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Steady flows of incompressible Newtonian fluids with threshold slip boundary conditions

C. Le Roux*

Department of Mathematics, Faculty of Natural and Agricultural Sciences, University of Pretoria, Pretoria 0002, South Africa

慶應義塾大学・理工学部　谷　温之 (A. Tani)†

Department of Mathematics, Faculty of Science and Technology, Keio University, 3-14-1 Hiyoshi, Kohoku-ku, Yokohama 223-8522, Japan

Abstract

We give some wellposedness results for the time-independent Navier-Stokes equations with threshold slip boundary conditions in bounded domains. The boundary condition is a generalization of Navier's slip condition and a friction-like condition: for wall slip to occur the magnitude of the tangential traction must exceed a prescribed threshold, independent of the normal stress, and where slip occurs the tangential traction is equal to a prescribed, possibly nonlinear, function of the slip velocity. In addition, a Dirichlet condition is imposed on a component of the boundary if the domain is rotationally symmetric.

1 Introduction

We consider the Navier-Stokes equations for steady flows of incompressible fluids,

$$-\nu\triangle \boldsymbol{v}+\boldsymbol{v}\cdot\nabla\boldsymbol{v}+\nabla p=\boldsymbol{f} \quad \text{in } \Omega, \tag{1.1}$$

$$\operatorname{div}\boldsymbol{v}=0 \quad \text{in } \Omega, \tag{1.2}$$

with the impermeability boundary condition

$$v_n=0 \quad \text{on } \Gamma \tag{1.3}$$

and the slip boundary condition

$$\left.\begin{array}{l} |(\boldsymbol{Tn})_\tau|\le g\Rightarrow \boldsymbol{v}_\tau=(\boldsymbol{v}_w)_\tau, \\ |(\boldsymbol{Tn})_\tau|>g\Rightarrow \boldsymbol{v}_\tau\neq(\boldsymbol{v}_w)_\tau,\ (\boldsymbol{Tn})_\tau=-(g+h(|(\boldsymbol{v}-\boldsymbol{v}_w)_\tau|))\dfrac{(\boldsymbol{v}-\boldsymbol{v}_w)_\tau}{|(\boldsymbol{v}-\boldsymbol{v}_w)_\tau|} \end{array}\right\} \text{ on } \Gamma. \tag{1.4}$$

*christiaan.leroux@up.ac.za

†tani@math.keio.ac.jp

Here Ω is the flow region, a bounded domain in $\mathbb{R}^2$ or $\mathbb{R}^3$, ν is the kinetic viscosity, $\boldsymbol{v}$ is the velocity, p is the modified pressure, and $\boldsymbol{f}$ is the external body force per unit mass. Thus $\nu = \mu/\rho$ and $p = \tilde{p}/\rho$, where μ is the viscosity coefficient, ρ is the density and $\tilde{p}$ is the pressure. Furthermore, $\boldsymbol{n}$ is the outward unit normal on the boundary $\partial\Omega$ of Ω, $\Gamma \subset \partial\Omega$, $v_n := \boldsymbol{v}\cdot\boldsymbol{n}$, $\boldsymbol{v}_\tau := \boldsymbol{v} - v_n\boldsymbol{n}$ is the tangential component of the velocity, $\boldsymbol{T} := -\tilde{p}\boldsymbol{I} + 2\mu\boldsymbol{D}(\boldsymbol{v})$ is the Cauchy stress tensor with $\boldsymbol{D}(\boldsymbol{v}) := \frac{1}{2}[\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T]$, $(\boldsymbol{Tn})_\tau = \boldsymbol{Tn} - (\boldsymbol{n}\cdot\boldsymbol{Tn})\boldsymbol{n} = 2\mu(\boldsymbol{D}(\boldsymbol{v})\boldsymbol{n})_\tau$ is the tangential traction, $\boldsymbol{v}_w$ is the tangential velocity of the wall surface at Γ, $g : \Gamma \to (0,\infty)$ and $h : \Gamma \times [0,\infty) \to [0,\infty)$, with $h(|\boldsymbol{v}_\tau|)(\cdot) := h(\cdot, |\boldsymbol{v}_\tau(\cdot)|)$ on Γ. Condition (1.4) means that the fluid slips at a point on the boundary if and only if the magnitude of the tangential traction exceeds the *slip threshold* g at that point, in which case the tangential traction is a (not necessarily invertible) function of the slip velocity.

We assume that $\partial\Omega$ consists of two disjoint parts, Γ and Σ, such that $|\Gamma| > 0$, where $|\cdot|$ denotes the surface measure (curve measure, if $\Omega \subset \mathbb{R}^2$), and

1. $\overline{\Gamma} \cap \overline{\Sigma} = \emptyset$, i.e. dist $(\Gamma, \Sigma) > 0$, if $|\Sigma| > 0$;

2. $|\Sigma| > 0$ if Ω is rotationally symmetric.

For brevity, we will refer to these geometric assumptions as *condition (C)*. If $|\Sigma| > 0$, we impose the Dirichlet condition

$$\boldsymbol{v} = \boldsymbol{v}_* \quad \text{on } \Sigma, \tag{1.5}$$

where $\boldsymbol{v}_*$ is such that $\int_\Sigma \boldsymbol{v}_*\cdot\boldsymbol{n} = 0$. Thus, Γ is an impermeable solid surface along which the fluid may slip, and Σ is a porous or artificial boundary where the flow is prescribed. If for every $\boldsymbol{x} \in \Gamma$, $h(\boldsymbol{x}, v) = 0$ if and only if $v = 0$, then condition (1.4) is equivalent to

$$\left.\begin{aligned} |(\boldsymbol{Tn})_\tau| &\le g + h(|(\boldsymbol{v} - \boldsymbol{v}_w)_\tau|), \\ (\boldsymbol{Tn})_\tau\cdot(\boldsymbol{v} - \boldsymbol{v}_w)_\tau &= -(g + h(|(\boldsymbol{v} - \boldsymbol{v}_w)_\tau|))|(\boldsymbol{v} - \boldsymbol{v}_w)_\tau| \end{aligned}\right\} \text{ on } \Gamma. \tag{1.6}$$

Slip boundary conditions of this kind have been used to model flows of polymer melts during extrusion, flows of yield stress fluids, and flows of Newtonian fluids with a moving contact line. It can be viewed as generalization of the following three slip boundary conditions:

- The slip condition of Navier [1],

$$(\boldsymbol{Tn})_\tau = -k\boldsymbol{v}_\tau \quad \text{on } \Gamma, \tag{1.7}$$

  where $k : \Gamma \to (0,\infty)$ is given, is the special case of (1.4) in which $g(\boldsymbol{x}) = 0$, $h(\boldsymbol{x}, u) = k(\boldsymbol{x})u$ for all $\boldsymbol{x} \in \Gamma$ and $u \ge 0$ (if we ignore the assumption $g(\boldsymbol{x}) > 0$), and $\boldsymbol{v}_w$ can be described by a single rigid body motion. Navier's slip condition has been applied in a wide variety of fluid problems. The wellposedness of a number of boundary-value problems (e.g. [2, 3]), initial-boundary-value problems (e.g. [4]–[7]), free surface problems (e.g. [8]–[10]) and control problems (e.g. [11]–[13]) for the Navier-Stokes equations with Navier's slip condition has been established. In particular, Tani *et al.* [4]–[6] consider the general Navier slip condition

$$\theta(\boldsymbol{Tn})_\tau = -(1-\theta)\boldsymbol{v}_\tau \quad \text{on } \Gamma, \tag{1.8}$$

where $\theta : \Gamma \times [0,T) \to [0,1]$ is a prescribed function of position and time. Thus, the extremes of no-slip ($\theta = 0$) and free slip ($\theta = 1$) are possible in condition (1.8).

- Nonlinear Navier-type slip conditions of the form

$$(\boldsymbol{v} - \boldsymbol{v}_w)_\tau = -\hat{h}(|(\boldsymbol{Tn})_\tau|)\frac{(\boldsymbol{Tn})_\tau}{|(\boldsymbol{Tn})_\tau|} \quad \text{on } \Gamma \tag{1.9}$$

or

$$(\boldsymbol{Tn})_\tau = -h(|(\boldsymbol{v} - \boldsymbol{v}_w)_\tau|)\frac{(\boldsymbol{v} - \boldsymbol{v}_w)_\tau}{|(\boldsymbol{v} - \boldsymbol{v}_w)_\tau|} \quad \text{on } \Gamma, \tag{1.10}$$

where $\hat{h}, h : \Gamma \times [0,\infty) \to [0,\infty)$ are given functions, are often used to model the wall slip of non-Newtonian fluids. See [14] for more detail and some related references.

- The threshold slip condition

$$\left.\begin{aligned} &|(\boldsymbol{Tn})_\tau| \le g, \\ &|(\boldsymbol{Tn})_\tau| < g \Rightarrow \boldsymbol{v}_\tau = (\boldsymbol{v}_w)_\tau, \\ &|(\boldsymbol{Tn})_\tau| = g \Rightarrow |(\boldsymbol{v} - \boldsymbol{v}_w)_\tau|(\boldsymbol{Tn})_\tau = -g(\boldsymbol{v} - \boldsymbol{v}_w)_\tau \end{aligned}\right\} \text{ on } \Gamma \tag{1.11}$$

or, equivalently,

$$\left.\begin{aligned} |(\boldsymbol{Tn})_\tau| &\le g, \\ (\boldsymbol{Tn})_\tau \cdot (\boldsymbol{v} - \boldsymbol{v}_w)_\tau &= -g|(\boldsymbol{v} - \boldsymbol{v}_w)_\tau| \end{aligned}\right\} \text{ on } \Gamma, \tag{1.12}$$

where $g : \Gamma \to (0,\infty)$ is the prescribed slip threshold, corresponds to the special case of condition (1.4) when $h = h(\boldsymbol{x})$, $\boldsymbol{x} \in \Gamma$. Fujita *et al.* [15]–[23] studied the existence, numerical approximation and regularity of stationary and non-stationary solutions to the Stokes equations with condition (1.11) (with $\boldsymbol{v}_w \equiv \boldsymbol{0}$), which they call slip of the "friction type". Fujita [15] also established the solvability of the time-independent Navier-Stokes equations with this boundary condition. Hence, we will refer to condition (1.11) as *Fujita slip*, to condition (1.4) as *nonlinear Navier-Fujita slip*, and to problem (1.1)–(1.5) (or problem (1.1)–(1.4), if $\partial\Omega = \Gamma$) as *problem (NNF)*.

The outline of the rest of the paper is as follows. First we define the notation (Section 2). Then we formulate problem (NNF) as a variational inequality and give results which establish the existence and uniqueness of a weak solution and its continuous dependence on the data (Section 3). Lastly, we consider the case when $h(\boldsymbol{x}, \cdot)$ is a linear function (Section 4). We do not provide any proofs here; the proofs can be found in [24, 25].

2 Notation

Ω denotes a bounded domain in $\mathbb{R}^d$, $d \in \{2,3\}$, with a boundary, $\partial\Omega$, consisting of two disjoint parts, Γ and Σ, which satisfy condition (C). For the strong formulation of problem (NNF) we assume that $\partial\Omega$ is of class $C^{2,1}$, and for the variational formulations we assume that $\partial\Omega$ is of class $C^{1,1}$.

For $1 \le q \le \infty$, $L^q(\Omega)$ and $L^q(\partial\Omega)$ are the usual Lebesgue spaces, with norms denoted by $\|\cdot\|_q$ and $\|\cdot\|_{q,\partial\Omega}$, respectively. For $m \in \mathbb{N}$ and $1 < q < \infty$, $W^{m,q}(\Omega)$ is the standard Sobolev space with norm $\|\cdot\|_{m,q}$, and for $\partial\Omega \in C^{m-1,1}$, $W^{m-1/q,q}(\partial\Omega)$ is the associated trace space with norm $\|\cdot\|_{m-1/q,q,\partial\Omega}$. We also define

$$
\begin{aligned}
\bar{L}^q(\Omega) &:= \{p \in L^q(\Omega) : \int_\Omega p = 0\},\\
\bar{W}^{m,q}(\Omega) &:= W^{m,q}(\Omega) \cap \bar{L}^q(\Omega),\\
W_0^{m,q}(\Omega) &:= \{v \in W^{m,q}(\Omega) : v = 0 \text{ on } \partial\Omega\}.
\end{aligned}
$$

Here, and in what follows, the boundary values are to be understood in the sense of traces. We omit the trace operators where the meaning is clear; otherwise we denote the traces by $v|_{\partial\Omega}$, $v|_\Gamma$, etc. For $k = 1,2,3$, the inner products in the spaces $L^2(\Omega)^k$, $L^2(\partial\Omega)^k$ and $W^{m,2}(\Omega)^k$ are denoted by $(\cdot,\cdot)$, $(\cdot,\cdot)_{\partial\Omega}$ and $(\cdot,\cdot)_m$, respectively. The product spaces with $k = d$ are denoted by bold letters: $\boldsymbol{W}^{m,q}(\Omega) := W^{m,q}(\Omega)^d$, $\boldsymbol{W}^{m-1/q,q}(\partial\Omega) := W^{m-1/q,q}(\partial\Omega)^d$, etc. In addition, for $\partial\Omega \in C^{m,1}$, $m \in \mathbb{N}$ and $1 < q < \infty$,

$$
\begin{aligned}
\boldsymbol{V}^{m,q}(\Omega) &:= \{\boldsymbol{v} \in \boldsymbol{W}^{m,q}(\Omega) : v_n = 0 \text{ on } \Gamma\},\\
\boldsymbol{V}_0^{m,q}(\Omega) &:= \{\boldsymbol{v} \in \boldsymbol{V}^{m,q}(\Omega) : \boldsymbol{v} = \boldsymbol{0} \text{ on } \Sigma\},\\
\boldsymbol{V}_*^{m,q}(\Omega) &:= \{\boldsymbol{v} \in \boldsymbol{V}^{m,q}(\Omega) : \boldsymbol{v} = \boldsymbol{v}_* \text{ on } \Sigma\},\\
\boldsymbol{W}_\sigma^{m,q}(\Omega) &:= \{\boldsymbol{v} \in \boldsymbol{W}^{m,q}(\Omega) : \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega\},\\
\boldsymbol{V}_\sigma^{m,q}(\Omega) &:= \boldsymbol{V}^{m,q}(\Omega) \cap \boldsymbol{W}_\sigma^{m,q}(\Omega),\\
\boldsymbol{V}_{0,\sigma}^{m,q}(\Omega) &:= \boldsymbol{V}_0^{m,q}(\Omega) \cap \boldsymbol{W}_\sigma^{m,q}(\Omega),\\
\boldsymbol{V}_{*,\sigma}^{m,q}(\Omega) &:= \boldsymbol{V}_*^{m,q}(\Omega) \cap \boldsymbol{W}_\sigma^{m,q}(\Omega).
\end{aligned}
$$

In these definitions it is understood that $\boldsymbol{V}_0^{m,q}(\Omega)$ and $\boldsymbol{V}_*^{m,q}(\Omega)$ ($\boldsymbol{V}_{0,\sigma}^{m,q}(\Omega)$ and $\boldsymbol{V}_{*,\sigma}^{m,q}(\Omega)$, respectively) reduce to $\boldsymbol{V}^{m,q}(\Omega)$ ($\boldsymbol{V}_\sigma^{m,q}(\Omega)$, respectively) if $\partial\Omega = \Gamma$. The spaces $\boldsymbol{W}_0^{m,q}(\Omega)$, $\boldsymbol{V}^{m,q}(\Omega)$, $\boldsymbol{V}_0^{m,q}(\Omega)$, $\boldsymbol{W}_\sigma^{m,q}(\Omega)$, $\boldsymbol{V}_\sigma^{m,q}(\Omega)$ and $\boldsymbol{V}_{0,\sigma}^{m,q}(\Omega)$ are equipped with the norm $\|\cdot\|_{m,q}$ (inner product $(\cdot,\cdot)_m$ if $q = 2$, respectively); so defined, they are Banach spaces (Hilbert spaces, respectively). Similarly, the trace spaces

$$
\begin{aligned}
\boldsymbol{L}_\tau^q(\Gamma) &:= \{\boldsymbol{v} \in \boldsymbol{L}^q(\Gamma) : v_n = 0 \text{ on } \Gamma\},\\
\boldsymbol{W}_\tau^{m-1/q,q}(\Gamma) &:= \boldsymbol{W}^{m-1/q,q}(\Gamma) \cap \boldsymbol{L}_\tau^q(\Gamma),\\
\boldsymbol{V}_\sigma^{m-1/q,q}(\Sigma) &:= \{\boldsymbol{v} \in \boldsymbol{W}^{m-1/q,q}(\Sigma) : (v_n, 1)_\Sigma = 0\}
\end{aligned}
$$

are equipped with the norms (and, if $q = 2$, inner products) of $\boldsymbol{L}^q(\Gamma)$, $\boldsymbol{W}^{m-1/q,q}(\Gamma)$ and $\boldsymbol{W}^{m-1/q,q}(\Sigma)$, respectively.

For any two Banach spaces X and Y, $a \in X$ and $r > 0$, $\overline{B}(X,a,r) := \{x \in X : \|x - a\|_X \le r\}$, $\mathcal{L}(X,Y)$ denotes the space of bounded linear operators $L : X \to Y$, equipped with the usual norm $\|L\| = \sup\{\|Lx\|_Y : \|x\|_X \le 1\}$, and X' denotes the dual space $\mathcal{L}(X, \boldsymbol{R})$. In particular, if $1 < q < \infty$ and $1/q + 1/q' = 1$, we denote the corresponding norms of $[\boldsymbol{W}_0^{1,q'}(\Omega)]' \equiv \boldsymbol{W}^{-1,q}(\Omega)$, $[\boldsymbol{V}_0^{1,q'}(\Omega)]'$ and $[\boldsymbol{V}_{0,\sigma}^{1,q'}(\Omega)]'$ by $\|\cdot\|_{-1,q}$, $\|\cdot\|_{V,-1,q}$ and $\|\cdot\|_{V,\sigma,-1,q}$, respectively.

Lastly, by $C_1(\Omega, q)$, $C_K(\Omega)$, etc. we will denote positive constants which depend at most on the quantities in brackets. In order to define some constants that appear in the results, we recall some auxiliary results. We denote the conjugate of an exponent $r \in [1, \infty]$ by r', i.e. $1/r + 1/r' = 1$, and define

$$I_2 := [1,\infty),\ I_2' := (1,\infty],\ I_3 := [1,4],\ I_3' := [4/3,\infty],\ K_2 := [1,\infty),\ K_3 := [1,4).$$

If $\partial\Omega \in C^{1,1}$ and $q \in I_d$, then $W^{1,2}(\Omega) \hookrightarrow L^q(\Gamma)$: there is a constant $C_1 = C_1(\Omega, q)$ such that $\|u\|_{q,\Gamma} \le C_1\|u\|_{1,2}$ for all $u \in W^{1,2}(\Omega)$. The imbedding is compact if $q \in K_d$.

Let $a(\cdot,\cdot)$ be the bilinear form defined by

$$a(\boldsymbol{u},\boldsymbol{v}) = 2\nu(\boldsymbol{D}(\boldsymbol{u}),\boldsymbol{D}(\boldsymbol{v})) = \frac{\nu}{2}\sum_{i,j=1}^{d}(u_{i,j}+u_{j,i}, v_{i,j}+v_{j,i}) \quad \text{for all } \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{W}^{1,2}(\Omega).$$

Then condition (C) and the following versions of Korn's inequality imply that $a(\cdot,\cdot)$ is coercive in $\boldsymbol{V}_0^{1,2}(\Omega)$.

Lemma 2.1 *Suppose that Ω is a bounded domain in $\mathbb{R}^2$ or $\mathbb{R}^3$, $\partial\Omega \in C^{0,1}$, $\Sigma \subset \partial\Omega$ and $|\Sigma| > 0$.*
(a) *There is a constant $C_K = C_K(\Omega) > 0$ such that*

$$\|\boldsymbol{D}(\boldsymbol{v})\|_2^2 \ge C_K\|\boldsymbol{v}\|_{1,2}^2 \quad \textit{for all } \boldsymbol{v} \in \boldsymbol{W}^{1,2}(\Omega) \textit{ such that } \boldsymbol{v} = \boldsymbol{0} \textit{ on } \Sigma. \tag{2.1}$$

(b) *If Ω is not a body of rotation, there is a constant $C_K = C_K(\Omega) > 0$ such that*

$$\|\boldsymbol{D}(\boldsymbol{v})\|_2^2 \ge C_K\|\boldsymbol{v}\|_{1,2}^2 \quad \textit{for all } \boldsymbol{v} \in \boldsymbol{V}^{1,2}(\Omega). \tag{2.2}$$

Let $b(\cdot,\cdot,\cdot)$ be the trilinear form defined by

$$b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = (\boldsymbol{u}\cdot\nabla\boldsymbol{v},\boldsymbol{w}) = \sum_{i,j=1}^{d}(u_j v_{i,j}, w_i) \quad \text{for all } \boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{W}^{1,2}(\Omega).$$

Then there is a constant $C_2 = C_2(\Omega)$ such that

$$|b(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})| \le C_2\|\boldsymbol{u}\|_{1,2}\|\nabla\boldsymbol{v}\|_2\|\boldsymbol{w}\|_{1,2} \quad \text{for all } \boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{W}^{1,2}(\Omega). \tag{2.3}$$

Lemma 2.2 *Suppose that Ω, Γ, Σ satisfy condition (C), $|\Sigma| > 0$, $\partial\Omega \in C^{0,1}$ and $1 < q < \infty$.*
(a) *There exists a bounded linear mapping $\boldsymbol{W}_\tau^{1-1/q,q}(\Gamma) \to \boldsymbol{V}_{0,\sigma}^{1,q}(\Omega) : \boldsymbol{v}_w \mapsto \tilde{\boldsymbol{w}}$ such that $\tilde{\boldsymbol{w}}|_\Gamma = \boldsymbol{v}_w$ and*

$$\|\tilde{\boldsymbol{w}}\|_{1,q} \le C_3(\Omega, q)\|\boldsymbol{v}_w\|_{1-1/q,q,\Gamma}. \tag{2.4}$$

(b) *Suppose that* $\Sigma_1, \dots, \Sigma_k$ *are the connected components of* Σ. *Then, for every* $\varepsilon > 0$, *there exists a bounded linear mapping* $\boldsymbol{V}_\sigma^{1/2,2}(\Sigma) \to \boldsymbol{V}_{*,\sigma}^{1,2}(\Omega) : \boldsymbol{v}_* \mapsto \tilde{\boldsymbol{v}}$ *such that* $\tilde{\boldsymbol{v}}|_\Sigma = \boldsymbol{v}_*$, $\tilde{\boldsymbol{v}} = \mathbf{0}$ *in a neighborhood of* Γ *and*

$$\|\tilde{\boldsymbol{v}}\|_{1,2} \leq C_3(\Omega)\|\boldsymbol{v}_*\|_{1/2,2,\Sigma}, \tag{2.5}$$

$$|b(\boldsymbol{\psi}, \tilde{\boldsymbol{v}}, \boldsymbol{\psi})| \leq (\varepsilon + \Phi(\boldsymbol{v}_*))\|\nabla\boldsymbol{\psi}\|_2^2 \quad \textit{for all } \boldsymbol{\psi} \in \boldsymbol{V}_{0,\sigma}^{1,2}(\Omega), \tag{2.6}$$

where $\Phi(\boldsymbol{v}_*) := K_i|\Phi_i|$ *with* $\Phi_i := (\boldsymbol{v}_* \cdot \boldsymbol{n}, 1)_{\Sigma_i}$, $K_i = K_i(\Omega, \Sigma_i) > 0$, $i = 1, \dots, k$.

3 Navier-Stokes problem

3.1 Variational formulation

Suppose that

$$\begin{aligned} &\partial\Omega \in C^{2,1}, \quad \boldsymbol{f} \in \boldsymbol{L}^2(\Omega), \quad \boldsymbol{v}_* \in \boldsymbol{V}_\sigma^{3/2,2}(\Sigma) \text{ if } |\Sigma| > 0, \\ &\boldsymbol{v}_w \in \boldsymbol{W}_\tau^{3/2,2}(\Gamma), \quad g \in W^{1,2}(\Gamma), \quad g > 0 \text{ a.e. on } \Gamma, \end{aligned} \tag{3.1}$$

and that $h : \Gamma \times [0, \infty) \to [0, \infty)$ has the following properties:

1. $h(\boldsymbol{x}, \cdot)$ is continuously differentiable on $[0, \infty)$ for every $\boldsymbol{x} \in \Gamma$;
2. $h(\cdot, u)$ is continuously differentiable on Γ for all $u \in [0, \infty)$;
3. for almost every $\boldsymbol{x} \in \Gamma$, $h(\boldsymbol{x}, u) = 0$ if and only if $u = 0$.

Then $h(|\boldsymbol{v} - \boldsymbol{v}_w|) \in W^{1,2}(\Gamma)$ for all $\boldsymbol{v} \in \boldsymbol{W}^{2,2}(\Omega)$. Hence, with these hypotheses we can formulate the following strong form of problem (NNF):

Problem 3.1 (NNF) *Find* $(\boldsymbol{v}, p) \in \boldsymbol{V}_{*,\sigma}^{2,2}(\Omega) \times \bar{W}^{1,2}(\Omega)$ *that satisfies the Navier-Stokes equations* (1.1) *in the sense of distributions and the slip boundary condition* (1.6) *in the sense of traces.*

For the weak formulations of problem (NNF) we assume that

$$\begin{aligned} &\partial\Omega \in C^{1,1}, \quad \boldsymbol{f} \in [\boldsymbol{V}_0^{1,2}(\Omega)]', \quad \boldsymbol{v}_* \in \boldsymbol{V}_\sigma^{1/2,2}(\Sigma) \text{ if } |\Sigma| > 0, \\ &\boldsymbol{v}_w \in \boldsymbol{W}_\tau^{1/2,2}(\Gamma), \quad g \in L^r(\Gamma),\ r \in I_d', \quad g > 0 \text{ a.e. on } \Gamma, \end{aligned} \tag{3.2}$$

where $h : \Gamma \times [0, \infty) \to [0, \infty)$ has the following properties:

1. $h(\boldsymbol{x}, \cdot)$ is continuous on $[0, \infty)$ for almost every $\boldsymbol{x} \in \Gamma$;
2. $h(\cdot, u)$ is measurable on Γ for all $u \in [0, \infty)$;
3. for almost every $\boldsymbol{x} \in \Gamma$, $h(\boldsymbol{x}, u) = 0$ if and only if $u = 0$;
4. if $r < \infty$: there exist a nonnegative function $a_h \in L^r(\Gamma)$ and constants $b_h > 0$ and $q \in I_d$ such that for almost every $\boldsymbol{x} \in \Gamma$,

$$|h(\boldsymbol{x}, u)| \leq a_h(\boldsymbol{x}) + b_h|u|^{q/r} \quad \text{for all } u \geq 0; \tag{3.3}$$

if $r = \infty$: there exists a nonnegative function $a_h \in L^\infty(\Gamma)$ such that for almost every $\boldsymbol{x} \in \Gamma$,

$$|h(\boldsymbol{x}, u)| \leq a_h(\boldsymbol{x}) \quad \text{for all } u \geq 0. \tag{3.4}$$

Then we can extend h to $\Gamma \times \mathbb{R}$ so that the extension is a Carathéodory function and generates a superposition operator which maps $L^q(\Gamma)$ into $L^r(\Gamma)$. The superposition operator is continuous if $r < \infty$. Furthermore, for every $\boldsymbol{v} \in \boldsymbol{W}^{1,2}(\Omega)$, $|\boldsymbol{v} - \boldsymbol{v}_w| \in L^q(\Gamma)$ and $h(|\boldsymbol{v} - \boldsymbol{v}_w|) \in L^r(\Gamma)$. Thus, we can define a functional $j_2 : \boldsymbol{W}^{1,2}(\Omega)^2 \to [0,\infty)$ by $j_2(\boldsymbol{v};\boldsymbol{\phi}) := \rho^{-1}(g + h(|\boldsymbol{v}_\tau|), |\boldsymbol{\phi}_\tau|)_\Gamma$. With these hypotheses, we choose $\tilde{\boldsymbol{w}}$ as in Lemma 2.2 and consider the following weak and variational inequality formulations of problem (NNF):

Problem 3.2 (NNF-W) *Find* $(\boldsymbol{v}, p, \boldsymbol{\sigma}) \in \boldsymbol{V}^{1,2}_{*,\sigma}(\Omega) \times \bar{L}^2(\Omega) \times \boldsymbol{L}^r_\tau(\Gamma)$ *such that*

$$a(\boldsymbol{v},\boldsymbol{\psi}) + b(\boldsymbol{v},\boldsymbol{v},\boldsymbol{\psi}) - (p, \operatorname{div}\boldsymbol{\psi}) = \langle \boldsymbol{f},\boldsymbol{\psi}\rangle + \rho^{-1}(\boldsymbol{\sigma},\boldsymbol{\psi})_\Gamma \quad \text{if } \boldsymbol{\psi} \in \boldsymbol{V}^{1,2}_0(\Omega), \tag{3.5}$$

$$\left.\begin{aligned} |\boldsymbol{\sigma}| &\le g + h(|\boldsymbol{v} - \boldsymbol{v}_w|), \\ \boldsymbol{\sigma}\cdot(\boldsymbol{v} - \boldsymbol{v}_w) &= -(g + h(|\boldsymbol{v} - \boldsymbol{v}_w|))|\boldsymbol{v} - \boldsymbol{v}_w| \end{aligned}\right\} \text{ at a.e. point on } \Gamma. \tag{3.6}$$

Problem 3.3 (NNF-W$_\sigma$) *Find* $(\boldsymbol{v}, \boldsymbol{\sigma}) \in \boldsymbol{V}^{1,2}_{*,\sigma}(\Omega) \times \boldsymbol{L}^r_\tau(\Gamma)$ *such that*

$$a(\boldsymbol{v},\boldsymbol{\psi}) + b(\boldsymbol{v},\boldsymbol{v},\boldsymbol{\psi}) = \langle \boldsymbol{f},\boldsymbol{\psi}\rangle + \rho^{-1}(\boldsymbol{\sigma},\boldsymbol{\psi})_\Gamma \quad \text{for all } \boldsymbol{\psi} \in \boldsymbol{V}^{1,2}_{0,\sigma}(\Omega), \tag{3.7}$$

$$\left.\begin{aligned} |\boldsymbol{\sigma}| &\le g + h(|\boldsymbol{v} - \boldsymbol{v}_w|), \\ \boldsymbol{\sigma}\cdot(\boldsymbol{v} - \boldsymbol{v}_w) &= -(g + h(|\boldsymbol{v} - \boldsymbol{v}_w|))|\boldsymbol{v} - \boldsymbol{v}_w| \end{aligned}\right\} \text{ at a.e. point on } \Gamma. \tag{3.8}$$

Problem 3.4 (NNF-VI) *Find* $(\boldsymbol{v}, p) \in \boldsymbol{V}^{1,2}_{*,\sigma}(\Omega) \times \bar{L}^2(\Omega)$ *such that*

$$\begin{aligned} &a(\boldsymbol{v},\boldsymbol{\phi}-\boldsymbol{v}) + b(\boldsymbol{v},\boldsymbol{v},\boldsymbol{\phi}-\boldsymbol{v}) - (p, \operatorname{div}(\boldsymbol{\phi}-\boldsymbol{v})) - \langle \boldsymbol{f},\boldsymbol{\phi}-\boldsymbol{v}\rangle \\ &\qquad + j_2(\boldsymbol{v}-\tilde{\boldsymbol{w}};\boldsymbol{\phi}-\tilde{\boldsymbol{w}}) - j_2(\boldsymbol{v}-\tilde{\boldsymbol{w}};\boldsymbol{v}-\tilde{\boldsymbol{w}}) \ge 0 \quad \text{for all } \boldsymbol{\phi} \in \boldsymbol{V}^{1,2}_*(\Omega). \end{aligned} \tag{3.9}$$

Problem 3.5 (NNF-VI$_\sigma$) *Find* $\boldsymbol{v} \in \boldsymbol{V}^{1,2}_{*,\sigma}(\Omega)$ *such that for all* $\boldsymbol{\phi} \in \boldsymbol{V}^{1,2}_{*,\sigma}(\Omega)$,

$$\begin{aligned} &a(\boldsymbol{v},\boldsymbol{\phi}-\boldsymbol{v}) + b(\boldsymbol{v},\boldsymbol{v},\boldsymbol{\phi}-\boldsymbol{v}) - \langle \boldsymbol{f},\boldsymbol{\phi}-\boldsymbol{v}\rangle \\ &\qquad + j_2(\boldsymbol{v}-\tilde{\boldsymbol{w}};\boldsymbol{\phi}-\tilde{\boldsymbol{w}}) - j_2(\boldsymbol{v}-\tilde{\boldsymbol{w}};\boldsymbol{v}-\tilde{\boldsymbol{w}}) \ge 0. \end{aligned} \tag{3.10}$$

We will not copy out the corresponding assumptions and problem formulations for the case when $\partial\Omega = \Gamma$: simply omit the statements involving $\boldsymbol{v}_*$, replace $\boldsymbol{V}^{2,2}_{*,\sigma}(\Omega)$ by $\boldsymbol{V}^{2,2}_\sigma(\Omega)$ and $\boldsymbol{V}^{1,2}_0(\Omega)$ by $\boldsymbol{V}^{1,2}(\Omega)$ in problem (NNF-W), etc. The same applies to the formulations of the results to follow.

Unless we state otherwise, we will henceforth assume that the data (i.e. Ω, Γ, Σ, $\boldsymbol{f}$, $\boldsymbol{v}_w$, $\boldsymbol{v}_*$, g, h, r) satisfy the hypotheses of problem (NNF-W): conditions (C) and (3.2)–(3.4).

Theorem 3.6 (a) *Assume the hypotheses of problem (NNF) and let* $r \in I'_d$. *In addition, assume that* $g \in L^r(\Gamma)$ *and that* h *satisfies condition 4 of the hypotheses of problem (NNF). (These additional assumptions are redundant if* $r \in I_d \cap I'_d$.) *If*

$(\boldsymbol{v}, p)$ is a solution of problem (NNF) and $\boldsymbol{\sigma}$ is the associated tangential traction $(\boldsymbol{T}\boldsymbol{n})_\tau \in \boldsymbol{W}_\tau^{1/2,2}(\Gamma)$, then $(\boldsymbol{v}, p, \boldsymbol{\sigma})$ is a solution of problem (NNF-W). Conversely, if $(\boldsymbol{v}, p, \boldsymbol{\sigma})$ is a solution of problem (NNF-W) and $(\boldsymbol{v}, p) \in \boldsymbol{V}_{*,\sigma}^{2,2}(\Omega) \times \bar{W}^{1,2}(\Omega)$, then $(\boldsymbol{v}, p)$ is a solution of problem (NNF) and $\boldsymbol{\sigma}$ is the associated tangential traction $(\boldsymbol{T}\boldsymbol{n})_\tau \in \boldsymbol{W}_\tau^{1/2,2}(\Gamma)$.

(b) *Problems (NNF-W), (NNF-W_σ), (NNF-VI) and (NNF-VI_σ) are equivalent.*

The assertion in part (b) means that if $(\boldsymbol{v}, p, \boldsymbol{\sigma})$ is a solution of (NNF-W) then $(\boldsymbol{v}, p)$ is a solution of (NNF-VI); if $(\boldsymbol{v}, p)$ is a solution of (NNF-VI) then there exists a $\boldsymbol{\sigma}$ such that $(\boldsymbol{v}, p, \boldsymbol{\sigma})$ is a solution of (NNF-W); etc.

3.2 Existence

Assume that the data satisfy the hypotheses of problem (NNF-W) and let $\tilde{\boldsymbol{v}}$ and $\tilde{\boldsymbol{w}}$ be as in Lemma 2.2. Then $\tilde{\boldsymbol{v}}|_\Gamma = \boldsymbol{0}$ and $(\boldsymbol{\phi} - \tilde{\boldsymbol{w}})|_\Gamma = (\boldsymbol{\phi} - \boldsymbol{v})|_\Gamma + (\boldsymbol{v} - \tilde{\boldsymbol{v}} - \tilde{\boldsymbol{w}})|_\Gamma$ for all $\boldsymbol{\phi}, \boldsymbol{v} \in \boldsymbol{V}_{*,\sigma}^{1,2}(\Omega)$. Thus $\boldsymbol{v}$ is a solution of problem (NNF-VI$_\sigma$) if and only if $\boldsymbol{V} := \boldsymbol{v} - \tilde{\boldsymbol{v}} - \tilde{\boldsymbol{w}}$ is a solution of the following problem:

Problem 3.7 (NNF$_0$-VI) *Find $\boldsymbol{V} \in \boldsymbol{V}_{0,\sigma}^{1,2}(\Omega)$ such that for all $\boldsymbol{\psi} \in \boldsymbol{V}_{0,\sigma}^{1,2}(\Omega)$,*

$$\begin{aligned} a(\boldsymbol{V} + \tilde{\boldsymbol{v}} + \tilde{\boldsymbol{w}}, \boldsymbol{\psi}) + b(\boldsymbol{V} + \tilde{\boldsymbol{v}} + \tilde{\boldsymbol{w}}, \boldsymbol{V} + \tilde{\boldsymbol{v}} + \tilde{\boldsymbol{w}}, \boldsymbol{\psi}) - \langle \boldsymbol{f}, \boldsymbol{\psi} \rangle \\ + j_2(\boldsymbol{V}; \boldsymbol{V} + \boldsymbol{\psi}) - j_2(\boldsymbol{V}; \boldsymbol{V}) \geq 0. \end{aligned} \tag{3.11}$$

By definition, $\boldsymbol{V}$ depends on $\tilde{\boldsymbol{v}}$, which depends on the choice of ε in Lemma 2.2(b). We will fix ε in the next theorem. For brevity, we let

$$\Psi(\boldsymbol{v}_w) := C_3(\Omega)\|\boldsymbol{v}_w\|_{1/2,2,\Gamma}, \quad \Psi(\boldsymbol{v}_*) := C_3(\Omega)\|\boldsymbol{v}_*\|_{1/2,2,\Sigma}. \tag{3.12}$$

Theorem 3.8 *Suppose that $C_2(\Omega)\Psi(\boldsymbol{v}_w) + \Phi(\boldsymbol{v}_*) < 2\nu C_K(\Omega)$, fix $\theta \in (0,1)$ and set*

$$\varepsilon = (1-\theta)\big(2\nu C_K(\Omega) - C_2(\Omega)\Psi(\boldsymbol{v}_w) - \Phi(\boldsymbol{v}_*)\big). \tag{3.13}$$

(a) *If $\boldsymbol{V}$ is a solution of problem (NNF$_0$-VI) then $\|\boldsymbol{V}\|_{1,2} \leq E_0$, where*

$$E_0 := \frac{\|\boldsymbol{f}\|_{V,\sigma,-1,2} + 2\nu\big(\Psi(\boldsymbol{v}_w) + \Psi(\boldsymbol{v}_*)\big) + C_2(\Omega)\big(\Psi(\boldsymbol{v}_w) + \Psi(\boldsymbol{v}_*)\big)^2}{\theta\big(2\nu C_K(\Omega) - C_2(\Omega)\Psi(\boldsymbol{v}_w) - \Phi(\boldsymbol{v}_*)\big)}. \tag{3.14}$$

(b) *If $d = 3$, suppose also that $r > 4/3$. Then problem (NNF$_0$-VI) has a solution.*

Remark 3.9 (a) The proof of Theorem 3.8(b) is based on a fixed-point argument in which we employ an existence result for elliptic variational inequalities of the second kind, the Galerkin method and the Leray-Schauder principle. See [25] for the detail.

(b) The pressure field $p \in \bar{L}^2(\Omega)$ in a solution of problems (NNF-W) and (NNF-VI) is constructed from the corresponding velocity field $\boldsymbol{v} = \boldsymbol{V} + \tilde{\boldsymbol{v}} + \tilde{\boldsymbol{w}}$ in the same manner as for the Dirichlet problem. In particular, there is a constant $C_4 = C_4(\Omega) > 0$ such that $\|p\|_2 \leq C_4(\Omega)\|Z\|_{-1,2}$, where $Z \in \boldsymbol{W}^{-1,2}(\Omega)$ is defined by

$\langle Z, \psi\rangle := a(\boldsymbol{v}, \boldsymbol{\psi}) + b(\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{\psi}) - \langle \boldsymbol{f}, \boldsymbol{\psi}\rangle$ for all $\boldsymbol{\psi} \in \boldsymbol{W}_0^{1,2}(\Omega)$. Thus, if $C_2(\Omega)\Psi(\boldsymbol{v}_w) + \Phi(\boldsymbol{v}_*) < 2\nu C_K(\Omega)$, we have the *a priori* estimates

$$\|\boldsymbol{v}\|_{1,2} \le E_1 := E_0 + \Psi(\boldsymbol{v}_*) + \Psi(\boldsymbol{v}_w), \tag{3.15}$$

$$\|p\|_2 \le C_4(\Omega)\big(\|\boldsymbol{f}\|_{-1,2} + 2\nu\|\boldsymbol{v}\|_{1,2} + C_2(\Omega)\|\boldsymbol{v}\|_{1,2}^2\big). \tag{3.16}$$

Moreover, the boundary condition $(3.6)_1$ and the acting condition (3.3) (or (3.4)) imply that

$$\|\boldsymbol{\sigma}\|_{r,\Gamma} \le \|g\|_{r,\Gamma} + \|a_h\|_{r,\Gamma} + b_h C_1(\Omega, q)^{q/r}\|\boldsymbol{V}\|_{1,2}^{q/r}. \tag{3.17}$$

3.3 Uniqueness

Assume that the data satisfy the hypotheses of problem (NNF-W).

Theorem 3.10 *Suppose that $h(\boldsymbol{x}, \cdot)$ is monotone increasing on $[0, \infty)$ for almost every $\boldsymbol{x} \in \Gamma$.*

(a) *If $C_2(\Omega)\Psi(\boldsymbol{v}_w) + \Phi(\boldsymbol{v}_*) < 2\nu C_K(\Omega)$, $\boldsymbol{V}$ is a solution of problem (NNF$_0$-VI) and*

$$\|\boldsymbol{V}\|_{1,2} < \theta\big(2\nu C_K(\Omega) - C_2(\Omega)\Psi(\boldsymbol{v}_w) - \Phi(\boldsymbol{v}_*)\big)/C_2(\Omega), \tag{3.18}$$

then $\boldsymbol{V} + \tilde{\boldsymbol{v}} + \tilde{\boldsymbol{w}}$ is the only solution of problem (NNF-VI$_\sigma$).

(b) *If $\boldsymbol{v}$ is a solution of problem (NNF-VI$_\sigma$) and*

$$\|\boldsymbol{v}\|_{1,2} < 2\nu C_K(\Omega)/C_2(\Omega), \tag{3.19}$$

then $\boldsymbol{v}$ is the only solution of problem (NNF-VI$_\sigma$).

(c) *Suppose that $C_2(\Omega)\Psi(\boldsymbol{v}_w) + \Phi(\boldsymbol{v}_*) < 2\nu C_K(\Omega)$ and*

$$\theta^{-1}C_2E_0 + C_2\Psi(\boldsymbol{v}_w) + \Phi(\boldsymbol{v}_*) < 2\nu C_K(\Omega), \tag{3.20}$$

where $E_0 = E_0(\nu, \Omega, \boldsymbol{f}, \Gamma, \boldsymbol{v}_w, \Sigma, \boldsymbol{v}_*)$ is as in (3.14). In addition, if $d = 3$ assume that $r > 4/3$. Then problem (NNF-VI$_\sigma$) has a unique solution.

Remark 3.11 By virtue of Theorem 3.6 and the fact that p and $\boldsymbol{\sigma}$ are uniquely determined by $\boldsymbol{v}$, Theorem 3.10 also applies to problems (NNF-W), (NNF-W$_\sigma$) and (NNF-VI). So too does Theorem 3.12 below.

Now consider the case when $h(\boldsymbol{x}, \cdot)$ is not necessarily a monotone function. For $r < \infty$ we define

$$M_{q,r}[h, R] := \sup\{\|h(|w|)\|_{r,\Gamma} : w \in \overline{B}(L^q(\Gamma), 0, R)\},$$

$$N_{q,r}[h, R] := \inf\{\|a\|_{r,\Gamma} + bR^{q/r} : a \in L^r(\Gamma), b \ge 0 \text{ such that for a.e. } \boldsymbol{x} \in \Gamma,$$
$$h(\boldsymbol{x}, u) \le a(\boldsymbol{x}) + bu^{q/r} \text{ for all } u \ge 0\}$$

for all $R > 0$. Similarly, for $r = \infty$ we define

$$M_{q,\infty}[h, R] := \sup\{\|h(|w|)\|_{\infty,\Gamma} : w \in \overline{B}(L^q(\Gamma), 0, R)\},$$
$$N_{q,\infty}[h, R] := \inf\{\|a\|_{\infty,\Gamma} : a \in L^\infty(\Gamma) \text{ such that for a.e. } \boldsymbol{x} \in \Gamma, \; h(\boldsymbol{x}, u) \leq a(\boldsymbol{x}) \text{ for all } u \geq 0\}$$

for all $R >$. Then, in both cases,

$$M_{q,r}[h, R] \leq N_{q,r}[h, R] \quad \text{for all } R > 0. \tag{3.21}$$

In addition to the hypotheses of problem (NNF-W), suppose that the function h satisfies a Lipschitz condition: for almost every $\boldsymbol{x} \in \Gamma$,

$$|h(\boldsymbol{x}, u_1) - h(\boldsymbol{x}, u_2)| \leq k(\boldsymbol{x}, v)|u_1 - u_2| \quad \text{for all } v > 0 \text{ and all } u_1, u_2 \in [0, v], \tag{3.22}$$

where the function $k : \Gamma \times [0, \infty) \to [0, \infty)$ has the following properties:

1. $k(\boldsymbol{x}, \cdot)$ is continuous on $[0, \infty)$ for almost every $\boldsymbol{x} \in \Gamma$;
2. $k(\cdot, u)$ is measurable on Γ for all $u \in [0, \infty)$;
3. there exist constants $q_* \in I_d$, $r_* \in I'_d$, $r_* \leq \min(r, q_*)$, with $q_*/r_* \geq q/r$ if $r < \infty$, a nonnegative function $a_k \in L^{s_*}(\Gamma)$, where $s_* := \infty$ if $q_* = r_*$ and $s_* := q_*r_*/(q_* - r_*)$ otherwise, and a constant $b_k \geq 0$, with $b_k = 0$ if $q_* = r_*$, such that for a.e. $\boldsymbol{x} \in \Gamma$,

$$|k(\boldsymbol{x}, v)| \leq a_k(\boldsymbol{x}) + b_k|v|^{q_*/s_*} \quad \text{for all } v \geq 0. \tag{3.23}$$

Then the superposition operator generated by h maps $L^{q_*}(\Gamma)$ into $L^{r_*}(\Gamma)$ and is locally Lipschitz continuous in these spaces: for every $R > 0$,

$$\|h(|w_1|) - h(|w_2|)\|_{r_*,\Gamma} \leq L(R)\|w_1 - w_2\|_{q_*,\Gamma} \quad \text{if } w_1, w_2 \in \overline{B}(L^{q_*}(\Gamma), 0, R), \tag{3.24}$$

where $L(R) := M_{q_*,s_*}[k, R] \leq N_{q_*,s_*}[k, R] \leq \|a_k\|_{s_*} + b_k R^{q_*/s_*}$. For brevity, we let

$$N_k(E) := \rho^{-1}C_1(\Omega, q_*)C_1(\Omega, r'_*)N_{q_*,s_*}\big[k, C_1(\Omega, q_*)E\big], \quad E > 0. \tag{3.25}$$

Theorem 3.12 *Suppose that h satisfies the Lipschitz condition* (3.22).
(a) *If $E > 0$ and*

$$\theta^{-1}N_k(E) + \theta^{-1}C_2E + C_2\Psi(\boldsymbol{v}_w) + \Phi(\boldsymbol{v}_*) < 2\nu C_K(\Omega), \tag{3.26}$$

then problem (NNF_0-VI) has at most one solution $\boldsymbol{V}$ such that $\|\boldsymbol{V}\|_{1,2} \leq E$.
(b) *Suppose that $C_2(\Omega)\Psi(\boldsymbol{v}_w) + \Phi(\boldsymbol{v}_*) < 2\nu C_K(\Omega)$ and*

$$\theta^{-1}N_k(E_0) + \theta^{-1}C_2E_0 + C_2\Psi(\boldsymbol{v}_w) + \Phi(\boldsymbol{v}_*) < 2\nu C_K(\Omega), \tag{3.27}$$

where $E_0 = E_0(\nu, \Omega, \boldsymbol{f}, \Gamma, \boldsymbol{v}_w, \Sigma, \boldsymbol{v}_*)$ is as in (3.14). *In addition, assume that $r > 4/3$ if $d = 3$. Then problem (NNF-VI_σ) has a unique solution.*

The next theorem extends Theorem 3.12(b) to the case when $d = 3$ and $r = 4/3$ under slightly different restrictions on the size of the data. Inequalities (3.27)–(3.29) hold if ν is sufficiently large, since $E_0 \longrightarrow \theta^{-1}\big(\Psi(\boldsymbol{v}_w) + \Psi(\boldsymbol{v}_*)\big)/C_K(\Omega)$ as $\nu \longrightarrow \infty$.

Theorem 3.13 *Suppose that h satisfies the Lipschitz condition* (3.22), $C_2(\Omega)\Psi(\boldsymbol{v}_w) + \Phi(\boldsymbol{v}_*) < 2\nu C_K(\Omega)$ *and*

$$C_2(\Omega)\big(E_0 + (2-\theta)\Psi(\boldsymbol{v}_w) + 2\Psi(\boldsymbol{v}_*)\big) \le 2(1-\theta)\nu C_K(\Omega) + \theta\Phi(\boldsymbol{v}_*), \qquad (3.28)$$

$$N_k(E_0) + 2C_2(\Omega)\big(E_0 + \Psi(\boldsymbol{v}_w) + \Psi(\boldsymbol{v}_*)\big) < 2\nu C_K(\Omega). \qquad (3.29)$$

Then problem (NNF-VI$_\sigma$) has a unique solution.

Inequality (3.27) is equivalent to $N_b(E_0)+N_k(E_0) < 2\nu C_K(\Omega)$, and inequality (3.28) is equivalent to $2C_2(\Omega)(E_0+\Psi(\boldsymbol{v}_w)+\Psi(\boldsymbol{v}_*)) \le N_b(E_0)$. Thus, (3.27) and (3.28) imply (3.29). Hence, Theorem 3.13 yields the following extension of Theorem 3.12(b):

Corollary 3.14 *Suppose that h satisfies the Lipschitz condition* (3.22), $C_2(\Omega)\Psi(\boldsymbol{v}_w) + \Phi(\boldsymbol{v}_*) < 2\nu C_K(\Omega)$ *and inequality* (3.27) *holds. In addition, if $d = 3$ assume that $r > 4/3$ or that inequality* (3.28) *holds. Then problem (NNF-VI$_\sigma$) has a unique solution.*

3.4 Continuous dependence on data

Let Ω, Γ, Σ, ν, μ, ρ, $\boldsymbol{f}$, $\boldsymbol{v}_w$, $\boldsymbol{v}_*$, g, h, r, q satisfy the hypotheses of Theorem 3.10(c) or Corollary 3.14 and let $(\boldsymbol{v}, p, \boldsymbol{\sigma})$ be the solution of problem (NNF-W). Furthermore, for the same Ω, Γ, Σ, suppose that for every i in some parameter set, ν_i, μ_i, ρ_i, $\boldsymbol{f}_i$, $\boldsymbol{v}_w^i$, $\boldsymbol{v}_*^i$, g_i, h_i, r_i, q_i satisfy the hypotheses of problem (NNF-W) and $(\boldsymbol{v}_i, p_i, \boldsymbol{\sigma}_i)$ is a solution of the corresponding problem (NNF-W). (We do not assume that $h_i(\boldsymbol{x}, \cdot)$ is monotone or Lipschitz continuous.)

Theorem 3.15 *Suppose that there exist fixed constants $q_0 \in I_d$ and $r_0 \in I_d'$ such that $\max(q, q_i) \le q_0$ and $r_0 \le \min(r, r_i)$ for all i, and that $|\nu_i - \nu|$, $|\rho_i - \rho|$, $\|\boldsymbol{f}_i - \boldsymbol{f}\|_{V,-1,2}$, $\|\boldsymbol{v}_w^i - \boldsymbol{v}_w\|_{1/2,2,\Gamma}$, $\|\boldsymbol{v}_*^i - \boldsymbol{v}_*\|_{1/2,2,\Sigma}$, $\|g_i - g\|_{r_0,\Gamma}$ and $N_{q_0,r_0}[h_i - h, 2C_1(\Omega, q_0)E_0]$ converge to zero as i passes to some limit. Then $\|\boldsymbol{v}_i - \boldsymbol{v}\|_{1,2}$ and $\|p_i - p\|_2$ converge to zero, and $\boldsymbol{\sigma}_i$ converges weakly to $\boldsymbol{\sigma}$ in $\boldsymbol{L}^t(\Gamma)$ for every $t \in (1, r_0] \cap (1, \infty)$.*

4 Example

In view of Navier's slip condition (1.7), let us consider the case when $h(\boldsymbol{x}, \cdot)$ is linear, i.e., $h(\boldsymbol{x}, u) = k(\boldsymbol{x})u$ for some function $k : \Gamma \to [0, \infty)$. As in [24, 25], we will call the corresponding slip condition (1.4) or (1.6) *Navier-Fujita slip* and denote the corresponding problems by (NF), (NF-W), etc.

1. First suppose that $\partial\Omega$, $\boldsymbol{f}$, $\boldsymbol{v}_*$, $\boldsymbol{v}_w$, g satisfy conditions (3.1) and that $k(\boldsymbol{x}) > 0$ for almost every $\boldsymbol{x} \in \Gamma$. In addition, assume that $k \in W^{1,2}(\Gamma)$ if $d = 2$, and assume that $k \in W^{1,s}(\Gamma)$ for some $s > 2$ if $d = 3$. Then $k \in L^\infty(\Gamma)$ and $k|\boldsymbol{v} - \boldsymbol{v}_w| \in W^{1,2}(\Gamma)$ for all $\boldsymbol{v} \in \boldsymbol{W}^{2,2}(\Omega)$. Thus, we can formulate problem (NF) in the same manner as Problem 3.1.

2. For the weak versions of problem (NF), suppose that $\partial\Omega$, $\boldsymbol{f}$, $\boldsymbol{v}_*$, $\boldsymbol{v}_w$, g satisfy conditions (3.2), $r \in I_d \cap I'_d$ and $k(\boldsymbol{x}) > 0$ for almost every $\boldsymbol{x} \in \Gamma$. In addition, assume that $k \in L^s(\Gamma)$ for some $s \in (r, \infty]$ if $d = 2$, and assume that $k \in L^s(\Gamma)$, $s := 4r/(4-r) \in [2, \infty]$, if $d = 3$. Furthermore, let $q := sr/(s-r)$ if $d = 2$ (thus $q = r$ if $s = \infty$) and $q := 4$ if $d = 3$. Then $q \in I_d$, $r \leq q$ and $1/q + 1/s = 1/r$. Thus, $k|\boldsymbol{v} - \boldsymbol{v}_w| \in L^r(\Gamma)$ for all $\boldsymbol{v} \in \boldsymbol{W}^{1,2}(\Omega)$. Hence, we can define the functional j_2 and formulate problems (NF-W), (NF-W$_\sigma$), (NF-VI) and (NF-VI$_\sigma$) in the same manner as Problems 3.2–3.5 (omit the terms involving the trilinear form).

The hypotheses of problem (NF) do not imply that $k(\cdot)u$ is continuously differentiable on Γ for all $u \in [0, \infty)$, and the hypotheses of problem (NF-W) with $s < \infty$ do not imply that $h(\boldsymbol{x}, u) = k(\boldsymbol{x})u$ satisfies the acting condition (3.3). Nonetheless, in both problem (NF) and problems (NF-W)–(NF-VI$_\sigma$), $k|\boldsymbol{v} - \boldsymbol{v}_w|$ belongs to the same space as g, and we can show the following:

- The assertions of Theorem 3.6 hold for problems (NF) and (NF-W)–(NF-VI$_\sigma$).
- The assertions of Theorem 3.8 hold for problem (NF$_0$-VI). Moreover, the solutions of problem (NF-W) satisfy the *a priori* estimates (3.15)–(3.16), and estimate (3.17) becomes $\|\boldsymbol{\sigma}\|_{r,\Gamma} \leq \|g\|_{r,\Gamma} + C_1(\Omega, q)\|k\|_{s,\Gamma}\|\boldsymbol{V}\|_{1,2}$.
- The assertions of Theorem 3.10 hold for problems (NF$_0$-VI) and (NF-VI$_\sigma$). The additional assumption in part (c) is not necessary in this case.
- Inequality (3.22) holds with $k(\boldsymbol{x}, \cdot) = k(\boldsymbol{x})$, and we may take $q_* = q$, $r_* = r$, $s_* = s$, $a_k = k$ and $b_k = 0$ in (3.23)–(3.25). Moreover, $M_{q,s}[k, R] = N_{q,s}[k, R] = \|k\|_{s,\Gamma}$ for all $R > 0$.
- The assertions of Theorems 3.12–3.13 and Corollary 3.14 also hold for problem (NF-VI$_\sigma$).
- A continuity result similar to Theorem 3.15 (with continuous dependence on k instead of h) holds for problem (NF-W); see [25].

5 Stokes problem

Analogues of all the preceding results hold for the corresponding Stokes boundary-value problem. The simplifications are similar to those in the Dirichlet case: we can weaken the smallness conditions and sharpen the *a priori* estimates. See [24] for the precise formulations and proofs.

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