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Hyperbolic balance laws and entropy

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0. Introduction

The aim of this note is to give a definition of the entropy for hyperbolic balance laws in d space dimensions:

$$\text{(HB)} \qquad w_t + \sum_{j=1}^{d} f^j(w)_{x_j} = g(w),$$

where w is an N-vector. The notion of mathematical entropy was first introduced by Godunov [3] and Friedrichs and Lax [2] for hyperbolic conservation laws:

$$\text{(HC)} \qquad w_t + \sum_{j=1}^{d} f^j(w)_{x_j} = 0,$$

and the entropy plays as a symmetrizer of the system (HC). We give a brief reveiw of this theory in Sect. 1.

In Sect. 2, we discuss the entropy for viscous conservation laws:

$$\text{(VC)} \qquad w_t + \sum_{j=1}^{d} f^j(w)_{x_j} = \sum_{i,j=1}^{d} (G^{ij}(w)w_{x_j})_{xi},$$

which was introduced in [8]. We also discuss the global well posedness for (VC) under the stability condition formulated in [11].

Sect. 3 is the main part of this note and it is based on the recent joint work [10] with Wen-An Yong of the University of Heidelberg. We give a definition of the entropy for hyperbolic balance laws (HB). Our definition is different from the previous one given by Chen, Levermore and Liu [1] but is closely related to the one adopted by Yong [12]. We see that our definition of the entropy is suitable not only for 1) *global well posedness* but also for 2) *application of the Chapman-Enskog theory.* This definition is based on the observation of the Boltzmann H-function in discrete kinetic theory and gives a reasonable generalization of the H-function for a class of hyperbolic balance laws (HB) which includes the discrete Boltzmann equation. We also discuss the global well posedness for (HB) under the stability condition in [11].

Finally in Sect. 4, we apply the Chapman-Enskog theory to hyperbolic balance laws (HB) and derive the corresponding Navier-Stokes equation which is written in the form of (VC). We discuss some mathematical structure of this Navier-Stokes equation in connection with the original hyperbolic balance laws (HB).

1. Hyperbolic conservation laws

We briefly review on the entropy for hyperbolic conservation laws (HC).

Definition 1.1. ([3], [2]) A function $\eta(w)$ is called an *entropy* for hyperbolic conservation laws (HC) if the following two conditions are satisfied:
(i) $\eta(w)$ is strictly convex for any w.
(ii) $D_w f^j(w)(D_w^2\eta(w))^{-1}$ is symmetric for any w and $j = 1, \cdots, d$.

Let us consider a diffeomorphism $w = w(u)$ and rewrite (HC) as

$$\text{(HC)}' \qquad A^0(u)u_t + \sum_{j=1}^{d} A^j(u)u_{x_j} = 0,$$

where

$$A^0(u) := D_u w(u),$$

$$A^j(u) := D_u f^j(w(u)) = D_w f^j(w(u))D_u w(u), \quad j = 1, \cdots, d.$$

Definition 1.2. The system (HC)$'$ is called *symmetric* if the following two conditions are satisfied:
(i) $A^0(u)$ is real symmetric and positive definite for any w.
(ii) $A^j(u)$ is real symmetric for any w and $j = 1, \cdots, d$.

Theorem 1.1. ([3], [2]) *The system* (HC) *admits an entropy if and only if* (HC) *is symmetrizable by using a diffeomorphism.*

The outline of the proof of this theorem is as follows. Suppose that (HC) has an entropy $\eta(w)$. Then the desired symmerization is given by the diffeomorphism defined by

$$u = (D_w \eta(w))^T,$$

where the superscript T denotes the transposed. Conversely, we suppose that (HC) is symmerizable by using a diffeomorphism $w = w(u)$. Then there exist functions $\tilde{\eta}(u)$ and $\tilde{q}^j(u)$ such that

$$D_u\tilde{\eta}(u) = w(u)^T, \qquad D_u\tilde{q}^j(u) = f^j(w(u))^T, \quad j = 1, \cdots, d.$$

The desired entropy and the corresponding flux are then given by the formulas

$$\eta(w(u)) = < w(u), u > - \tilde{\eta}(u),$$

$$q^j(w(u)) = < f^j(w(u)),\, u > - \tilde{q}^j(u), \quad j = 1, \cdots, d,$$

where $<\ ,\ >$ denotes the standard inner product in $\mathbf{R}^N$. This completes the proof.

As a corollary of this theorem we can prove the local well posedness for hyperbolic conservation laws (HC) for initial data in $H^s(\mathbf{R}^d)$ with $s \geq [d/2]+2$.

2. Viscous conservation laws

The notion of the entropy was generalized in [8] for a class of viscous conservation laws (VC). Here we review the main results of [8].

Definition 2.1. ([8]) A function $\eta(w)$ is called an *entropy* for viscous conservation laws (VC) if the following four conditions are satisfied:

(i) and (ii) are the same as in Definition 1.1.

(iii) $\{G^{ij}(w)(D_w^2\eta(w))^{-1}\}^T = G^{ji}(w)(D_w^2\eta(w))^{-1}$ for any w and $i, j = 1, \cdots, d$.

(iv) $\sum_{ij} G^{ij}(w)(D_w^2\eta(w))^{-1}\omega_i\omega_j$ is real symmetric and nonnegative definite for any w and $\omega \in S^{d-1}$, where the sum is taken over all $i, j = 1, \cdots, d$.

Let $w = w(u)$ be a diffeomorphism. Then (VC) is rewritten as

$$\text{(VC)}' \qquad A^0(u)u_t + \sum_{j=1}^{d} A^j(u)u_{x_j} = \sum_{i,j=1}^{d} (B^{ij}(u)u_{x_j})_{x_i},$$

where $A^0(u)$ and $A^j(u)$ are the same as in (HC)$'$ and

$$B^{ij}(u) := G^{ij}(w(u))D_u w(u), \quad i, j = 1, \cdots, d.$$

Definition 2.2. ([8]) The system (VC)$'$ is called *symmetric* if the following four conditions are satisfied:

(i) and (ii) are the same as in Definition 1.2.

(iii) $B^{ij}(u)^T = B^{ji}(u)$ for any u and $i, j = 1, \cdots, d$.

(iv) The viscosity matrix $B(u, \omega) := \sum_{ij} B^{ij}(u)\omega_i\omega_j$ is real symmmetric and nonnegative definite for any u and $\omega \in S^{d-1}$, where the sum is taken over all $i, j = 1, \cdots, d$.

Theorem 2.1. ([8]) *The system* (VC) *admits an entropy if and only if* (VC) *is symmetrizable by using a diffeomorphism.*

The proof of this theorem is analogous to that of Theorem 1.1. Here we note that the entropy $\eta(w)$ for (VC) satisfies

$$\eta(w)_t + \sum_{j=1}^{d} q^j(w)_{x_j} = \sum_{i,j=1}^{d} (< u, B^{ij}(u)u_{x_j} >)_{x_i} - \sum_{i,j=1}^{d} < u_{x_i}, B^{ij}(u)u_{x_j} >,$$

where $q^j(w)$ is the corresponding entropy flux and $u = (D_w\eta(w))^T$.

The symmetization in Theorem 2.1 is not sufficient to show the local well posedness for (VC). But this symmetrization together with the following condition (#) formulated in [8] gives the local well posedness for initial data in $H^s(\mathbf{R}^d)$ with $s \geq [d/2] + 2$ (see [6], [8]):

(#) $\mathcal{N}(B(u, \omega))$ is independent of u and $\omega \in S^{d-1}$,

where $\mathcal{N}(B(u, \omega))$ denotes the null space of the viscosity matrix $B(u, \omega)$.

Furthermore, we can prove the global well posedness for viscous conservation laws (VC) under the following *stability condition* (∗) formulated in [11].

(∗) Let $\lambda A^0(u)z + A(u, \omega)z = 0$ and $B(u, \omega)z = 0$ for some $z \in \mathbf{R}^N$, $\lambda \in \mathbf{R}$, $\omega \in S^{d-1}$. Then $z = 0$.

Here $A(u, \omega) = \sum_j A^j(u)\omega_j$. In fact we have:

Theorem 2.2. ([6], [7]) *Suppose that the system* (VC) *admits an entropy and satisfies* (#) *and* (∗). *Then* (VC) *is globally well posed for initial data in a small* $H^s(\mathbf{R}^d)$*-neighborhood of a given constant state* $\bar{w}$*, where* $s \geq [d/2] + 2$.

3. Hyperbolic balance laws

Let us give a definition of the entropy for hyperbolic balance laws (HB). To this end, we introduce:

$$\mathcal{M} := \{\psi \in \mathbf{R}^N ; < \psi, g(w) > = 0 \text{ for any } w\}.$$

$\mathcal{M}$ is a subspace of $\mathbf{R}^N$. Obviously, we have $g(w) \in \mathcal{M}^\perp$ for any w. In discrete kinetic theory, $\mathcal{M}$ is called the space of collision invariants.

Definition 3.1. ([10]) A function $\eta(w)$ is called an *entropy* for hyperbolic balance laws (HB) if the following four conditions are satisfied:
(i) and (ii) are the same as in Definition 1.1.
(iii) $g(w) = 0$ holds if and only if $(D_w\eta(w))^T \in \mathcal{M}$.
(iv) Let w^* be such that $g(w^*) = 0$. Then the matrix $-D_w g(w)(D_w^2\eta(w))^{-1}$ evaluated at $w = w^*$ is real symmetric and nonnegative definite. Moreover, its null space coincides with $\mathcal{M}$.

We note that the Boltzmann H-function for the discrete Boltzmann equation satisfies all these conditions in Definition 3.1.

Let $w = w(u)$ be a diffeomorphism and we rewrite (HB) as

$$\text{(HB)}' \qquad A^0(u)u_t + \sum_{j=1}^{d} A^j(u)u_{x_j} = g(w(u)),$$

where $A^0(u)$ and $A^j(u)$ are the same as in (HC)$'$.

Definition 3.2. ([10]) The system (HB)$'$ is called *symmetric dissipative* if the following four conditions are satisfied:
(i) and (ii) are the same as in Definition 1.2.
(iii) $g(w(u)) = 0$ holds if and only if $u \in \mathcal{M}$.
(iv) For any $u^* \in \mathcal{M}$, the matrix $L(u) := -D_u g(w(u)) = -D_w g(w(u)) D_u w(u)$ evaluated at $u = u^*$ is real symmmetric and nonnegative definite. Moreover, the null space $\mathcal{N}(L(u^*))$ coincides with $\mathcal{M}$.

In discrete kinetic theory the matrix $L(u^*)$ is called the linearized collision operator.

Theorem 3.1. ([10]) *The system* (HB) *admits an entropy if and only if* (HB) *is put into a symmetric dissipative system by using a diffeomorphism.*

The proof of this theorem is analogous to that of Theorem 1.1. Here we note that the entropy $\eta(w)$ for (HB) satisfies

$$\eta(w)_t + \sum_{j=1}^{d} q^j(w)_{x_j} = < u, \, g(w(u)) >,$$

where $u = (D_w \eta(w))^T$.

To develop the global existence theory for (HB), we need to examine the term $g(w(u))$ carefully. Let $\bar{u} \in \mathcal{M}$. We write $g(w(u))$ in the form

$$g(w(u)) = -L(\bar{u})u + r(u).$$

Claim 3.2. *Suppose that (iii) and (iv) of Definition 3.2 hold true. Let $\bar{u} \in \mathcal{M}$. Then we have $r(u) \in \mathcal{M}^{\perp}$ for any u. Moreover, there are positive constants δ and C such that*

$$|r(u)| \leq C|u - \bar{u}||(I - P)u|$$

for any u with $|u - \bar{u}| \leq \delta$, where P is the orthogonal projection onto $\mathcal{M}$.

An important consequence of Claim 3.2 is the following qualitative estimate for the entropy production term: There are constants δ, $c > 0$ such that

$$< u, \, g(w(u)) > \leq -c|(I - P)u|^2$$

for any u with $|u - \bar{u}| \le \delta$.

By virtue of Claim 3.2, we can prove the global well posedness for hyperbolic balance laws (HB) under the following stability condition (**) formulated in [11]. Let $\bar{u} \in \mathcal{M}$.

(**) Let $\lambda A^0(\bar{u})\varphi + A(\bar{u}, \omega)\varphi = 0$ and $L(\bar{u})\varphi = 0$ (i.e., $\varphi \in \mathcal{M}$) for some $\varphi \in \mathbf{R}^N$, $\lambda \in \mathbf{R}$, $\omega \in S^{d-1}$. Then $\varphi = 0$.

Our global existence theorem for (HB) is a modified version of the one obtained by Yong [12] and is regarded as a generalization of the global existence result in [5], [11] for the discrete Boltzmann equation.

Theorem 3.3. *Suppose that the system* (HB) *admits an entropy and satisfies* (**) *at a constant state* $\bar{u} \in \mathcal{M}$. *Then* (HB) *is globally well posed for initial data in a small* $H^s(\mathbf{R}^d)$*-neighborhood of* $\bar{w} = w(\bar{u})$, *where* $s \ge [d/2] + 2$.

We remark that a similar global existence result has been obtained by Hanouzet and Natalini [4] in one space dimension ($d = 1$).

4. The Chapman-Enskog expansion

The Chapman-Enskog theory was developed in [1] for hyperbolic balance laws. Here we follow the traditional approach (see [9]) and derive the Navier-Stokes equation corresponding to the hyperbolic balance laws

$$\text{[HB]} \qquad W_t + \sum_{j=1}^{d} F^j(W)_{x_j} = G(W),$$

where W is an N-vector; capital letters are used to describe the hyperbolic balance laws in this section.

Let $\mathcal{M}$ be the subspace defined by $G(W)$:

$$\mathcal{M} := \{\psi \in \mathbf{R}^N ; < \psi, G(W) > = 0 \text{ for any } W\}.$$

We assume that $\dim \mathcal{M} = n$ and write $\mathcal{M} = \mathrm{span}\{\psi^{(1)}, \cdots, \psi^{(n)}\}$, where $\{\psi^{(1)}, \cdots, \psi^{(n)}\}$ is a basis of $\mathcal{M}$. Let us introduce the moment vector w in the usual way:

$$w = (w_1, \cdots, w_n)^T, \qquad w_k = < \psi^{(k)}, W >, \quad k = 1, \cdots, n.$$

If we use the $N \times n$ matrix $\Psi := (\psi^{(1)}, \cdots, \psi^{(n)})$, we can write

$$w = \Psi^T W.$$

We assume that the hyperbolic balance law [HB] has an entropy $H(W)$ in the sense of Definition 3.1. Then we can apply the traditional Chapman-Enskog expansion (see [9]) to [HB] and obtain the corresponding Navier-Stokes equation in the form of the viscous conservation laws:

$$\text{[VC]} \qquad w_t + \sum_{j=1}^{d} f^j(w)_{x_j} = \sum_{i,j=1}^{d} (g^{ij}(w)w_{x_j})_{x_i},$$

where w is the moment vector; small letters are used to describe our Navier-Stokes equation.

The symmetric form associated with [HB] is written as

$$\text{[HB]}' \qquad A^0(U)U_t + \sum_{j=1}^{d} A^j(U)U_{x_j} = G(W(U)),$$

where $U = (D_W H(W))^T$, and this defines a diffeomorphism $W = W(U)$. We see that $G(W(U)) = 0$ holds if and only if $U \in \mathcal{M}$. Such a vector $U = U^*$ is characterized in term of an n-vector $u = (u, \cdots, u_n)^T$ as

$$U^* = \sum_{k=1}^{n} u_k \psi^{(k)} = \Psi u.$$

Furthermore we see that $w \to u$ is a diffeomorphism and our Navier-Stokes equation [VC] can be symmetrizable by using this diffeomorphism as

$$\text{[VC]}' \qquad a^0(u)u_t + \sum_{j=1}^{d} a^j(u)u_{x_j} = \sum_{i,j=1}^{d} (b^{ij}(u)u_{x_j})_{x_i}.$$

Here the coefficient matrices are given explicitly in terms of the coefficient matrices in [HB]'. In particular,

$$a^0(u) = \Psi^T A^0(\Psi u)\Psi,$$

$$a^j(u) = \Psi^T A^j(\Psi u)\Psi, \quad j = 1, \cdots, d.$$

Also, the null space of the viscosity matrix $b(u,\omega) = \sum_{ij} b^{ij}(u)\omega_i\omega_j$ is given as

$$\mathcal{N}(b(u,\omega)) = \{z \in \mathbf{R}^n;\ A^0(\Psi u)^{-1} A(\Psi u, \omega)\Psi z \in \mathcal{M}\ \}.$$

This null space depends, in general, upon u and $\omega \in S^{d-1}$ and therefore we must impose the condition (#) in Sect. 2 in order to ensure the local well posedness of the Navier-Stokes equation [VC].

Our Navier-Stokes equation [VC] is symmetrizable so that it has an entropy by Theorem 2.1. This entropy $\eta(w)$ is given explicitly in terms of the entropy $H(W)$ for [HB]. In fact we have:

Theorem 4.1. ([10]) *The entropies for* [HB] *and* [VC] *are related as*

$$\eta(w(u)) = H(W(\Psi u)), \qquad q^j(w(u)) = Q^j(W(\Psi u)), \quad j = 1, \cdots, d,$$

where $Q^j(W)$ *and* $q^j(w)$ *are the corresponding entropy fluxes for* [HB] *and* [VC], *respectively.*

This is a refinement of the similar result obtained in [1]. This relationship between entropies is known in discrete kinetic theory (see [9]).

The stability conditions for [HB] and [VC] are formulated as

[**] Let $\lambda A^0(U)\varphi + A(U,\omega)\varphi = 0$ and $\varphi \in \mathcal{M}$ for some $\lambda \in \mathbf{R}$, $\omega \in S^{d-1}$. Then $\varphi = 0$.

[*] Let $\lambda a^0(u)z + a(u,\omega)z = 0$ and $b(u,\omega)z = 0$ for some $z \in \mathbf{R}^n$, $\lambda \in \mathbf{R}$, $\omega \in S^{d-1}$. Then $z = 0$.

As in the discrete kinetic theory, these two stability conditions are equivalent to each other (see [9]).

Theorem 4.2. ([10]) *The hyperbolic balance law* [HB] *satisfies the stability condition* [**] *at* $U = \Psi u$ *if and only if the corresponding Navier-Stokes equation* [VC] *satisfies the stability condition* [*].

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