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Incompressible ideal fluid motion with free boundary far from equilibrium

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1. Introduction

We study the motion of an incompressible ideal fluid with free boundary. The fluid occupies a semi-infinite domain $\Omega(t), t>0$, in the two-dimensional space:

$$\Omega(t)=\{z=(z_1,z_2);\ -h+b(z_1)<z_2<\eta(t,z_1),\ z_1\in\mathbf{R}^1\},\quad h>0.$$

Here the domain is bounded by the bottom Γ_b and the free surface $\Gamma_s(t)$:

$$\Gamma_b=\{z=(z_1,z_2);\ z_2=-h+b(z_1),\ z_1\in\mathbf{R}^1\},$$

$$\Gamma_s(t)=\{z=(z_1,z_2);\ z_2=\eta(t,z_1),\ z_1\in\mathbf{R}^1\}.$$

We consider the free boundary problem

$$\rho\left(\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla_z)\mathbf{v}\right)+\nabla_z p=-\rho(0,g)\quad \text{in } \Omega(t),\ t>0, \tag{1.1}$$

$$\nabla_z\cdot\mathbf{v}=0\quad \text{in } \Omega(t),\ t>0, \tag{1.2}$$

$$p=p_e\quad \text{on } \Gamma_s(t),\ t>0, \tag{1.3}$$

$$\frac{\partial\eta}{\partial t}+v_1\frac{\partial\eta}{\partial z_1}-v_2=0\quad \text{on } \Gamma_s(t),\ t>0, \tag{1.4}$$

$$\mathbf{v}\cdot\mathbf{n}=0\quad \text{on } \Gamma_b,\ t>0, \tag{1.5}$$

$$\eta(0,z_1)=\eta_0(z_1),\ \mathbf{v}(0,z)=\mathbf{v}_0(z)\quad \text{on } \Omega\equiv\Omega(0), \tag{1.6}$$

where ρ is density (constant), $\mathbf{v}=(v_1,v_2)$ is the velocity, p is the pressure, g is a gravitational positive constant, p_e is an atmospheric pressure (constant) and $\mathbf{n}$ is the unit outer normal to Γ_b.

In this paper, the unique solvability of problem (1.1) – (1.6) will be shown. For this purpose, put

$$P=\frac{p-p_e}{\rho}+gz_2$$

and transform problem (1.1) – (1.6) by the Lagrangian coordinates (t,x),

$$z=x+\int_0^t\mathbf{u}(\tau,x)d\tau\equiv\Phi_{\mathbf{u}}(x;t),\quad \mathbf{u}(t,x)=\mathbf{v}(t,\Phi_{\mathbf{u}}(x;t)).$$

Then we obtain the fixed boundary problem

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla_{\mathbf{u}} q = 0 \qquad \text{in } \Omega,\ t > 0, \tag{1.7}$$

$$\nabla_{\mathbf{u}} \cdot \mathbf{u} = 0 \qquad \text{in } \Omega,\ t > 0, \tag{1.8}$$

$$q = g\left(x_2 + \int_0^t u_2(\tau, x)\mathrm{d}\tau\right) \qquad \text{on } \Gamma_s \equiv \Gamma_s(0),\ t > 0, \tag{1.9}$$

$$\mathbf{u} \cdot \mathbf{n}(\Phi_{\mathbf{u}}(x;t)) = 0 \qquad \text{on } \Gamma_b,\ t > 0, \tag{1.10}$$

$$\mathbf{u}|_{t=0} = \mathbf{v}_0 \qquad \text{on } \Omega, \tag{1.11}$$

where $q(t,x) = P(t, \Phi_{\mathbf{u}}(x;t)), \nabla_{\mathbf{u}} = A_{\mathbf{u}} \nabla_x$ and $A_{\mathbf{u}} = {}^t(\partial \Phi_{\mathbf{u}}/\partial x)^{-1}$.

Since it holds that

$$\mathbf{v}(t,z) = \mathbf{u}(t, \Phi_{\mathbf{u}}^{-1}(z;t)), \quad P(t,z) = q(t, \Phi_{\mathbf{u}}^{-1}(z;t)), \quad \Omega(t) = \Phi_{\mathbf{u}}(\Omega;t),$$

we will construct the solution of problem (1.7) - (1.11).

Several papers addressed the well-posedness for the problem of water waves. In [6], [12] and [13], the unique existence of solution to this problem was shown under the assumption that the boundaries of the domain were almost flat and the initial velocity was sufficiently small. Recently, in [10], [11], Wu removed these restrictions for the problem in case of infinite depth. Moreover, the problem of capillary-gravity waves with a bottom and the large initial data was treated by Iguchi [4].

On the other hand, the well-posedness of the problem describing the dynamics of vortical surface waves was shown in [5], [7], [8], [9]. However, the assumptions for the boundaries and the initial velocity as above are necessary to prove the well-posedness in these articles. Then we address the well-posedness for the free boundary problem when the flow is rotational and the initial surface and the bottom are uneven.

Here we state our main result.

Theorem. *Let $s \geq 4$. There exists a positive constant δ such that if*

$$\begin{cases} \eta_0 \in H^{s+2}(\mathbf{R}^1), \quad b \in H^{s+3}(\mathbf{R}^1), \quad \mathbf{v}_0 \in H^{s+3/2}(\Omega), \\ \inf\{\eta_0(x_1) - (-h + b(x_1))\} > 0, \\ \|\mathbf{v}_0\|_{H^{2+1/2}(\Omega)} + \|\omega_0\|_{H^{2+1/2}(\Omega)} \leq \delta, \end{cases}$$

where $\omega_0 = \nabla_x^\perp \cdot \mathbf{v}_0, \nabla_x^\perp = (-\partial/\partial x_2, \partial/\partial x_1)$, and $\mathbf{v}_0$ satisfies the compatibility conditions, then problem (1.7) - (1.11) *has a unique solution* $(\mathbf{u}, q)$ *on some time interval* $[0,T]$ *satisfying*

$$\begin{cases} \mathbf{u} \in C^j([0,T]; H^{s+3/2-j/2}(\Omega)), \quad j = 0,1,2,3, \\ q \in C^j([0,T]; H^{s+2-j/2}(\Omega)), \quad j = 1,2. \end{cases}$$

Now we explain the outline of the proof. At first, we introduce the function X by

$$X(t,x) = \int_0^t \mathbf{u}(\tau,x)\mathrm{d}\tau, \qquad x \in \Omega, \tag{1.12}$$

and denote the restrictions of X to the boundaries by

$$\begin{cases} \bar{X}(t,x_1) = X(t,x_1,\eta_0(x_1)), \\ \check{X}(t,x_1) = X(t,x_1,-h+b(x_1)). \end{cases} \tag{1.13}$$

Then it follows from $(1.1),(1.3)$ that

$$\left(1+\frac{\partial \bar{X}_1}{\partial x_1}\right)\frac{\partial^2 \bar{X}_1}{\partial t^2} + \left(\frac{\mathrm{d}\eta_0}{\mathrm{d}x_1}+\frac{\partial \bar{X}_2}{\partial x_1}\right)\left(g+\frac{\partial^2 \bar{X}_2}{\partial t^2}\right) = 0 \qquad \text{for } t \geq 0. \tag{1.14}$$

On the other hand, for the vorticity $\nabla^{\perp}\cdot \mathbf{v} = \omega$, the Helmholtz theorem implies that

$$\nabla_{\mathbf{u}}^{\perp}\cdot \mathbf{u} = \omega_0 \qquad \text{in } \Omega, \quad t \geq 0. \tag{1.15}$$

Hence, by $(1.8),(1.15)$, we see that

$$\bar{X}_{2t} = K\bar{X}_{1t} + H \qquad \text{for } t \geq 0 \tag{1.16}$$

with an operator $K = K(\bar{X})$ and a function $H = H(X,\check{X},\omega_0)$.

If the functions X and $\check{X}$ are given, we obtain H. Then assuming that an H is given, we solve the Cauchy problem $(1.14),(1.16)$ for $\bar{X}$ with the initial conditions determined by $(1.12),(1.13)_1$. Next, for a given $\bar{X}$, we find $\mathbf{u}$ by solving the boundary value problem

$$\begin{cases} \nabla_{\mathbf{u}}\cdot \mathbf{u} = 0, \quad \nabla_{\mathbf{u}}^{\perp}\cdot \mathbf{u} = \omega_0 & \text{in } \Omega,\ t \geq 0, \\ u_1 = \bar{X}_{1t} & \text{on } \Gamma_s,\ t \geq 0, \\ \mathbf{u}\cdot \mathbf{n}(\Phi_{\mathbf{u}}(x;t)) = 0 & \text{on } \Gamma_b,\ t \geq 0. \end{cases}$$

Moreover, for a given $\mathbf{u}$, the functions X and $\check{X}$ are determined through (1.12) and $(1.13)_2$, respectively. By repeating this procedure, the iteration method gives the solution $(\bar{X},\mathbf{u},X,\check{X})$.

In order to obtain q, we solve the boundary value problem

$$\begin{cases} \Delta q = -\nabla\cdot(A_{\mathbf{u}}^{-1}\mathbf{u}_t) & \text{in } \Omega,\ t \geq 0, \\ q = g\left(x_2 + \int_0^t u_2(\tau,x)\mathrm{d}\tau\right) & \text{on } \Gamma_s,\ t \geq 0, \\ \dfrac{\partial q}{\partial \mathbf{n}(\Phi_{\mathbf{u}})} = -(\mathbf{u}\cdot\nabla_{\mathbf{u}})\mathbf{u}\cdot\mathbf{n}(\Phi_{\mathbf{u}}) & \text{on } \Gamma_b,\ t \geq 0. \end{cases}$$

Then the proof is complete.

In Section 3, we will give the explicit form of K and H. In Section 4, the properties of K are investigated. Even if the free surface is uneven, we can obtain the same estimates for K as those in the previous articles. Moreover we will see that the initial value problem for (1.14), (1.16) is well-posed.

The details of the proof for the main theorem will appear elsewhere.

2. Notations

Let j be a nonnegative integer, $0 < T < \infty$ and B a Banach space. We say that $u \in C^j([0,T];B)$ if u is a j-times continuously differentiable function on $[0,T]$ with values in B. By $H^s(D)$, $s \in \mathbf{R}^1$, $D \subset \mathbf{R}^n$, we denote the Sobolev space. Moreover the adjoint operator of A is denoted by A^*.

Let η_0 be the Lipschitz continuous function. We introduce the non-tangential cones $C^{\pm}(P)$, $P = (y_1, \eta_0(y_1)) \in \Gamma_s$,

$$\begin{cases} C^+(P) = \{(x_1,x_2) \in \mathbf{R}^2; x_2 - \eta_0(y_1) > M|x_1 - y_1|\}, \\ C^-(P) = \{(x_1,x_2) \in \mathbf{R}^2; x_2 - \eta_0(y_1) < -M|x_1 - y_1|\}, \end{cases}$$

where $\|\eta_0'\|_{L^\infty(\mathbf{R}^1)} < M$. Then for a function ν on $\mathbf{R}^2 \setminus \Gamma_s$, the maximal functions and the non-tangential limits of ν are given by

$$\nu_*^{\pm}(P) = \sup_{X \in C^{\pm}(P)} |\nu(X)| \quad \text{for } P \in \Gamma_s,$$

$$\nu^{\pm}(P) = \lim_{X \to P, X \in C^{\pm}(P)} \nu(X) \quad \text{for } P \in \Gamma_s,$$

respectively.

Further we use integral operators $\mathcal{L}_i(u)$, $L_i(u)$, $i = 1,2$ and $\mathcal{M}(u) = (\mathcal{M}_1(u), \mathcal{M}_2(u))$, defined by

$$\begin{cases} \mathcal{L}_1(u)(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \frac{\eta_0(y_1) - x_2 - \eta_0'(y_1)(y_1 - x_1)}{(y_1 - x_1)^2 + (\eta_0(y_1) - x_2)^2} u(y_1) \mathrm{d}y_1, \\ \mathcal{L}_2(u)(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \frac{y_1 - x_1 + \eta_0'(y_1)(\eta_0(y_1) - x_2)}{(y_1 - x_1)^2 + (\eta_0(y_1) - x_2)^2} u(y_1) \mathrm{d}y_1, \quad x \in \mathbf{R}^2 \setminus \Gamma_s, \end{cases}$$

$$\begin{cases} L_1(u)(x_1) = \dfrac{1}{2\pi}\text{v.p.}\displaystyle\int_{-\infty}^{\infty} \frac{\eta_0(y_1) - \eta_0(x_1) - \eta_0'(y_1)(y_1 - x_1)}{(y_1 - x_1)^2 + (\eta_0(y_1) - \eta_0(x_1))^2} u(y_1) \mathrm{d}y_1, \\ L_2(u)(x_1) = \dfrac{1}{2\pi}\text{v.p.}\displaystyle\int_{-\infty}^{\infty} \frac{y_1 - x_1 + \eta_0'(y_1)(\eta_0(y_1) - \eta_0(x_1))}{(y_1 - x_1)^2 + (\eta_0(y_1) - \eta_0(x_1))^2} u(y_1) \mathrm{d}y_1, \quad x_1 \in \mathbf{R}^1, \end{cases}$$

$$\begin{cases} \mathcal{M}_1(u)(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \dfrac{y_1 - x_1}{(y_1 - x_1)^2 + (\eta_0(y_1) - x_2)^2} u(y_1)\mathrm{d}y_1, \\ \mathcal{M}_2(u)(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \dfrac{\eta_0(y_1) - x_2}{(y_1 - x_1)^2 + (\eta_0(y_1) - x_2)^2} u(y_1)\mathrm{d}y_1, \quad x \in \mathbf{R}^2 \setminus \Gamma_s. \end{cases}$$

3. Representation of K and H

Throughout this section, let the time $t \geq 0$ be arbitrarily fixed. We regard the plane $\mathbf{R}^2_{z_1,z_2}$ as the complex space of $z = z_1 + iz_2$. Then $\Gamma_s(t)$ and Γ_b are given by

$$\begin{cases} \Gamma_s(t) : w_s(x_1) = x_1 + \bar{X}_1(x_1) + i(\eta_0(x_1) + \bar{X}_2(x_1)), \\ \Gamma_b \quad : w_b(x_1) = x_1 + i(-h + b(x_1)), \qquad -\infty < x_1 < \infty. \end{cases}$$

Moreover, we regard the function $\mathbf{v}$ as the complex function and put

$$\begin{cases} F = v_1 - iv_2, \\ f(x_1) = F(w_s(x_1)), \\ g(x_1) = F(w_b(x_1)). \end{cases}$$

Since

$$\nabla \cdot \mathbf{v} = 0, \quad \nabla^\perp \cdot \mathbf{v} = \omega \quad \text{in } \Omega(t),$$

Cauchy integral formula implies that

$$\begin{aligned} F(z^0) = &-\frac{1}{2\pi i}\int_{\Gamma_s(t)} \frac{f(y_1)}{w_s(y_1) - z^0}\frac{\mathrm{d}w_s(y_1)}{\mathrm{d}y_1}\mathrm{d}y_1 + \frac{1}{2\pi i}\int_{\Gamma_b} \frac{g(y_1)}{w_b(y_1) - z^0}\frac{\mathrm{d}w_b(y_1)}{\mathrm{d}y_1}\mathrm{d}y_1 \\ &+ i\iint_{\Omega(t)} \omega\frac{\partial E(z - z^0)}{\partial z_1}\mathrm{d}z_1\mathrm{d}z_2 - \iint_{\Omega(t)} \omega\frac{\partial E(z - z^0)}{\partial z_2}\mathrm{d}z_1\mathrm{d}z_2. \end{aligned} \tag{3.1}$$

Here $z^0 \in \Omega(t)$ and E is the fundamental solution for Laplace's equation in two-dimensional space:

$$E(z) = \frac{1}{2\pi}\log|z|.$$

We notice that $\mathrm{Re} f = v_1|_{\Gamma_s(t)}$, $\mathrm{Im} f = -v_2|_{\Gamma_s(t)}$, $\mathrm{Re} g = v_1|_{\Gamma_b}$ and $\mathrm{Im} g = -v_2|_{\Gamma_b}$. Therefore, by taking z^0 to $w_s^0 = w_s(x_1)$ on $\Gamma_s(t)$ non-tangentially, the imaginary part of (3.1) leads to the relation $\bar{X}_{2t} = K\bar{X}_{1t} + H$ with

$$K = -\left(\frac{1}{2} - A_1\right)^{-1} A_2,$$

$$H = -\left(\frac{1}{2} - A_1\right)^{-1} (-B_2\check{X}_{1t} + B_1\check{X}_{2t} + H_1),$$

where

$$\begin{cases} A_1 u(x_1) \\ = \dfrac{1}{2\pi}\text{v.p.}\displaystyle\int_{-\infty}^{\infty} \Big\{(1+\bar{X}_1'(y_1))(\eta_0(y_1)+\bar{X}_2(y_1)-\eta_0(x_1)-\bar{X}_2(x_1)) \\ \qquad\qquad -(\eta_0'(y_1)+\bar{X}_2'(y_1))(y_1+\bar{X}_1(y_1)-x_1-\bar{X}_1(x_1))\Big\} \\ \times\Big\{(y_1+\bar{X}_1(y_1)-x_1-\bar{X}_1(x_1))^2+(\eta_0(y_1)+\bar{X}_2(y_1)-\eta_0(x_1)-\bar{X}_2(x_1))^2\Big\}^{-1} u(y_1)\mathrm{d}y_1, \\ A_2 u(x_1) \\ = \dfrac{1}{2\pi}\text{v.p.}\displaystyle\int_{-\infty}^{\infty} \Big\{(1+\bar{X}_1'(y_1))(y_1+\bar{X}_1(y_1)-x_1-\bar{X}_1(x_1)) \\ \qquad\qquad +(\eta_0'(y_1)+\bar{X}_2'(y_1))(\eta_0(y_1)+\bar{X}_2(y_1)-\eta_0(x_1)-\bar{X}_2(x_1))\Big\} \\ \times\Big\{(y_1+\bar{X}_1(y_1)-x_1-\bar{X}_1(x_1))^2+(\eta_0(y_1)+\bar{X}_2(y_1)-\eta_0(x_1)-\bar{X}_2(x_1))^2\Big\}^{-1} u(y_1)\mathrm{d}y_1, \\ B_1 u(x_1) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \frac{-h+b(y_1)-\eta_0(x_1)-\bar{X}_2(x_1)-b'(y_1-x_1-\bar{X}_1(x_1))}{(y_1-x_1-\bar{X}_1(x_1))^2+(-h+b(y_1)-\eta_0(x_1)-\bar{X}_2(x_1))^2} u(y_1)\mathrm{d}y_1, \\ B_2 u(x_1) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \frac{y_1-x_1-\bar{X}_1(x_1)+b'(-h+b(y_1)-\eta_0(x_1)-\bar{X}_2(x_1))}{(y_1-x_1-\bar{X}_1(x_1))^2+(-h+b(y_1)-\eta_0(x_1)-\bar{X}_2(x_1))^2} u(y_1)\mathrm{d}y_1, \\ H_1 = \displaystyle\iint_{\Omega(t)} \omega(z)\frac{\partial E(z-w_s^0)}{\partial z_1}\mathrm{d}z_1\mathrm{d}z_2. \end{cases}$$

We can divide the operators A_1 and A_2 as follows:

$$\begin{cases} A_1 = B_3 + B_5, \\ A_2 = B_4 - B_6, \end{cases}$$

where

$$\begin{cases} B_3u(x_1) = L_1(u)(x_1), \\ B_4u(x_1) = L_2(u)(x_1), \\ B_5u(x_1) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \operatorname{Im}\log\{1+ \\ \quad +\Big\{(y_1-x_1)(\bar{X}_1(y_1)-\bar{X}_1(x_1)) + (\eta_0(y_1)-\eta_0(x_1))(\bar{X}_2(y_1)-\bar{X}_2(x_1)) \\ \qquad -i\{(\eta_0(y_1)-\eta_0(x_1))(\bar{X}_1(y_1)-\bar{X}_1(x_1)) - (y_1-x_1)(\bar{X}_2(y_1)-\bar{X}_2(x_1))\}\Big\} \\ \quad \times\Big\{(y_1-x_1)^2 + (\eta_0(y_1)-\eta_0(x_1))^2\Big\}^{-1}\}u'(y_1)\mathrm{d}y_1, \\ B_6u(x_1) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \operatorname{Re}\log\{1+ \\ \quad +\Big\{(y_1-x_1)(\bar{X}_1(y_1)-\bar{X}_1(x_1)) + (\eta_0(y_1)-\eta_0(x_1))(\bar{X}_2(y_1)-\bar{X}_2(x_1)) \\ \qquad -i\{(\eta_0(y_1)-\eta_0(x_1))(\bar{X}_1(y_1)-\bar{X}_1(x_1)) - (y_1-x_1)(\bar{X}_2(y_1)-\bar{X}_2(x_1))\}\Big\} \\ \quad \times\Big\{(y_1-x_1)^2 + (\eta_0(y_1)-\eta_0(x_1))^2\Big\}^{-1}\}u'(y_1)\mathrm{d}y_1. \end{cases}$$

Therefore the operator K has the form

$$\begin{aligned} K &= -\left(\frac{1}{2} - B_3 - B_5\right)^{-1}(B_4 - B_6) \\ &= -\left(\frac{1}{2} - B_3 - B_5\right)^{-1}\left(\frac{1}{2}i\operatorname{sgnD} - B_7 - B_6\right) \\ &= -i\operatorname{sgnD} + 2(-B_7 - B_6) \\ &\quad + 2(-B_3 + B_5)\left(\frac{1}{2} - B_3 + B_5\right)^{-1}\left(\frac{1}{2}i\operatorname{sgnD} + B_7 + B_6\right) \\ &=: -i\operatorname{sgnD} + K_1, \end{aligned} \tag{3.2}$$

where

$$\mathrm{D} = -i\partial/\partial x_1, \quad B_7u(x_1) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \log\{1 + \left(\frac{\eta_0(y_1)-\eta_0(x_1)}{y_1 - x_1}\right)^2\}^{1/2}u'(y_1)\mathrm{d}y_1.$$

4. Problem on the surface

By [2], [12], we can show

Lemma 4.1. *Suppose that* $\inf\{\eta_0(x_1) - (-h + b(x_1))\} > 0$.

(1) *Let* $\eta_0, \bar{X}, \bar{X}^0 \in H^s(\mathbf{R}^1)$, $s \geq 0$, *b the Lipschitz continuous function and* $\|\bar{X}\|_{H^s(\mathbf{R}^1)}$, $\|\bar{X}^0\|_{H^s(\mathbf{R}^1)} \leq d$ *for some* $d > 0$. *It holds that*

$$\begin{cases} \|B_j(\bar{X})u\|_{H^s(\mathbf{R}^1)} \leq C\|u\|_{H^0(\mathbf{R}^1)}, \\ \|B_j(\bar{X})u - A_j(\bar{X}^0)u\|_{H^s(\mathbf{R}^1)} \leq C\|\bar{X} - \bar{X}^0\|_{H^s(\mathbf{R}^1)}\|u\|_{H^0(\mathbf{R}^1)}, \quad j = 1, 2, \end{cases}$$

where $C = C(s, d, \|\eta_0\|_{H^s(\mathbf{R}^1)}, \|b'\|_{L^\infty(\mathbf{R}^1)}) > 0$.

(2) *Let* $\eta_0 \in H^s(\mathbf{R}^1)$, $s, s_0 > 3/2$. *It holds that*

$$\|B_j u\|_{H^s(\mathbf{R}^1)} \leq C\|u\|_{H^{s_0}(\mathbf{R}^1)}, \quad j = 3, 7, \ C = C(s, s_0, \|\eta_0\|_{H^s(\mathbf{R}^1)}) > 0.$$

(3) *Let* η_0 *be the Lipschitz continuous function and* $\eta_0 \in H^{s+3/2}(\mathbf{R}^1)$, $s \geq 0$. *It holds that*

$$\|B_j u\|_{H^s(\mathbf{R}^1)} \leq C\|u\|_{H^0(\mathbf{R}^1)}, \quad j = 3, 7, \ C = C(s, \|\eta_0\|_{H^{s+3/2}(\mathbf{R}^1)}, \|\eta_0'\|_{L^\infty(\mathbf{R}^1)}) > 0.$$

(4) *There exists a positive constant c such that if* $\eta_0'' \in L^\infty(\mathbf{R}^1)$, $\eta_0, \bar{X}, \bar{X}^0 \in H^s(\mathbf{R}^1)$, $s \geq 2$ *and* $\|\bar{X}\|_{H^2(\mathbf{R}^1)}, \|\bar{X}^0\|_{H^2(\mathbf{R}^1)} \leq c$, $\|\bar{X}\|_{H^s(\mathbf{R}^1)}, \|\bar{X}^0\|_{H^s(\mathbf{R}^1)} \leq d$ *for some* $d > 0$, *then it holds that*

$$\begin{cases} \|B_j(\bar{X})u\|_{H^s(\mathbf{R}^1)} \leq C\|\bar{X}\|_{H^s(\mathbf{R}^1)}\|u\|_{H^{s_0}(\mathbf{R}^1)}, \\ \|B_j(\bar{X})u - B_j(\bar{X}^0)u\|_{H^s(\mathbf{R}^1)} \leq C\|\bar{X} - \bar{X}^0\|_{H^s(\mathbf{R}^1)}\|u\|_{H^{s_0}(\mathbf{R}^1)}, j = 5, 6, \ s_0 > 3/2, \end{cases}$$

where $C = C(s, s_0, c, d, \|\eta_0\|_{H^s(\mathbf{R}^1)}, \|\eta_0''\|_{L^\infty(\mathbf{R}^1)}) > 0$.

In order to show the invertibility of the operator $\frac{1}{2} - B_3 - B_5$, the following proposition is useful.

Proposition 4.1. *Suppose that A is a bounded linear operator in* $L^2(\mathbf{R}^1)$ *and satisfies*

$$\|Au\|_{L^2(\mathbf{R}^1)} \geq C\|u\|_{L^2(\mathbf{R}^1)}, \quad \|A^*u\|_{L^2(\mathbf{R}^1)} \geq C\|u\|_{L^2(\mathbf{R}^1)} \tag{4.1}$$

for any $u \in L^2(\mathbf{R}^1)$, *where* $C > 0$. *Then the operator A is invertible in* $L^2(\mathbf{R}^1)$.

By [1] and [3], we have

Lemma 4.2.

(1) $L_1(u)(x_1)$, $L_2(u)(x_1)$ *exist for almost every* $x_1 \in \mathbf{R}^1$ *and*

$$\|L_i(u)\|_{L^2(\mathbf{R}^1)} \leq C\|u\|_{L^2(\mathbf{R}^1)}, \qquad i = 1, 2,$$

where $C = C(\|\eta_0'\|_{L^\infty(\mathbf{R}^1)}) > 0$.

(2) *The maximal functions* $(\mathcal{L}_i(u))^{\pm}_*, i = 1, 2$, *satisfy*

$$\|(\mathcal{L}_i(u))^{\pm}_*\|_{L^2(\mathbf{R}^1)} \leq C\|u\|_{L^2(\mathbf{R}^1)}, \qquad i = 1, 2,$$

where $C = C(\|\eta_0'\|_{L^\infty(\mathbf{R}^1)}) > 0$. *Moreover, the non-tangential limits* $(\mathcal{L}_i(u))^{\pm}(x_1)$, $i = 1, 2$, *exist for almost every* $x_1 \in \mathbf{R}^1$ *and*

$$\begin{cases} (\mathcal{L}_1(u))^{\pm}(x_1) = \mp\frac{1}{2}u(x_1) + L_1(u)(x_1), \\ (\mathcal{L}_2(u))^{\pm}(x_1) = L_2(u)(x_1) \quad \text{for a.e. } x_1 \in \mathbf{R}^1. \end{cases}$$

Moreover the divergence theorem yields

Lemma 4.3. *Let* η_0 *be the Lipschitz continuous function. Suppose that*

(1) $\mathbf{v} = (\mathrm{v}_1, \mathrm{v}_2)$ *satisfies* $\nabla \cdot \mathbf{v} = 0$ *and* $\nabla^\perp \cdot \mathbf{v} = 0$ *in* $\mathbf{R}^2 \setminus \Gamma_s$,

(2) *The maximal functions* $\mathbf{v}^{\pm}_* = \sup_{X \in C^{\pm}(P)} |\mathbf{v}(X)|, P \in \Gamma_s$, *belong to* $L^2(\mathbf{R}^1)$,

(3) *The non-tangential limits* $\mathbf{V}^{\pm} = (\mathrm{V}_1^{\pm}, \mathrm{V}_2^{\pm}) = \lim_{X \to P, X \in C^{\pm}(P)} \mathbf{v}(X), P \in \Gamma_s$, *exist for almost every* P,

(4) $\mathbf{v}(x) = O(|x|^{-1}) \quad$ *as* $|x| \to \infty$.

If we denote the normal vector and the tangential vector to Γ_s *by* $\mathbf{N} = (N_1, N_2)$, $\mathbf{T} = (N_2, -N_1)$, *respectively, then the norms* $\|\mathrm{V}_1\|_{L^2(\mathbf{R}^1)}, \|\mathrm{V}_2\|_{L^2(\mathbf{R}^1)}, \|\mathbf{N} \cdot \mathbf{V}\|_{L^2(\mathbf{R}^1)}$ *and* $\|\mathbf{T} \cdot \mathbf{V}\|_{L^2(\mathbf{R}^1)}$ *are equivalent, where* $\mathbf{V} = \mathbf{V}^+$ *or* $\mathbf{V}^-$.

Lemma 4.4. *The operator* $\frac{1}{2} - B_3 : L^2(\mathbf{R}^1) \to L^2(\mathbf{R}^1)$ *is invertible. Moreover, it holds that*

$$\|(\frac{1}{2} - B_3)^{-1}u\|_{L^2(\mathbf{R}^1)} \leq C\|u\|_{L^2(\mathbf{R}^1)}$$

with $C = C(\|\eta_0'\|_{L^\infty(\mathbf{R}^1)}) > 0$.

Proof. Let us first consider the layer potentials

$$\mathrm{v}_1 = \mathcal{L}_1(u), \quad \mathrm{v}_2 = -\mathcal{L}_2(u)$$

for $u \in L^2(\mathbf{R}^1)$. By Lemma 4.2, we see that

$$\mathrm{V}_1^{\pm} = \mp\frac{1}{2}u + B_3 u, \quad \mathrm{V}_2^{\pm} = -B_4 u. \tag{4.2}$$

Moreover, $\mathbf{v}$ satisfies $\nabla \cdot \mathbf{v} = 0$ and $\nabla^\perp \cdot \mathbf{v} = 0$. Hence it follows from (4.2) and Lemma 4.3 that

$$\|(\frac{1}{2} + B_3)u\|_{L^2(\mathbf{R}^1)} \leq C\|(\frac{1}{2} - B_3)u\|_{L^2(\mathbf{R}^1)}.$$

Therefore it holds that

$$\|u\|_{L^2(\mathbf{R}^1)} \leq C\|(\frac{1}{2} - B_3)u\|_{L^2(\mathbf{R}^1)}. \tag{4.3}$$

Next, we consider the layer potentials

$$\tilde{\mathrm{v}}_1 = \mathcal{M}_1(u), \quad \tilde{\mathrm{v}}_2 = \mathcal{M}_2(u)$$

for $u \in L^2(\mathbf{R}^1)$. Then for the non-tangential limits $\tilde{\mathbf{V}}^{\pm}$ of $\tilde{\mathbf{v}}$, Lemma 4.2 implies that

$$\mathbf{N} \cdot \tilde{\mathbf{V}}^{\pm} = N_2(\mp\frac{1}{2}u - B_3^* u), \quad \mathbf{T} \cdot \tilde{\mathbf{V}}^{\pm} = -N_2 B_4^* u.$$

Again Lemma 4.3 leads to

$$\|(\frac{1}{2} + B_3^*)u\|_{L^2(\mathbf{R}^1)} \leq C\|(\frac{1}{2} - B_3^*)u\|_{L^2(\mathbf{R}^1)},$$

hence we see that

$$\|u\|_{L^2(\mathbf{R}^1)} \leq C\|(\frac{1}{2} - B_3^*)u\|_{L^2(\mathbf{R}^1)}. \tag{4.4}$$

Thus estimates (4.3), (4.4) give our assertion. □

Lemma 4.5. *Suppose that $\eta_0 \in H^{s+3/2}(\mathbf{R}^1)$, $\bar{X}, \bar{X}^0 \in H^s(\mathbf{R}^1)$, $\|\eta_0\|_{H^{s+3/2}(\mathbf{R}^1)} \leq \kappa$ and $s \geq 2$. There exists a positive constant c such that if $\|\bar{X}\|_{H^2(\mathbf{R}^1)}, \|\bar{X}^0\|_{H^2(\mathbf{R}^1)} \leq c$, then the operator $\frac{1}{2} - B_3 - B_5 : H^s(\mathbf{R}^1) \to H^s(\mathbf{R}^1)$ is invertible. Moreover it holds that*

$$\begin{cases} \|(\frac{1}{2} - B_3 - B_5)^{-1}u\|_{H^s(\mathbf{R}^1)} \leq C\|u\|_{H^s(\mathbf{R}^1)}, \\ \|(\frac{1}{2} - B_3 - B_5)^{-1}(\bar{X})u - (\frac{1}{2} - B_3 - B_5)^{-1}(\bar{X}^0)u\|_{H^s(\mathbf{R}^1)} \\ \qquad \leq C\|\bar{X} - \bar{X}^0\|_{H^s(\mathbf{R}^1)}\|u\|_{H^s(\mathbf{R}^1)}, \end{cases}$$

where $C = C(s, \kappa) > 0$.

Proof. Using Lemma 4.4, we easily see that the operator $\frac{1}{2} - B_3$ is invertible in $H^s(\mathbf{R}^1), s \geq 0$. Moreover, we define the inverse operator $(\frac{1}{2} - B_3 - B_5)^{-1}$ by

$$(\frac{1}{2} - B_3 - B_5)^{-1} = \sum_{n=0}^{\infty} \left(-(\frac{1}{2} - B_3)^{-1}B_5\right)^n (\frac{1}{2} - B_3)^{-1}.$$

Then by the proof for [12, Lemma 4.22(4)], the above assertions are obtained. □

It follows from (3.2) and Lemmas 4.1, 4.5 that

Lemma 4.6. *There exists a positive constant c such that if $\eta_0 \in H^s(\mathbf{R}^1) \cap H^{s_1+3/2}(\mathbf{R}^1)$, $\bar{X}, \bar{X}^0 \in H^s(\mathbf{R}^1)$, $s \geq 2$, $s_0, s_1 > 3/2$ and $\|\eta_0\|_{H^s(\mathbf{R}^1)}, \|\eta_0\|_{H^{s_1+3/2}(\mathbf{R}^1)} \leq \kappa$, $\|\bar{X}\|_{H^2(\mathbf{R}^1)}$, $\|\bar{X}^0\|_{H^2(\mathbf{R}^1)} \leq c$, $\|\bar{X}\|_{H^s(\mathbf{R}^1)}, \|\bar{X}^0\|_{H^s(\mathbf{R}^1)} \leq d$ for some $d > 0$, then it holds that*

$$\begin{cases} \|K_1(\bar{X})u\|_{H^s(\mathbf{R}^1)} \leq C\|u\|_{H^{s_0}(\mathbf{R}^1)}, \\ \|K_1(\bar{X})u - K_1(\bar{X}^0)u\|_{H^s(\mathbf{R}^1)} \leq C\|\bar{X} - \bar{X}^0\|_{H^s(\mathbf{R}^1)}\|u\|_{H^{s_0}(\mathbf{R}^1)}, \end{cases}$$

where $C = C(s, s_0, c, d, \kappa) > 0$.

Now, for a given H, we solve the initial value problem

$$\left(1+\frac{\partial \bar{X}_1}{\partial x_1}\right)\frac{\partial^2 \bar{X}_1}{\partial t^2}+\left(\frac{d\eta_0}{dx_1}+\frac{\partial \bar{X}_2}{\partial x_1}\right)\left(g+\frac{\partial^2 \bar{X}_2}{\partial t^2}\right)=0 \qquad \text{for } t\geq 0, \tag{4.5}$$

$$\bar{X}_{2t}=K\bar{X}_{1t}+H \qquad \text{for } t\geq 0, \tag{4.6}$$

$$\bar{X}|_{t=0}=(0,0), \quad \bar{X}_{1t}|_{t=0}=u_{01}|_{\Gamma_s}. \tag{4.7}$$

Putting

$$Y=\bar{X}_{tt}, \quad Z=\bar{X}_{x_1}, \quad W=(\bar{X},Y,Z), \quad W'=(\bar{X},Y_1),$$

we reduce the above problem to the initial value problem for a quasi-linear system

$$\begin{cases} \bar{X}_{tt}=Y, \quad Y_{1tt}+a(W)|D|Y_1=f_1(W,W'_t,H), \\ Y_{2t}=f_2(W,W'_t,H), \quad Z_{1t}=f_3(W,W'_t,H), \quad Z_{2t}=f_4(W,W'_t,H), \\ W(0)=\widetilde{W}=(\widetilde{\bar{X}},\tilde{Y},\tilde{Z}), \quad W'_t(0)=\widetilde{W'_t}=(\widetilde{\bar{X}_t},\widetilde{Y_{1t}}), \end{cases} \tag{4.8}$$

where f_i, $i=1,2,3,4$, are the lower order terms. The initial data $\widetilde{W}$ and $\widetilde{W'_t}$ should be determined by (4.5) – (4.7).

Here we mention the inverse operator $\{1+Z_1+(\eta'_0+Z_2)K\}^{-1}$ in f_1. Since $1-\eta'_0(\frac{1}{2}-B_3)^{-1}B_4$ can be expressed by the non-tangential limits of some layer potentials, we define the inverse operator $\{1-\eta'_0(\frac{1}{2}-B_3)^{-1}B_4\}^{-1}$ by the same way as in Lemma 4.4. Moreover, $1+\eta'_0K=1-\eta'_0(\frac{1}{2}-B_3-B_5)^{-1}(B_4-B_6)$ and $\{1+Z_1+(\eta'_0+Z_2)K\}^{-1}$ are defined as in Lemma 4.5 without the assumption for the almost flatness of the boundary.

Then the arguments in [5], [6], [8], [12] show that the initial value problem (4.8) is uniquely solvable. Furthermore, we see that

Theorem 4.1. *There exists a positive constant ε such that if $s\geq 3+1/2$, $0<T_1<\infty$ and $\eta_0, u_{01}|_{\Gamma_s}$, H satisfy the conditions*

$$\begin{cases} \eta_0\in H^{s+2}(\mathbf{R}^1), \quad u_{01}|_{\Gamma_s}\in H^{s+1}(\mathbf{R}^1), \\ \|u_{01}|_{\Gamma_s}\|_{H^2(\mathbf{R}^1)}\leq \varepsilon/2, \end{cases}$$

$$\begin{cases} H\in C^j([0,T_1];H^{s+3/2-j/2}(\mathbf{R}^1)), \quad j=1,3, \\ \|H(0)\|_{H^2(\mathbf{R}^1)}+\|H_t(0)\|_{H^2(\mathbf{R}^1)}\leq \varepsilon/2, \end{cases}$$

then there exists $T\in(0,T_1]$ such that problem (4.5) – (4.7) has a unique solution

$$\bar{X}\in C^j([0,T];H^{s+3/2-j/2}(\mathbf{R}^1)), \quad j=1,2,3,4.$$

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