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A survey of undecidability problems of rings of totally real algebraic integers

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Abstract Let $\mathbb{Z}^{tr}$ be the ring of all totally real algebraic integers in $\mathbb{C}$. We consider (un)decidability of its subrings of infinite degree over $\mathbb{Q}$. Julia Robinson [Ro] proved that $\mathbb{Z}$ is first order definable (without parameters) in $\mathbb{Z}^{tr}$, thus showed that it is undecidable. Moreover she showed undecidability of the rings of (algebraic) integers of any subfield of $\mathbb{Q}(\{\sqrt{p} \mid p \text{ prime}\})$ also by showing the definability of $\mathbb{Z}$ in those rings. From her remark in [Ro], it seems that we may conjecture that all subrings of $\mathbb{Z}^{tr}$ are undecidable. We survey recent progress on this problem. We note that rings of algebraic integers of finite degree over $\mathbb{Q}$ are undecidable. This fact is also proved in [Ro].

1 A method of Julia Robinson

Let $R \subset \mathbb{Z}^{tr}$ be a ring of totally real integers. To a formula $\varphi(x, \bar{y})$ (where $\bar{y} = (y_1, \ldots, y_n)$) in the ring language L we can define a family $\{\varphi(x, \bar{r}) \mid r \in R^n\}$ of subsets R where $\varphi(x, \bar{y}) = \{s \in R \mid R \models \varphi(s, \bar{r}\}$. In her 1962 paper *On the decision problem for algebraic rings* [Ro], she proved the following.

Proposition 1. *Let $R \subset \mathbb{Z}^{tr}$ be a ring and suppose that there is a family as above containing finite sets of arbitrary large size. Then $\mathbb{Z}$ is first order definable (without parameters) in R.*

For details see [Ro] and [JV].

In order to define such family, she used the following Siegel's theorem.

For an algebraic number x, x is totally positive iff x is a sum of four squares in $\mathbb{Q}(x)$.

An algebraic number is said to be totally positive if each conjugate of x is positive.

Corollary 2. *Let $R \subset \mathbb{Z}^{tr}$ be a ring and suppose that there is a smallest interval $(0, s)$, s real or ∞, which contains infinitely many sets f conjugates of integers of R. Then $\mathbb{Z}$ is definable in R, hence R is undecidable.*

She applied this corollary to the following cases.

For $R = \mathbb{Z}^{tr}$ she put $0 \ll y_1 x \ll y_2$ as $\varphi(x, y_1, y_2)$ where

$$x \ll y \Leftrightarrow \exists t, u, v, w, z[t^2(y - x) = u^2 + v^2 + w^2 + z^2 \wedge t \neq 0].$$

This means that $y - x$ is totally non-negative, which is first order definable in R by the Siegel's result. It follows from a theorem of Kronecker that the interval $(0, 4)$ contains infinitely many sets of conjugates of totally real algebraic integers and no sub-intervals does. We can take positive integers y_1, y_2 so that y_2/y_1 is as close as we

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like to but less than 4. Then this family contains finite sets of arbitrary large size. Thus $\mathbb{Z}$ is first order definable (without parameters) in $R = \mathbb{Z}^{tr}$.

For the rings of integers R of any subfield of $\mathbb{Q}(\{\sqrt{p}\,|\,p \text{ prime}\})$ she put $0 \ll x \ll y$ as $\varphi(x, y)$. It can be shown that this family contains finite sets of arbitrary large size. Thus $\mathbb{Z}$ is definable in R.

2 Julia Robinson number

We noticed that intervals Julia Robinson used are $(0, 4)$ and $(0, +\infty)$. In [Ro], after Corollary 2, she remarked *"This condition may in fact hold for all totally real algebraic integer rings"*.

Unfortunately, up to 2015, no rings $R \subset \mathbb{Z}^{tr}$ satisfying this condition with the intervals $(0, s), s \neq 4, +\infty$ are known.

Vidaux and Videla [VV] defined *Julia Robinson number* of R.

For $r \in R$ and $a, b \in \mathbb{R} \cup \{\pm\infty\}$, let $a \prec r \prec b$ mean that r and all its conjugates are strictly between a and b. For $t \in \mathbb{R}$ positive, write

$$R_t = \{r \in R \,|\, 0 \prec r \prec t\}.$$

They define the *Julia Robinson number* of R to be

$$\mathrm{JR}(R) = \inf A(R),$$

where

$$A(R) = \{t \in \mathbb{R} \cup \{\pm\infty\} \mid R_t \text{ is infinite}\}.$$

We notice that $A(R)$ is either the singleton $\{+\infty\}$ or an interval: $A(\mathbb{Z}^{tr})$ is the interval $[4, +\infty)$ and $A(R_0) = +\infty$ where R_0 is the ring of integers of $\mathbb{Q}(\{\sqrt{p} \mid p \text{ prime}\})$. R is said to have the Julia Robinson Property if $\mathrm{JR}(R) \in A(R)$, that is, if $A(R)$ is a closed interval $[\mathrm{JR}(R), +\infty)$ or $\{+\infty\}$. Thus $\mathrm{JR}(\mathbb{Z}^{tr}) = 4$ and $\mathrm{JR}(R_0) = +\infty$.

If a ring $R \subset \mathbb{Z}^{tr}$ has the Julia Robinson Property, then we can prove that $\mathbb{Z}$ is definable by the arguments of Julia Robinson.

Vidaiux and Videla, in their 2015 paper *Definability of the natural numbers in totally real towers of nested square roots* [VV], constructed an infinite family of subrings of such rings for which JR number is strictly between 4 and $+\infty$, thus they are undecidable.

Remark.

1. Also in 2015, they [VV2] proved that the compositum of all totally real abelian extensions of $\mathbb{Q}$ of bounded degree d is undecidable, showing that its JR number is $+\infty$.

2. In 2008, Jarden and Videla [JV] proved that certain families of subrings of $\mathbb{Z}^{tr}$ are undecidable showing that the theory of finite graphs is interpretable in those rings. (The theory of finite graph is undecidable.)

References

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