \lambda) + \frac{1}{\lambda}(LK_L(., y; \lambda), LK(., x))_H = \frac{1}{\lambda}K(x, y).$$  \hfill (2)

**Theorem 1** The best approximation $f_{\lambda, g, f_0}^*$ in the sense, for any $f_0 \in H_K$ and for any $g \in H$

$$\inf_{f \in H_K} \{ \lambda \| f - f_0 \|_{H_K}^2 + \| Lf - g \|_H^2 \}$$

exists uniquely and it is represented by

$$f_{\lambda, g, f_0}^*(x) = \lambda(f_0(\cdot), K_L(\cdot, x; \lambda))_{H_K} + (g(\cdot), LK_L(\cdot, x; \lambda))_{H_K}.$$  \hfill (4)

As simple and typical reproducing kernel Hilbert spaces, we shall consider the Sobolev Hilbert spaces $H_{K_1}$ and $H_{K_2}$ admitting the reproducing kernels

$$K_1(x, y) = \frac{1}{2} e^{-|x-y|} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\xi(x-y)}}{\xi^2 + 1} d\xi$$  \hfill (5)

and

$$K_2(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\xi(x-y)}}{\xi^4 + \xi^2 + 1} d\xi.$$  \hfill (6)

The norms in $H_{K_1}$ and $H_{K_2}$ are given by

$$\| f \|_{H_{K_1}}^2 = \int_{-\infty}^{\infty} (|f'(x)|^2 + |f(x)|^2) dx$$

and

$$\| f \|_{H_{K_2}}^2 = \int_{-\infty}^{\infty} (|f''(x)|^2 + |f'(x)|^2 + |f(x)|^2) dx,$$

respectively. We shall examine the best approximation in (3) for some typical Hilbert spaces $H$ and bounded linear operators $L$. In general, we are interested in the behaviours of the best approximation functions for $\lambda$ tending to zero from the viewpoint of the Tikhonov regularization. So, we wish to illustrate the behaviours of the best approximations for $\lambda$ tending to zero.
2 Typical Examples

See [3] for many concrete reproducing kernel forms for which Theorem 1 is applied. We can see a general example and a general approach for simultaneous linear partial differential equations in N. Aronszajn [1] who discussed deeply Green's functions in connection with reproducing kernels. We shall give typical examples.

2.1 Let

$$G(x, y) = \frac{1}{2} e^{-|x-y|}. \quad (7)$$

Then $G(x, y)$ is the reproducing kernel for the Hilbert Sobolev space $H_G$ comprising all absolutely continuous functions $f(x)$ on $\mathbb{R}$ with finite norms

$$\left\{ \int_{-\infty}^{\infty} (|f'(x)|^2 + |f(x)|^2) \, dx \right\}^{\frac{1}{2}} < \infty. \quad (8)$$

Hence, we can examine the best approximation problem as follows: For any given $F_1, F_2 \in L_2(\mathbb{R})$,

$$\inf_{f \in H_G} \int_{-\infty}^{\infty} (|F_1(x) - f'(x)|^2 + |F_2(x) - f(x)|^2) \, dx. \quad (9)$$

2.2 For the first order Sobolev Hilbert space $H_{K_1}$ we shall consider the two bounded linear operators $L_1 : H_{K_1} \to L_1 f = f \in L_2(\mathbb{R})$ and $L_2 : H_{K_1} \to L_2 f = f' \in L_2(\mathbb{R})$. Then, the associated reproducing kernels $K_{1,1}(x, y; \lambda)$ and $K_{1,2}(x, y; \lambda)$ for the RKHSs with the norms

$$\lambda \| f \|_{H_{K_1}}^2 + \| f \|_{L_2(\mathbb{R})}^2$$

and

$$\lambda \| f \|_{H_{K_1}}^2 + \| f' \|_{L_2(\mathbb{R})}^2$$

are given by

$$K_{1,1}(x, y; \lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi(x-y)} \frac{1}{\lambda \xi^2 + (\lambda + 1)} \, d\xi$$

$$= \frac{1}{2\sqrt{\lambda(\lambda + 1)}} \exp \left\{ -\sqrt{\frac{\lambda + 1}{\lambda}} |x - y| \right\} \quad (10)$$

and

$$K_{1,2}(x, y; \lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi(x-y)} \frac{1}{(\lambda + 1)\xi^2 + \lambda} \, d\xi$$
\[
\exp \left\{ -\sqrt{\frac{\lambda}{\lambda + 1}} |x - y| \right\}, \quad (11)
\]
respectively. Hence, the best approximate functions \( f_{1,1}^{*}(x; \lambda, g) \) and \( f_{1,2}^{*}(x; \lambda, g) \) in the senses, for any \( g \in L_2(\mathbb{R}) \)
\[
\inf_{f \in H_{K_1}} \left\{ \lambda \|f\|_{H_{K_1}}^2 + \|f - g\|_{L_2(\mathbb{R})}^2 \right\}
= \lambda \|f_{1,1}^{*}(\cdot; \lambda, g)\|_{H_{K_1}}^2 + \|f_{1,1}^{*}(\cdot; \lambda, g) - g\|_{L_2(\mathbb{R})}^2 \quad (12)
\]
and
\[
\inf_{f \in H_{K_1}} \left\{ \lambda \|f\|_{H_{K_1}}^2 + \|f' - g\|_{L_2(\mathbb{R})}^2 \right\}
= \lambda \|f_{1,2}^{*}(\cdot; \lambda, g)\|_{H_{K_1}}^2 + \|f_{1,2}^{*}(\cdot; \lambda, g) - g\|_{L_2(\mathbb{R})}^2 \quad (13)
\]
are given by
\[
f_{1,1}^{*}(x; \lambda, g) = \int_{-\infty}^{\infty} g(\xi) \frac{1}{2\sqrt{\lambda(\lambda + 1)}} \exp \left\{ -\sqrt{\frac{\lambda + 1}{\lambda}} |\xi - x| \right\} d\xi \quad (14)
\]
and
\[
f_{1,2}^{*}(x; \lambda, g) = \int_{-\infty}^{\infty} g(\xi) \frac{1}{2\sqrt{\lambda(\lambda + 1)}} \frac{\partial}{\partial \xi} \exp \left\{ -\sqrt{\frac{\lambda}{\lambda + 1}} |\xi - x| \right\} d\xi, \quad (15)
\]
respectively. Note that \( f_{1,2}^{*}(x; \lambda, g) \) can be considered as an approximate and generalized solution of the differential equation
\[
y' = g(x) \quad \text{on} \quad \mathbb{R} \quad (16)
\]
in the first order Sobolev Hilbert space \( H_{K_1} \). See Figure 3.
2.3 For the second order Sobolev Hilbert space $H_{K_2}$ we shall consider the three bounded linear operators into $L_2(\mathbb{R})$ defined by

$$L_1 : f \rightarrow f$$

$$L_2 : f \rightarrow f'$$

and

$$L_3 : f \rightarrow f''.$$  

Then, the reproducing kernels $K_{2,1}(x, y; \lambda)$, $K_{2,2}(x, y; \lambda)$ and $K_{2,3}(x, y; \lambda)$ for the Hilbert spaces with the norms

$$\lambda \|f\|_{H_{K_2}}^2 + \|f\|_{L_2(\mathbb{R})}^2,$$

$$\lambda \|f\|_{H_{K_2}}^2 + \|f'\|_{L_2(\mathbb{R})}^2,$$

and

$$\lambda \|f\|_{H_{K_2}}^2 + \|f''\|_{L_2(\mathbb{R})}^2,$$

are given by

$$K_{2,1}(x, y; \lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\xi(x-y)}}{\lambda \xi^4 + \lambda \xi^2 + (\lambda + 1)} d\xi,$$ (17)

$$K_{2,2}(x, y; \lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\xi(x-y)}}{\lambda \xi^4 + (\lambda + 1) \xi^2 + \lambda} d\xi.$$ (18)
Figure 2: Graphs of $f_{1,1}^*(x; \lambda, g)$ in (14) (top) and $f_{2,1}^*(x; \lambda, g)$ in (20) (bottom) for $g(x) = \chi_{[-1,1]}$. 
Figure 3: Graphs of $f_{1,2}^*(x; \lambda, g)$ in (15) (top) and $f_{2,2}^*(x; \lambda, g)$ in (21) (bottom) for $g(x) = \chi_{[-1,1]}$. 
Figure 4: Graphs of $f^*_{2,3}(x; \lambda, g)$ in (22) for $g(x) = \chi_{[-1,1]}$.

Figure 5: Examples of approximated functions in (25). (a) $F_1(x) = F_2(x) = F_3(x) = \chi_{[-1,1]}$ (top bold curve). (b) $F_1(x) = \chi_{[-1,1]}$ and $F_2(x) = F_3(x) = 0$ (the second bold curve) (c) $F_1(x) = F_2(x) = 0$ and $F_3(x) = \chi_{[-1,1]}$ (the thin curve). (d) $F_1(x) = F_3(x) = 0$ and $F_2(x) = \chi_{[-1,1]}$ (the rest curve).
and
\[ K_{2,3}(x, y; \lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\xi(x-y)}}{(\lambda+1)\xi^4 + \lambda\xi^2 + \lambda} d\xi. \] (19)

Then, the corresponding best approximate functions \( f_{2,1}^*(x; \lambda, g) \), \( f_{2,2}^*(x; \lambda, g) \), and \( f_{2,3}^*(x; \lambda, g) \) are given by, for any \( g \in L_2(\mathbb{R}) \)
\[ f_{2,1}^*(x; \lambda, g) = \int_{-\infty}^{\infty} g(\xi) \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\eta(\xi-x)}}{\lambda\eta^4 + \lambda\eta^2 + (\lambda+1)} d\eta, \] (20)
\[ f_{2,2}^*(x; \lambda, g) = \int_{-\infty}^{\infty} g(\xi) \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{-i\eta \cdot e^{-i\eta(\xi-x)}}{\lambda\eta^4 + (\lambda+1)\eta^2 + \lambda} d\eta, \] (21)
and
\[ f_{2,3}^*(x; \lambda, g) = \int_{-\infty}^{\infty} g(\xi) \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{-\eta^2 \cdot e^{-i\eta(\xi-x)}}{(\lambda+1)\eta^4 + \lambda\eta^2 + \lambda} d\eta, \] (22)
respectively. We shall give another type applications of Theorem 1. Note that
\[ K(x, y) = \frac{1}{4}e^{-|x-y|} \{1 + |x - y|\} \] (23)
is the reproducing kernel of the Sobolev space \( H_K \) with finite norms
\[ \left\{ \int_{-\infty}^{\infty} \left( |f''(x)|^2 + 2|f'(x)|^2 + |f(x)|^2 \right) dx \right\}^{\frac{1}{2}} < \infty. \] (24)
Therefore, we can examine the approximate problem as follows:
\[ \inf_{f \in H_K} \int_{-\infty}^{\infty} \left( |F_1(x) - f''(x)|^2 + 2|F_2(x) - f'(x)|^2 + |F_3(x) - f(x)|^2 \right) dx. \] (25)
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