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Every strongly definable C^rG vector bundle admits a unique strongly definable $C^\infty G$ vector bundle structure

Tomohiro KAWAKAMI*

Abstract

Let G be a compact subgroup of $GL_n(\mathbb{R})$. We prove that every strongly definable C^rG vector bundle over an affine definable $C^\infty G$ manifold admits a unique strongly definable $C^\infty G$ vector bundle structure up to definable $C^\infty G$ vector bundle isomorphism $(0 \leqq r < \infty)$.

1 Introduction

By [12], if s is a non-negative integer, then every C^s Nash map between affine Nash manifolds is approximated in the definable C^s topology by Nash maps. This definable C^s topology is a new topology defined in [12].

In this paper, G denotes a compact subgroup of $GL_n(\mathbb{R})$, every definable map is continuous and any manifold does not have boundary, unless otherwise stated. Under our assumption, G is a compact algebraic subgroup of $GL_n(\mathbb{R})$ (e.g. 2.2 [10]). We consider an equivariant definable version of the above theorem in an o-minimal expansion $\mathcal{M} = (\mathbb{R}, +, \cdot, <, \dots)$ of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of the field $\mathbb{R}$ of real numbers. General references on o-minimal structures are [1], [3], see also [13]. Further properties and constructions of them are studied in [2], [4], [11].

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*Department of Mathematics, Faculty of Education, Wakayama University, Sakaedani Wakayama 640-8510, Japan.

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We consider strongly definable $C^\infty G$ vector bundle structures of strongly definable $C^r G$ vector bundles ($0 \leqq r < \infty$).

Everything is considered in $\mathcal{M}$ and the term "definable" is used throughout in the sense of "definable with parameters in $\mathcal{M}$", each definable map is assumed to be continuous.

2 Preliminaries

An ordered structure $(R, <)$ with a dense linear order $<$ without endpoints is *o-minimal* (*order minimal*) if every definable set of R is a finite union of open intervals and points, where open interval means $(a, b), -\infty \leq a < b \leq \infty$.

If $(R, +, \cdot, <)$ is a real closed field, then it is o-minimal and the collection of definable sets coincides that of semialgebraic sets.

The topology of R is the interval topology and the topology of R^n is the product topology.

Let $X \subset R^n$ and $Y \subset R^m$ be definable sets. A continuous map $f : X \to Y$ is *definable* if the graph of f ($\subset X \times Y \subset R^n \times R^m$) is a definable set. A definable map $f : X \to Y$ is a *definable homeomorphism* if there exists a definable map $f' : Y \to X$ such that $f \circ f' = id_Y, f' \circ f = id_X$.

A group G is a *definable group* if G is a definable set and the group operations $G \times G \to G$ and $G \to G$ are definable.

Let G be a definable group. A pair (X, ϕ) consisting a definable set X and a G action $\phi : G \times X \to X$ is a *definable G set* if ϕ is definable. We simply write X instead of (X, ϕ) and gx instead of $\phi(g, x)$.

A definable map $f : X \to Y$ between definable G sets is a *definable G map* if for any $x \in X, g \in G$, $f(gx) = gf(x)$. A definable G map is a *definable G homeomorphism* if it is a homeomorphism.

Definition 1 A topological fiber bundle $\eta = (E, p, X, F, K)$ is called a *definable fiber bundle* over X with fiber F and structure group K if the following two conditions are satisfied:

(1) The total space E is a definable space, the base space X is a definable set, the structure group K is a definable group, the fiber F is a definable set with an effective definable K action, and the projection $p : E \to X$ is a definable map.

(2) There exists a finite family of local trivializations $\{U_i, \phi_i : p^{-1}(U_i) \to U_i \times F\}_i$ of η such that each U_i is a definable open subset of X, $\{U_i\}_i$ is a finite

open covering of X. For any $x \in U_i$, let $\phi_{i,x} : p^{-1}(x) \to F$, $\phi_{i,x}(z) = \pi_i \circ \phi_i(z)$, where π_i stands for the projection $U_i \times F \to F$. For any i and j with $U_i \cap U_j \neq \emptyset$, the transition function $\theta_{ij} := \phi_{j,x} \circ \phi_{i,x}^{-1} : U_i \cap U_j \to K$ is a definable map. We call these trivializations *definable*. Definable fiber bundles with compatible definable local trivializations are identified.

(3) A definable fiber bundle is a *definable vector bundle* if $F = \mathbb{R}^n$ and $K = GL(n, \mathbb{R})$.

Definition 2 (1) Let $0 \leqq r \leqq \infty$. A Hausdorff space X is an *n-dimensional definnable C^r manifold* if there exist a finite open cover $\{U_i\}_{i=1}^k$ of X, finite open sets $\{V_i\}_{i=1}^k$ of $\mathbb{R}^n$, and a finite collection of homeomorphisms $\{\phi_i : U_i \to V_i\}_{i=1}^k$ such that for any i, j with $U_i \cap U_j \neq \emptyset$, $\phi_i(U_i \cap U_j)$ is definable and $\phi_j \circ \phi_i^{-1} : \phi(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is a definable C^r diffeomorphism. This pair $(\{U_i\}_{i=1}^k, \{\phi_i : U_i \to V_i\}_{i=1}^k)$ of sets and homeomorphisms is called a *definable C^r coordinate system*.

(2) A definable C^r manifold G is a *definable C^r group* if G is a group and the group operations $G \times G \to G, G \to G$ are definable C^r maps

(3) Let G be a definable group. A pair (X, ϕ) consisting a definable C^r manifold X and a G action $\phi : G \times X \to X$ is a *definable C^rG manifold* if ϕ is a definable C^r map. We simply write X instead of (X, ϕ) and gx instead of $\phi(g, x)$.

Definition 3 ([6]) Let G be a definable C^r group and $0 \leqq r \leqq \infty$.
(1) A *definable C^rG vector bundle* is a definable C^r vector bundle $\eta = (E, p, X)$ satisfying the following three conditions.

(a) The total space E and the base space X are definable C^rG manifolds.

(b) The projection $p : E \to X$ is a definable C^rG map.

(c) For any $x \in X$ and $g \in G$, the map $p^{-1}(x) \to p^{-1}(gx)$ is linear.

(2) Let η and ζ be definable C^rG vector bundles over X. A definable C^r vector bundle morphism $\eta \to \zeta$ is called a *definable C^rG vector bundle morphism* if it is a G map. A definable C^rG vector bundle morphism $f : \eta \to \zeta$ is said to be a *definable C^rG vector bundle isomorphism* if there exists a definable C^rG vector bundle morphism $h : \zeta \to \eta$ such that $f \circ h = id$ and $h \circ f = id$. If $r = 0$, then a definable C^0G vector bundle (resp. a definable C^0G vector bundle morphism, a definable C^0G vector bundle isomorphism) is simply called a *definable G vector bundle* (resp. a *definable G vector bundle morphism*, a *definable G vector bundle isomorphism*).

(3) A definable C^r section of a definable C^rG vector bundle is a *definable C^rG section* if it is a G map.

Definition 4 ([8], [6]) Let $0 \leqq r \leqq \infty$.

(1) A group homomorphism (resp. A group isomorphism) from G to $O_n(\mathbb{R})$ is a *definable group homomorphism* (resp. a *definable group isomorphism*) if it is a definable map (resp. a definable homeomorphism).

Note that a definable group homomorphism (resp. a definable group isomorphism) between G and $O_n(\mathbb{R})$ is a definable C^∞ map (resp. a definable C^∞ diffeomorphism) because G and $O_n(\mathbb{R})$ are Lie groups.

(2) An *n-dimensional representation* of G means $\mathbb{R}^n$ with the linear action induced by a definable group homomorphism from G to $O_n(\mathbb{R})$. In this paper, we assume that every representation of G is orthogonal.

(3) A definable C^r submanifold of a definable C^rG manifold X is called a *definable C^rG submanifold* of X if it is G invariant.

(4) A definable C^rG manifold is called *affine* if it is definably C^rG diffeomorphic (definably G homeomorphic if $r = 0$) to a definable C^rG submanifold of some representation of G.

(5) A *definable C^rG manifold with boundary* is defined similarly.

If $0 \leqq r < \infty$, then every definable C^r manifold is affine ([8], [7]) and if $\mathcal{M}$ is exponential, then each compact definable $C^\infty G$ manifold is affine [8].

Recall universal G vector bundles (e.g. [6]) and existence of a Nash G tubular neighborhood of a Nash G submanifold of a representation of G ([9]).

Let Ω be an n-dimensional representation of G induced by a definable group homomorphism $B : G \to O_n(\mathbb{R})$. Suppose that $M(\Omega)$ denotes the vector space of $n \times n$ matrices with the action $(g, A) \in G \times M(\Omega) \mapsto B(g)AB(g)^{-1} \in M(\Omega)$. For any positive integer k, we define the vector bundle $\gamma(\Omega, k) = (E(\Omega, k), u, G(\Omega, k)$ as follows:

$$G(\Omega, k) = \{A \in M(\Omega) | A^2 = A, {}^t A = A, TrA = k\},$$

$$E(\Omega, k) = \{(A, v) \in G(\Omega) \times \Omega | Av = v\},$$
$$u : E(\Omega, k) \to G(\Omega, k), u((A; v)) = A,$$

where ${}^tA$ denotes the transposed matrix of A and TrA stands for the trace of A. Then $\gamma(\Omega, k)$ is an algebraic vector bundle. Since the action on $\gamma(\Omega, k)$

is algebraic, it is an algebraic G vector bundle. We call it the *universal G vector bundle* associated with Ω and k. Remark that $G(\Omega, k) \subset M(\Omega)$ and $E(\Omega, k) \subset M(\Omega) \times \Omega$ are nonsingular algebraic G sets. In particular, they are Nash G submanifolds of $M(\Omega)$ and $M(\Omega) \times \Omega$, respectively.

Theorem 5 ([9]) *Every Nash G submanifold X of a representation Ω of G has a Nash G tubular neighborhood (U, θ) of X in Ω.*

Definition 6 ([6]) (1) Let G be a definable group. A definable G vector bundle $\eta = (E, p, X)$ over a definable G set X is called *strongly definable* if there exist a representation Ω of G and a definable G map $f : X \to G(\Omega, k)$ such that η is definably G vector bundle isomorphic to $f^*(\gamma(\Omega, k))$, where k denotes the rank of η.

(2) Let G be a definable C^r group and $0 \leqq r \leqq \infty$. A definable C^rG vector bundle $\eta = (E, p, X)$ over an affine definable C^rG manifold X is called *strongly definable* if there exist a representation Ω of G and a definable C^rG map $f : X \to G(\Omega, k)$ such that η is definably C^rG vector bundle isomorphic to $f^*(\gamma(\Omega, k))$, where k denotes the rank of η.

3 Our results

Theorem 7 ([5]) *If $0 \leqq s < \infty$ and M admits C^∞ cell decomposition and exponential, then every definable C^sG map between affine definable $C^\infty G$ manifolds is approximated in the definable C^s topology by definable $C^\infty G$ maps.*

Our main result is the following.

Theorem 8 ([5]) *Let X be an affine definable $C^\infty G$ manifold and M admits C^∞ cell decomposition and exponential. If $0 \leqq r < \infty$, then every strongly definable C^rG vector bundle over X admits a unique strongly definable $C^\infty G$ vector bundle structure up to definable $C^\infty G$ vector bundle isomorphism.*

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