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Definable sets in weakly o-minimal structures with the strong cell decomposition property

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Abstract

In this paper, we study the definable sets in weakly o-minimal structures with the strong cell decomposition property.

Throughout this paper, "definable" means "definable possibly with parameters" and we assume that a structure $\mathcal{M} = (M, <, \dots)$ is a dense linear ordering $<$ without endpoints.

A subset A of M is said to be *convex* if $a, b \in A$ and $c \in M$ with $a < c < b$ then $c \in A$. Moreover if $A = \emptyset$ or $\inf A, \sup A \in M \cup \{-\infty, +\infty\}$, then A is called an *interval* in M. We say that $\mathcal{M}$ is *o-minimal* (*weakly o-minimal*) if every definable subset of M is a finite union of intervals (convex sets), respectively. A theory T is said to be *weakly o-minimal* if every model of T is weakly o-minimal. The reader is assumed to be familiar with fundamental results of o-minimality and weak o-minimality; see, for example, [1], [2], [3], or [5].

For any subsets C, D of M, we write $C < D$ if $c < d$ whenever $c \in C$ and $d \in D$. A pair $\langle C, D\rangle$ of non-empty subsets of M is called a *cut* in M if $C < D, C \cup D = M$ and D has no lowest element. A cut $\langle C, D\rangle$ is said to be *definable* in $\mathcal{M}$ if the sets C, D are definable in $\mathcal{M}$. The set of all cuts

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definable in $\mathcal{M}$ will be denoted by $\overline{M}$. Note that we have $M = \overline{M}$ if $\mathcal{M}$ is o-minimal. We define a linear ordering on $\overline{M}$ by $\langle C_1, D_1\rangle < \langle C_2, D_2\rangle$ if and only if $C_1 \subsetneq C_2$. Then we may treat $(M, <)$ as a substructure of $(\overline{M}, <)$ by identifying an element $a \in M$ with the definable cut $\langle(-\infty, a], (a, +\infty)\rangle$.

We equip M ($\overline{M}$) with the *interval topology* (the open intervals form a base), and each product M^n ($\overline{M}^n$) with the corresponding product topology, respectively.

Recall the notion of definable functions from [5]. Let n be a positive integer and $A \subseteq M^n$ definable. A function $f : A \to \overline{M}$ is said to be *definable* if the set $\{\langle x, y\rangle \in M^{n+1} : x \in A,\ y < f(x)\}$ is definable. A function $f : A \to \overline{M} \cup \{-\infty, +\infty\}$ is said to be *definable* if f is a definable function from A to $\overline{M}$, $f(x) = -\infty$ for all $x \in A$, or $f(x) = +\infty$ for all $x \in A$.

Recall the notion of (refined) strong cells from [6].

Definition 1. Suppose that $\mathcal{M} = (M, <, \ldots)$ is a weakly o-minimal structure. For each positive integer n, we inductively define (*refined*) *strong cells* in M^n and their completions in $\overline{M}^n$.

(1) A one-element subset of M is called a *strong* 0-*cell* in M. If $C \subseteq M$ is a strong 0-cell, then its completion $\overline{C} := C$.

(2) A non-empty definable convex open subset of M is called a *strong* 1-*cell* in M. If $C \subseteq M$ is a strong 1-cell, then its completion $\overline{C} := \{x \in \overline{M} : (\exists a, b \in C)(a < x < b)\}$.

Assume that k is a non-negative integer, and strong k-cells in M^n and their completions in $\overline{M}^n$ are already defined.

(3) Let $C \subseteq M^n$ be a strong k-cell in M^n and $f : C \to M$ is a definable continuous function which has a continuous extension $\overline{f} : \overline{C} \to \overline{M}$. Then the graph $\Gamma(f)$ is called a *strong* k-*cell* in M^{n+1} and its completion $\overline{\Gamma(f)} := \Gamma(\overline{f})$.

(4) Let $C \subseteq M^n$ be a strong k-cell in M^n and $g, h : C \to \overline{M} \cup \{-\infty, +\infty\}$ are definable continuous functions which have continuous extensions $\overline{g}, \overline{h} : \overline{C} \to \overline{M} \cup \{-\infty, +\infty\}$ such that

(a) each of the functions g, h assumes all its values in one of the sets M, $\overline{M} \setminus M$, $\{\infty\}$, $\{-\infty\}$,

(b) $\overline{g}(x) < \overline{h}(x)$ for all $x \in \overline{C}$.

Then the set

$$(g, h)_C := \{\langle a, b\rangle \in C \times M : g(a) < b < h(a)\}$$

is called a *strong* $(k+1)$*-cell* in M^{n+1}. The completion of $(g, h)_C$ is defined as

$$\overline{(g, h)_C} := \{\langle a, b\rangle \in \overline{C} \times \overline{M} : \overline{g}(a) < b < \overline{h}(a)\}.$$

(5) Let C be a subset of M^n. The set C is called a *strong cell* in M^n if there exists some non-negative integer k such that C is a strong k-cell in M^n.

Let C be a strong cell of M^n. A definable function $f : C \to \overline{M}$ is said to be *strongly continuous* if f has a continuous extension $\overline{f} : \overline{C} \to \overline{M}$. A function which is identically equal to $-\infty$ or $+\infty$, and whose domain is a strong cell is also said to be *strongly continuous.*

Definition 2. Let $\mathcal{M} = (M, <, \ldots)$ be a weakly o-minimal structure. For each positive integer n, we inductively define a *strong cell decomposition* (or a *decomposition into strong cells*) in M^n of a non-empty definable set $A \subseteq M^n$.

(1) If $A \subseteq M$ is a non-empty definable set and $\mathcal{D} = \{C_1, \ldots, C_k\}$ is a partition of A into strong cells in M, then $\mathcal{D}$ is called a *decomposition of A into strong cells* in M.

(2) Suppose that $A \subseteq M^{n+1}$ is a non-empty definable set and $\mathcal{D} = \{C_1, \ldots, C_k\}$ is a partition of A into strong cells in M^{n+1}. Then $\mathcal{D}$ is called a *decomposition of A into strong cells* in M^{n+1} if $\{\pi(C_1), \ldots, \pi(C_k)\}$ is a decomposition of $\pi(A)$ into strong cells in M^n, where $\pi : M^{n+1} \to M^n$ is the projection on the first n coordinates.

Definition 3. Let $\mathcal{M} = (M, <, \ldots)$ be a weakly o-minimal structure and n a positive integer. Suppose that $A, B \subseteq M^n$ are definable sets, $A \neq \emptyset$ and $\mathcal{D}$ is a decomposition of A into strong cells in M^n. We say that $\mathcal{D}$ *partitions* B if for each strong cell $C \in \mathcal{D}$, we have either $C \subseteq B$ or $C \cap B = \emptyset$.

Definition 4. A weakly o-minimal structure $\mathcal{M} = (M, <, \ldots)$ is said to have the *strong cell decomposition property* if for any positive integers k, n and any definable sets $A_1, \ldots, A_k \subseteq M^n$, there exists a decomposition of M^n into strong cells partitioning each of the sets $A_1, \ldots, A_k$.

Let $\mathcal{C}$, $\mathcal{D}$ be strong cell decompositions of M^m. We denote $\mathcal{C} \prec \mathcal{D}$ if every strong cell of $\mathcal{D}$ is a subset of some strong cell of $\mathcal{C}$. Then, the relation $\prec$ is a partial order on the family of all strong cell decompositions of M^m.

Lemma 5 ([6, Fact 2.1]). *If $X_1, \ldots, X_k \subseteq M^m$ are definable sets, then there exists the smallest strong cell decomposition $\mathcal{C}$ of M^m partitioning each of $X_1, \ldots, X_k$.*

Definition 6 ([4, Definition 3.1]). Let X be a definable subset of M^m and $\mathcal{C}$ the smallest strong cell decomposition of M^m partitioning X. Then we set the completion of X in $\overline{M}^m$ as $\overline{X} := \bigcup\{\overline{C} : C \in \mathcal{C} \wedge C \subseteq X\}$.

Let $\mathcal{M} = (M, <, +, \ldots)$ be a weakly o-minimal expansion of an ordered abelian group $(M, <, +)$. Then, the weakly o-minimal structure $\mathcal{M}$ is said to be *non-valuational* if for any definable cut $\langle C, D\rangle$ we have $\inf\{d - c : c \in C,\ d \in D\} = 0$.

Then, the following facts hold.

Fact 7 ([5, Fact 2.5]). *Let $\mathcal{M} = (M, <, \ldots)$ be a weakly o-minimal structure with the strong cell decomposition property. Suppose that $X \subseteq M^n$ is definable and $f : X \to \overline{M}$ is definable. Then, there is a decomposition $\mathcal{D}$ of X into strong cells in M^n such that for every $D \in \mathcal{D}$,*

1. *$f|_D$ assumes all its values in one of the sets M, $\overline{M} \setminus M$,*

2. *$f|_D$ is strongly continuous.*

Fact 8 ([5, Corollary 2.16]). *Let $\mathcal{M} = (M, <, +, \dots)$ be a weakly o-minimal expansion of an ordered abelian group $(M, <, +)$. Then the following conditions are equivalent.*

1. *$\mathcal{M}$ is non-valuational.*

2. *$\mathcal{M}$ has the strong cell decomposition property.*

Let $\mathcal{M}$ be a weakly o-minimal structure with the strong cell decomposition property. For any strong cell $C \subseteq M^m$, we denote by $\overline{R}_C$ the m-ary relation determined by $\overline{C}$, i.e. if $a \in \overline{M}^m$, then $\overline{R}_C(a)$ holds iff $a \in \overline{C}$. We define the structure $\overline{\mathcal{M}} := (\overline{M}, <, (\overline{R}_C : C \text{ is a strong cell}))$. The following fact is known.

Fact 9 ([5]). *Let $\mathcal{M}$ be a weakly o-minimal structure with the strong cell decomposition property. Then, $\overline{\mathcal{M}}$ is o-minimal, and every set $X \subseteq \overline{M}^m$ definable in $\overline{\mathcal{M}}$ is a finite Boolean combination of completions of strong cells in M^m.*

Remark 10. *Let $\mathcal{M} = (M, <, \dots)$ be a weakly o-minimal structure with the strong cell decomposition property. Then, the following hold.*

1. *There exist strong cells C, D_1, D_2 such that $C = D_1 \cup D_2$ but $\overline{C} \neq \overline{D}_1 \cup \overline{D}_2$.*

2. *There exist strong cells C, D such that $C \subseteq D$ but $\overline{C} \not\subseteq \overline{D}$.*

Proposition 11. *Let $\mathcal{M} = (M, <, \dots)$ be a weakly o-minimal structure with the strong cell decomposition property. Then, there exist some strong cells C, D and some strongly continuous function $f : D \to \overline{M}$ such that $C \subseteq D$ and $f|_C : C \to \overline{M}$ is not strongly continuous.*

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