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Real Inversion Formulas of the Gaussian convolution

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Abstract

In this paper we introduce a unified method of solving inverse problems in some general linear differential equations numerically and as a prototype example we shall show a practical real inversion formula for the Gaussian convolution.

Keywords: Inverse problem, differential equation, Gaussian convolution, Weierstrass transform, approximation of functions, reproducing kernel, Tikhonov regularization, Sobolev space, generalized inverse, approximate inverse.

1 Introduction

Let $L : H_K \to H$ be a bounded linear operator of Hilbert spaces. We consider the best approximation problem

$$\inf_{f \in H_K} \|Lf - g\|_H, \tag{1}$$

where $g \in H$ is given. If there exists an element $f^* \in H_K$ which attains the infimum (1) then the problem (1) is called solvable otherwise it is called unsolvable. If H_K is a reproducing kernel Hilbert space admitting a reproducing kernel $K(p,q)$ on a set E then whether the problem (1) is solvable or not, the problem

$$\inf_{f\in H_K}\{\lambda\|f\|_{H_K}^2+\|Lf-g\|_H^2\} \tag{2}$$

is always solvable for all $\lambda>0$ and we can obtain a method of obtaining the member $f^*_{\lambda,g}$ of the smallest norm in H_K which attains the infimum (2). The problem (2) is called a *Tikhonov regularization* for the problem (1). If the problem (1) is solvable then $f^*_g := \lim_{\lambda\to 0} f^*_{\lambda,g}$ in H_K and f^*_g is the element of the minimum norm which attains the infimum (1)([9]). If the problem (1) is unsolvable then $\lim_{\lambda\to 0} f^*_{\lambda,g}$ does not exist in H_K. Even though the problem (1) is unsolvable, for a value of λ, we can think of $f^*_{\lambda,g}$ as a generalized solution of

$$Lf=g.$$

2 Background Theorems

Theorem 1 *([2,5]). Let H_K be a Hilbert space admitting the reproducing kernel $K(p,q)$ on a set E. Let $L: H_K\to\mathcal{H}$ be a bounded linear operator of Hilbert spaces. Then, for $g\in\mathcal{H}$ the problem*

$$\inf_{f\in H_K}\|Lf-g\|_{\mathcal{H}} \tag{3}$$

is solvable if and only if $L^*g\in H_k$, where H_k is the reproducing kernel Hilbert space admitting the RK $k(p,q)=\langle L^*LK(.,q),L^*LK(\cdot,p)\rangle_{H_K}$ on E. Furthermore, if the problem (3) is solvable then

$$f^*_g(p)=\langle L^*g, L^*LK(\cdot,p)\rangle_{H_k}$$

is the element of H_K with the smallest norm which attains the infimum (3).

Theorem 2 *([7]). Let H_K, L, $\mathcal{H}$ and E be as in Theorem 1 and let V be the underlying vector space of H_K. For $\lambda>0$ introduce a structure in V and call it H_{K_λ} as*

$$\langle f_1,f_2\rangle_{H_{K_\lambda}}=\lambda\langle f_1,f_2\rangle_{H_K}+\langle Lf_1,Lf_2\rangle_{\mathcal{H}} \tag{4}$$

then H_{K_λ} is the Hilbert space with the reproducing kernel $K_\lambda(p,q)$ on E satisfying the equation

$$K(\cdot,q) = (\lambda I + L^*L)K_\lambda(\cdot,q), \tag{5}$$

where L^* is the adjoint of $L : H_K \to \mathcal{H}$.

Theorem 3 *([7]). Let H_K, L, $\mathcal{H}$ and E be as in Theorem 1 and let K_λ be as in Theorem 2. Then, for any $\lambda > 0$ and for any $g \in \mathcal{H}$, the approximation problem*

$$\inf_{f\in H_K} \left(\lambda \|f\|^2_{H_K} + \|Lf - g\|^2_{\mathcal{H}}\right) \tag{6}$$

is solvable and

$$f^*_{\lambda,g}(p) = \langle g, LK_\lambda(.,p)\rangle_{\mathcal{H}} \tag{7}$$

is the member of H_K with the smallest H_K-norm which attains the infimum (6).

3 PDE and inverse problems

We will be able to apply our theory to various inverse problems to look for the whole data from local data of the domain or from some boundary data. Here, we will refer to these problems with a prototype example in order to show this basic idea, clearly, from [6].

We recall a Sobolev imbedding theorem ([1], pp. 18–19). In order to use the results in the framework of Hilbert spaces, we assume $p = q = 2$ there.

Let $W_2^\ell(G)$ ($\ell = 0,1,2,...$) be the Sobolev Hilbert space on G, where $G \subset \mathbf{R}^n$ is a bounded domain with a one piecewise-smooth continuously differentiable boundary Γ. We assume that

$$k \geq \ell - \frac{n}{2}. \tag{8}$$

Let $m = 0,1,2,...$ such that

$$m > n - 2(\ell - k), \quad 2 < \frac{2m}{n - 2(\ell - k)}. \tag{9}$$

Let $D \subseteq G \cup \Gamma$ be any ℓ times continuously differentiable manifold of dimension m. Then, for any $u \in W_2^\ell(G)$, the derivative $(D^\alpha u)(x) \in L_2(D)(x \in D)$, where $|\alpha| \leq k$, and we have the continuity of the imbedding operator

$$||D^\alpha u||_{L_2(D)} \leq M||u||_{W_2^\ell(G)}, \quad (M > 0; u \in W_2^\ell(G)). \tag{10}$$

Of course

$$||u||_{W_2^\ell(G)} \leq ||u||_{W_2^\ell(\mathbf{R}^n)}, \tag{11}$$

and we can construct the reproducing kernel for the space $W_2^\ell(\mathbf{R}^n)$ by using Fourier's integral for $2\ell > n$. Then, for any linear differential operator L with variable coefficients on G satisfying

$$||Lu||_{L_2(G)} < ||u||_{W_2^\ell(G)} \tag{12}$$

and for any linear (boundary) operator B with variable coefficients on D satisfying

$$||Bu||_{L_2(D)} < ||u||_{W_2^\ell(G)}, \tag{13}$$

we can discuss the best approximation: For any $F_1 \in L_2(G)$, for any $F_2 \in L_2(D)$ and for any $\lambda > 0$,

$$\inf_{u \in W_2^\ell(\mathbf{R}^n)} \left\{ \lambda ||u||^2_{W_2^\ell(\mathbf{R}^n)} + ||F_1 - Lu||^2_{L_2(G)} + ||F_2 - Bu||^2_{L_2(D)} \right\}. \tag{14}$$

If $F_1 = 0$ and λ is very close to zero then the problem may be interpreted that we wish to construct the solution u of the differential equation

$$Lu = 0 \quad \text{on} \quad G$$

satisfying

$$Bu = F_2 \quad \text{on} \quad D.$$

Our general theory gives a practical construction method for this inverse problem that from the observation F_2 on the part D, we construct u on the whole domain G satisfying the equation $Lu = 0$.

4 Gaussian Convolution

In this section we will consider the integral operator $L_t : H_S \to L_2(\mathbf{R})$ defined by

$$(L_t f)(x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} f(\xi) \exp[-\frac{(\xi - x)^2}{4t}] d\xi, \tag{15}$$

for given $t > 0$. Here H_S is the first order Sobolev Hilbert space on the whole real line with norm defined by

$$\|f\|_{H_S}^2 = \int_{-\infty}^{\infty} (f'(x)^2 + f(x)^2) dx \tag{16}$$

admitting the reproducing kernel

$$K(x, y) = \frac{1}{2} e^{-|x-y|} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\xi(x-y)}}{(1+\xi^2)} d\xi. \tag{17}$$

We now consider the best approximation problem: For any given $g \in L_2(\mathbf{R})$ and for $\lambda > 0$,

$$\inf_{f \in H_S} \left\{ \lambda \|f\|_{H_S}^2 + \|L_t f - g\|_{L_2(\mathbf{R})}^2 \right\}. \tag{18}$$

Then for the RKHS H_{K_λ} consisting of all the members of H_S with the norm

$$\|f\|_{H_{K_\lambda}}^2 = \lambda \|f\|_{H_S}^2 + \|L_t f\|_{L_2(\mathbf{R})}^2, \tag{19}$$

the reproducing kernel $K_\lambda(x, y)$ can be calculated directly by using Fourier's integrals as
follows:

$$K_\lambda(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\xi(x-y)} d\xi}{\lambda(1+\xi^2) + e^{-2\xi^2 t}}. \tag{20}$$

Hence the unique member of H_S with the minimum H_S-norm which attains the infimum (18) is given by

$$\begin{aligned} f_{\lambda,g}^*(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \Big\{ g(\xi) \\ &\quad \cdot \int_{-\infty}^{\infty} \frac{e^{-ip(\xi - x)} dp}{\lambda(p^2+1)e^{p^2 t} + e^{-p^2 t}} \Big\} d\xi. \end{aligned} \tag{21}$$

For $f \in H_S$ and for $g(\xi) = (L_t f)(\xi)$, we have the favourable formula

$$\lim_{\lambda \to 0} f_{\lambda,g}^*(x) = f(x)$$

uniformly on $\mathbf{R}$ ([7]).

Twenty years ago, the last author gave a surprise characterization of the image of (15) for $L_2(\mathbf{R}) = L_2(\mathbf{R}, dx)$ functions in terms of an analytic function and established a very simple complex inversion formula. The paper created a new method and many applications to general integral transforms in the framework of Hilbert spaces and various analytic extension formulas ([5]). However, in particular, its real inversion formulas are very involved and one might think that its real inversion formulas will be essentially involved for catching "*analyticity*" in terms of the data on the real line as in the real inversion formulas of the Laplace transform. This is a typical and famous ill-posed problem. See [7] for more details. For example, recall the classical real inversion of the Gaussian convolution formula: For a bounded and continuous function $f(x)$ and for $t = 1$,

$$e^{-D^2}(L_1 f(x)) = f(x) \quad \text{pointwisely} \quad \text{on} \quad \mathbf{R}$$

([4], p. 182).

The real inversion formula (21) will give a practical formula for the Gaussian convolution. We will show experimental results by computers in Figures 1 and 2. There, we will see that in order to overcome the high "ill-posedness" in the real inversion and in order to catch "analyticity" of the image of (15) we must work hardly; that is, we must take a very small λ and we must calculate the integral (21) hardly in the sense of numerical. Computers help us this hard work to calculate the integral for a very small λ.

Meanwhile, for any $\lambda > 0$ and any $t > 0$, we shall define a linear mapping

$$M_{\lambda,t} : L_2(\mathbf{R}) \to H_S$$

by $M_{\lambda,t}(g) = f^*_{\lambda,g}$. Now, we consider the composite operators $L_t M_{\lambda,t}$ and $M_{\lambda,t} L_t$. Using Fourier's integrals it can be shown that for $f \in H_S$,

$$\begin{aligned}(M_{\lambda,t} L_t f)(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big\{f(\xi) \\ &\quad \cdot \int_{-\infty}^{\infty}\frac{e^{-ip(\xi-x)}dp}{\lambda(p^2+1)e^{2p^2t}+1}\Big\}d\xi \qquad (22)\end{aligned}$$

and for $g \in L_2(\mathbf{R})$,

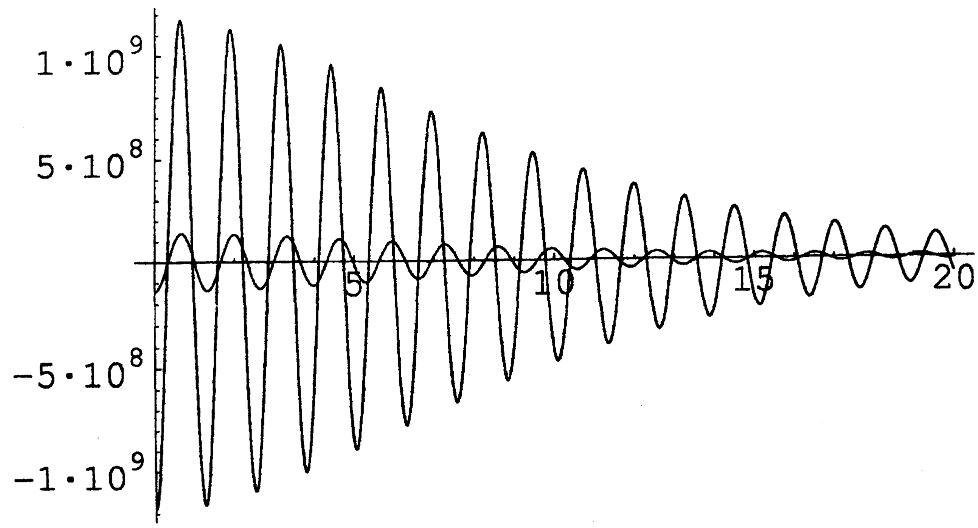

Figure 1: For $t = 1$ and for $g(x) = \chi_{[-1,1]}$, the inversion (21) for $\lambda = 10^{-21}$(the smaller one) and $\lambda = 10^{-23}$(the larger one).

$$\begin{aligned}(L_t M_{\lambda,t} g)(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big\{g(\xi) \\ &\quad \cdot \int_{-\infty}^{\infty}\frac{e^{-ip(\xi-x)}dp}{\lambda(p^2+1)e^{2p^2t}+1}\Big\}d\xi. \end{aligned} \tag{23}$$

Setting

$$\Delta_\lambda(x-\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-ip(\xi-x)}}{\lambda(p^2+1)e^{2p^2t}+1}dp$$

in (22) and (23), we have

$$(M_{\lambda,t}L_t f)(x) = \int_{-\infty}^{\infty} f(\xi)\Delta_\lambda(x-\xi)d\xi, \quad (f \in H_S) \tag{24}$$

and

$$(L_t M_{\lambda,t} g)(x) = \int_{-\infty}^{\infty} g(\xi)\Delta_\lambda(x-\xi)d\xi, \quad (g \in L_2(\mathbf{R})). \tag{25}$$

Then we obtain that

$$\lim_{\lambda\to 0}\Delta_\lambda(x-\xi) = \delta(x-\xi), \tag{26}$$

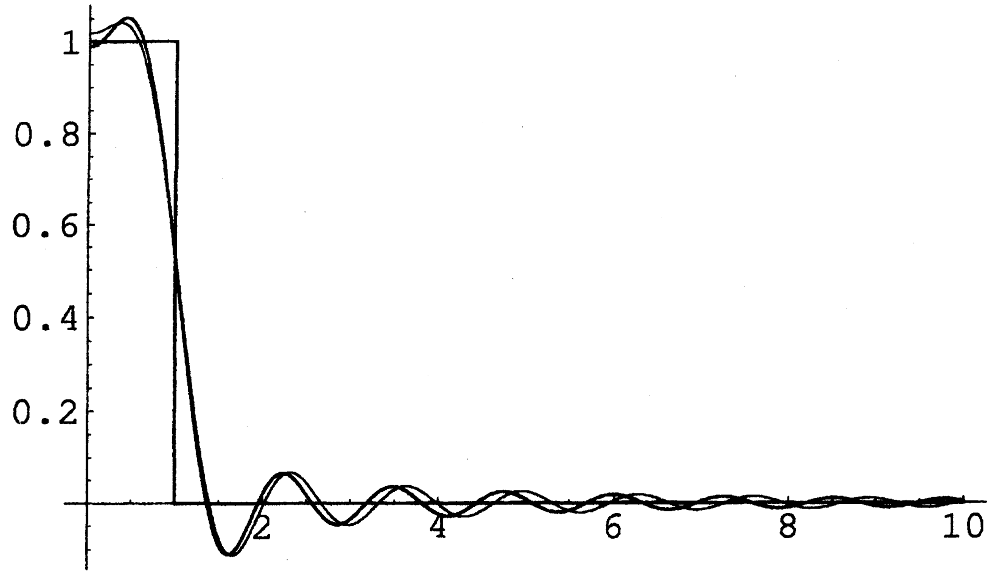

Figure 2: The images of Figure 1 by the Gaussian convolution: the bold curve corresponds to the larger curve of Figure 1 and the other one corresponds to the other one.

$$\lim_{\lambda\to 0} M_{\lambda,t}L_t = I \tag{27}$$

and

$$\lim_{\lambda\to 0} L_t M_{\lambda,t} = I. \tag{28}$$

The precise meaning of (26) and (27) is given as follows: For any $f \in H_S$

$$\lim_{\lambda\to 0}(M_{\lambda,t}L_t f)(x) = f(x)$$

uniformly on $\mathbf{R}$ ([7], Section 3). The precise meaning of (26) and (28) is given as follows: For any $g \in \mathcal{R}(L_t) + \mathcal{R}(L_t)^{\perp}$

$$\lim_{\lambda\to 0} L_t M_{\lambda,t} g = g$$

in $L_2(\mathbf{R})$ ([9]). See for example, [3] for the Tikhonov regularization.

In order to see (27) numerically, we consider an example: Let $f(x) = e^{-x^2}$ then $f \in H_S$. At $t = 1$, we see from Figure 3 that $\lim_{\lambda\to 0}(M_{\lambda,t}L_t f)(x) = f(x)$.

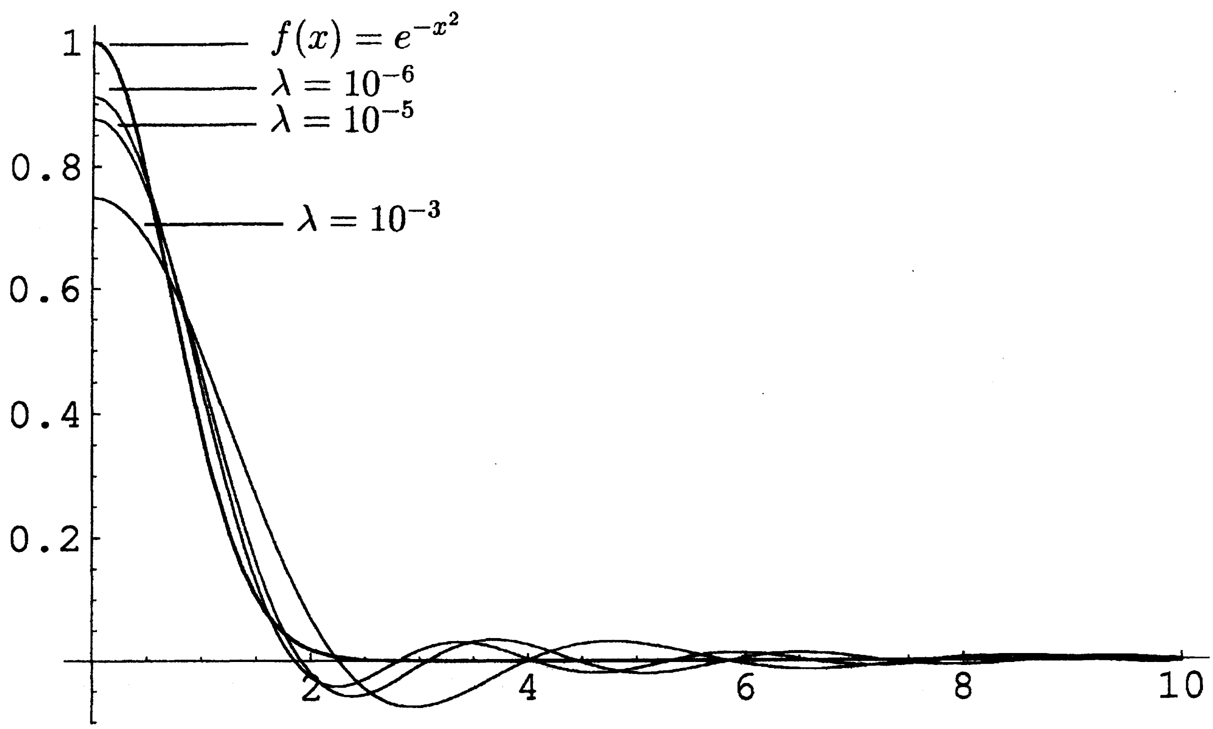

Figure 3: The figure shows how the graphs of $(M_{\lambda,t}L_t)f(x)$ approach to $f(x)$ as $\lambda \to 0$.

Now we give another experimental result to see the behaviour of

$$\lim_{\lambda \to 0} L_t M_{\lambda,t}$$

on $L_2(\mathbf{R}) \setminus R(L_t)$. Here, we consider $g(x) = \chi_{[-1,1]}$ then $g \in L_2(\mathbf{R}) \setminus R(L_t)$.

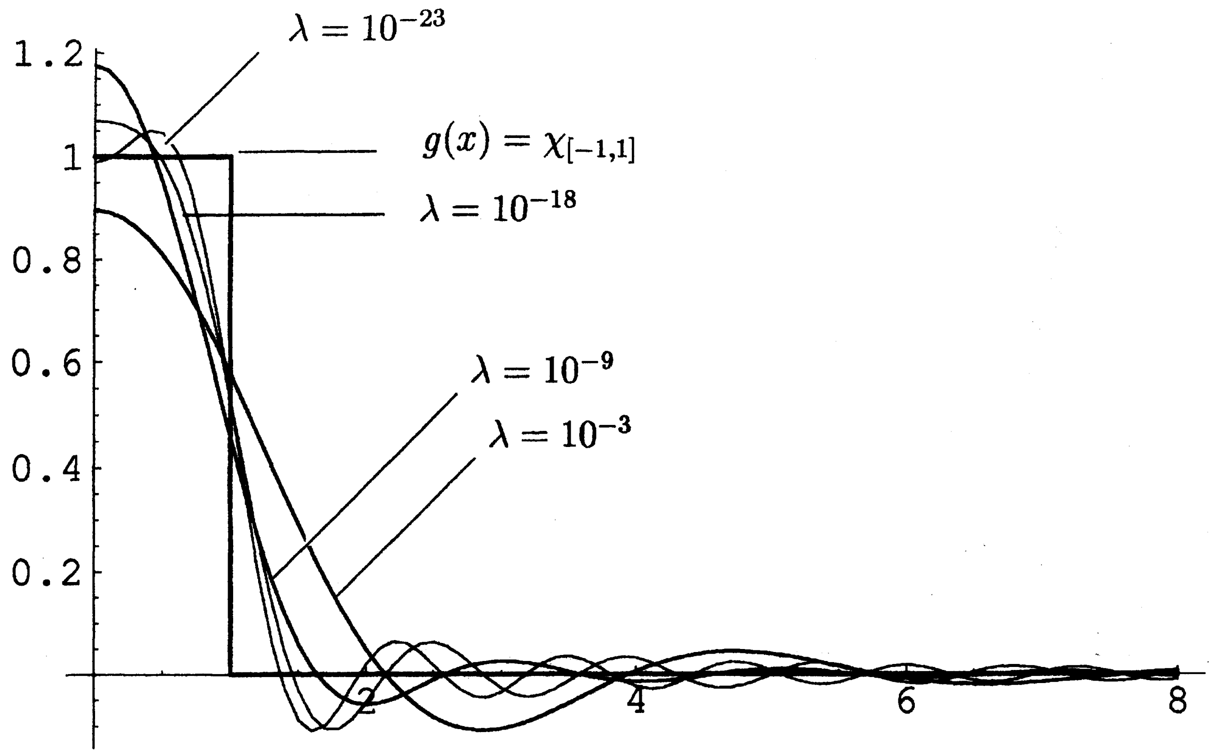

Figure 4: The figure shows the graphs of $L_t M_{\lambda,t} g(x)$ at $t = 1$ for different values of λ and $g(x)$.

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