Double of boundary singularity of stable map from 3-manifold with boundary to 2-manifold

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1 Introduction

We consider singularities of the smooth map obtained as the "double" of a stable map from a 3-manifold with boundary to a 2-manifold without boundary. Still, we restrict our attention to local theory, and hence take a map between Euclidean spaces. Let (x, y, z) be a coordinate system of $\mathbb{R}^3$, let $\mathbb{R}^3_{\geq 0}$ denote the half space $\{z \geq 0\}$ in $\mathbb{R}^3$, and let $f\colon \mathbb{R}^3_{\geq 0} \to \mathbb{R}^2$ be a smooth map. By the "double" of f, we mean the map $F\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined as $F(x, y, z) = f(x, y, z^2)$, which is clearly smooth. Note that, in the exterior of $\partial\mathbb{R}^3_{\geq 0}$, the transformation: $(x, y, z) \mapsto (x, y, z^2)$ is diffeomorphic at each point, and hence, the doubled map F inherits the types of singularities from the original map f. It might be naively hoped that, if a point p in $\partial\mathbb{R}^3_{\geq 0}$ is a stable boundary singular point of f, then p is a stable singular point of F. In this paper, we prove it for some types of stable boundary singular points, and disprove it for the other type.

Proposition 1. *With the above notation, we have the following.*

- *If p is a boundary regular point of f, then p is a regular point of F.*
- *If p is a boundary definite fold point of f, then p is a definite fold point of F.*
- *If p is a boundary indefinite fold point of f, then p is an indefinite fold point of F.*
- *If p is a boundary cusp point of f, then p is a cusp point of F.*
- *If p is a $\Sigma^{2,0}_{1,0}$ point of f, then p is an unstable singular point of F.*

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2 Preliminaries

In this section, we review standard definitions and facts with the following notation. Let M be a 3-dimensional C^∞ manifold possibly with boundary, N be a 2-dimensional

C^∞ manifold without boundary, and $f\colon M \to N$ be a C^∞ map. Let p be a point in M, and U be a sufficiently small neighborhood of p in M.

Singularity and boundary singularity are defined as follows. The point p is said to be a *regular point* of f if the differential $(df)_p : T_pM \to T_{f(p)}N$ is surjective, and a *singular point* of f otherwise. The point p is said to be a *boundary regular point* of f if $p \in \partial M$ and the differential $(d(f|_{\partial M}))_p : T_p\partial M \to T_{f(p)}N$ is surjective, and a *boundary singular point* of f otherwise. The set of singular points (resp. boundary singular points) of f is called the *singular set* (resp. the *boundary singular set*) of f, and denoted by $S(f)$ (resp. $S(f|_{\partial M})$).

Fold singularity is defined as follows. The point p is said to be a *fold point* of f if there are a local coordinate system (u, v, w) of M and one of N with respect to which $p = (0, 0, 0)$, $f(p) = (0, 0)$ and $f(u, v, w) = (u, v^2 + \varepsilon w^2)$ for $\varepsilon \in \{1, -1\}$. In particular, the fold point p is said to be *definite* if $\varepsilon = 1$, and *indefinite* if $\varepsilon = -1$. If p is an interior point of M, the singular set $S(f) \cap U$ is a regular arc which passes through p and consists only of fold points.

Cusp singularity is defined as follows. The point p is said to be a *cusp point* of f if there are a local coordinate system (u, v, w) of M and one of N with respect to which $p = (0, 0, 0)$, $f(p) = (0, 0)$ and $f(u, v, w) = (u, v^3 + uv + w^2)$. If p is an interior point of M, the singular set $S(f) \cap U$ is a regular arc which passes through p and consists only of fold points except for the cusp point p.

Boundary fold singularity is defined as follows. The point p is said to be a *boundary fold point* of f if there are a local coordinate system (u, v, w) of M and one of N with respect to which $p = (0, 0, 0)$, $f(p) = (0, 0)$, $M = \{w \geq 0\}$ and $f(u, v, w) = (u, v^2 + \varepsilon w)$ for $\varepsilon \in \{1, -1\}$. In particular, the boundary fold point p is said to be *definite* if $\varepsilon = 1$, and *indefinite* if $\varepsilon = -1$. Note that p is a boundary singular point but a regular point of f. The singular set $S(f) \cap U$ is empty, and the boundary singular set $S(f|_{\partial M}) \cap U$ is a regular arc which passes through p and consists only of boundary fold points.

Boundary cusp singularity is defined as follows. The point p is said to be a *boundary cusp point* of f if there are a local coordinate system (u, v, w) of M and one of N with respect to which $p = (0, 0, 0)$, $f(p) = (0, 0)$, $M = \{w \geq 0\}$ and $f(u, v, w) = (u, v^3 + uv + w)$. Note that p is a boundary singular point but a regular point of f. The singular set $S(f) \cap U$ is empty, and the boundary singular set $S(f|_{\partial M}) \cap U$ is the regular arc $\{3v^2 + u = w = 0\}$. This arc passes through p, and consists only of boundary fold points except for the boundary cusp point p.

$\Sigma^{2,0}_{1,0}$ singularity is defined as follows. The point p is said to be a $\Sigma^{2,0}_{1,0}$ *point* of f if there are a local coordinate system (u, v, w) of M and one of N with respect to which $p = (0, 0, 0)$, $f(p) = (0, 0)$, $M = \{w \geq 0\}$ and $f(u, v, w) = (u, v^2 + uw + \varepsilon w^2)$ for $\varepsilon \in \{1, -1\}$. Note that p is a boundary singular point and a fold point of f. The singular set $S(f) \cap U$ is a regular arc which has an endpoint at p and consists only of fold points. The boundary singular set $S(f|_{\partial M}) \cap U$ is a regular arc which passes through p and consists only of boundary fold points except for the $\Sigma^{2,0}_{1,0}$ point p.

The above singularities and boundary singularities are stable. Suppose that f is a stable map (see [1] for example). It is well known that any singular point of f is either a fold point or a cusp point. It follows from the results of Martins–Nabarro [2] and Shibata [5] that any boundary singular point of f is either a boundary fold point, a boundary cusp point, or a $\Sigma^{2,0}_{1,0}$ point. We refer the reader to [3] for more information.

Fold and cusp singularities can be recognized with the following criteria. Suppose that p is an interior point of M, and f has a local form: $f(x) = (f_1(x), f_2(x))$ for $x \in U$ such that $(df_1)_p \neq (0,0,0)$ and $(df_2)_p = (0,0,0)$. This implies that $\ker (df)_p$ has dimension 2. For C^∞ vector fields ξ_1 and ξ_2 on U, let $\mathbf{H}_{\xi_1,\xi_2} f_2$ denote the matrix

$$\begin{pmatrix} \xi_1\xi_1 f_2 & \xi_1\xi_2 f_2 \\ \xi_2\xi_1 f_2 & \xi_2\xi_2 f_2 \end{pmatrix}.$$

Provided that the vectors $(\xi_1)_p$ and $(\xi_2)_p$ are linearly independent, we regard $(\mathbf{H}_{\xi_1,\xi_2} f_2)_p$ as representing a linear transformation of $\left\langle (\xi_1)_p, (\xi_2)_p \right\rangle$ with respect to the basis $\left((\xi_1)_p, (\xi_2)_p \right)$. This allows us to treat $\ker (\mathbf{H}_{\xi_1,\xi_2} f_2)_p$ as a subspace of $\left\langle (\xi_1)_p, (\xi_2)_p \right\rangle$. Saji [4] gave criteria for recognizing general Morin singularities, and the following are those in special cases.

Theorem 2 (Saji). *The point p is a fold point of f if there exist C^∞ vector fields η_1 and η_2 on U such that*

- $\ker (df)_p = \left\langle (\eta_1)_p, (\eta_2)_p \right\rangle$,
- $\ker (\mathbf{H}_{\eta_1,\eta_2} f_2)_p = \{\mathbf{0}\}$.

Moreover, the fold point p is definite (resp. indefinite) if $(\mathbf{H}_{\eta_1,\eta_2} f_2)_p$ has eigenvalues of definite (resp. indefinite) sign.

Theorem 3 (Saji). *The point p is a cusp point of f if there exist C^∞ vector fields η_1 and η_2 on U such that*

- $\ker (df)_p = \left\langle (\eta_1)_p, (\eta_2)_p \right\rangle$,
- $(\eta_1)_q \in \ker (df)_q$ *for* $q \in S(f) \cap U$,
- $\ker (\mathbf{H}_{\eta_1,\eta_2} f_2)_p = \left\langle (\eta_1)_p \right\rangle$,
- $(d(\eta_1 f_2))_p \neq (0,0,0)$,
- $(\eta_1\eta_1\eta_1 f_2)_p \neq 0$.

3 Proof

In this section, we give a proof of Proposition 1. We use the notation of Introduction.

3.1 Regular case

The first assertion of the proposition can be proved almost immediately as follows. Suppose that p is a boundary regular point of f. By the definition, the differential $\left(d \left(f|_{\partial \mathbb{R}^3_{\geq 0}} \right) \right)_p$ is surjective. Since $\partial \mathbb{R}^3_{\geq 0} = \{z = 0\}$ and $F(x,y,z) = f(x,y,z^2)$, the maps f and F coincide in $\partial \mathbb{R}^3_{\geq 0}$, and hence $\left(d \left(F|_{\partial \mathbb{R}^3_{\geq 0}} \right) \right)_p$ is surjective. It implies that $(dF)_p$ is also surjective. Thus, p is a regular point of F.

3.2 Fold case

In this subsection, we give proofs of the second and third assertions of the proposition. Suppose that p is a boundary fold point of f.

The original map and the doubled map have local forms as follows. On one hand, there are local coordinate systems (u, v, w) and (s, t) of $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively, with respect to which $p = (0, 0, 0)$, $f(p) = (0, 0)$, $\mathbb{R}^3_{\geq 0} = \{w \geq 0\}$ and $f(u, v, w) = (u, v^2 + \varepsilon w)$, where $\varepsilon = 1$ if the boundary fold point p is definite and $\varepsilon = -1$ if indefinite. On the other hand, $F(x, y, z) = f(x, y, z^2)$ with respect to the coordinate system (x, y, z) of $\mathbb{R}^3$. We may suppose that $p = (0, 0, 0)$ with respect to (x, y, z). Suppose that F has a local form: $F(x, y, z) = (F_1(x, y, z), F_2(x, y, z))$ with respect to (x, y, z) and (s, t).

The relevant coordinate systems are related as follows. There is a coordinate transformation: $(x, y, z) \mapsto (u(x, y, z), v(x, y, z), w(x, y, z))$. Since $\{z \geq 0\} = \{w \geq 0\}$ and $p \in \{z = 0\} = \{w = 0\}$, the transformation satisfies the conditions that

$$\left(\frac{\partial u}{\partial x}\right)_p \left(\frac{\partial v}{\partial y}\right)_p - \left(\frac{\partial u}{\partial y}\right)_p \left(\frac{\partial v}{\partial x}\right)_p \neq 0,$$

$$\left(\frac{\partial w}{\partial x}\right)_p = \left(\frac{\partial w}{\partial y}\right)_p = \left(\frac{\partial^2 w}{\partial x^2}\right)_p = \left(\frac{\partial^2 w}{\partial y^2}\right)_p = \left(\frac{\partial^2 w}{\partial x \partial y}\right)_p = 0,$$

$$\left(\frac{\partial w}{\partial z}\right)_p > 0.$$

In particular, the top inequality implies that

$$\left(\left(\frac{\partial u}{\partial x}\right)_p, \left(\frac{\partial u}{\partial y}\right)_p\right) \neq (0, 0).$$

We calculate partial derivatives with respect to the coordinates as follows. Note that F_1 and F_2 have the local forms: $F_1(x, y, z) = u(x, y, z^2)$ and $F_2(x, y, z) = (v^2 + \varepsilon w)(x, y, z^2)$, respectively, under the coordinate transformation: $(x, y, z) \mapsto (u, v, w)$. By the chain rule, for example,

$$\begin{aligned}
&\frac{\partial F_2}{\partial z}(x, y, z) \\
&= \frac{\partial}{\partial z}\left((v^2 + \varepsilon w)(x, y, z^2)\right) \\
&= \left(\frac{\partial}{\partial z} x\right)\left(\left(\frac{\partial}{\partial x}(v^2 + \varepsilon w)\right)(x, y, z^2)\right) + \left(\frac{\partial}{\partial z} y\right)\left(\left(\frac{\partial}{\partial y}(v^2 + \varepsilon w)\right)(x, y, z^2)\right) \\
&\quad + \left(\frac{\partial}{\partial z} z^2\right)\left(\left(\frac{\partial}{\partial z}(v^2 + \varepsilon w)\right)(x, y, z^2)\right) \\
&= 2z\left(\left(\frac{\partial}{\partial z}(v^2 + \varepsilon w)\right)(x, y, z^2)\right) \\
&= 2z\left(\left(2v\frac{\partial v}{\partial z} + \varepsilon\frac{\partial w}{\partial z}\right)(x, y, z^2)\right),
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial^2 F_2}{\partial z^2}(x,y,z) \\
&= \frac{\partial}{\partial z}\left(2z\left(\left(2v\frac{\partial v}{\partial z}+\varepsilon\frac{\partial w}{\partial z}\right)(x,y,z^2)\right)\right) \\
&= 2\left(2v\frac{\partial v}{\partial z}+\varepsilon\frac{\partial w}{\partial z}\right)(x,y,z^2) + 2z\frac{\partial}{\partial z}\left(\left(2v\frac{\partial v}{\partial z}+\varepsilon\frac{\partial w}{\partial z}\right)(x,y,z^2)\right) \\
&= 2\left(2v\frac{\partial v}{\partial z}+\varepsilon\frac{\partial w}{\partial z}\right)(x,y,z^2) + 4z^2\left(\left(\frac{\partial}{\partial z}\left(2v\frac{\partial v}{\partial z}+\varepsilon\frac{\partial w}{\partial z}\right)\right)(x,y,z^2)\right) \\
&= 2\left(2v\frac{\partial v}{\partial z}+\varepsilon\frac{\partial w}{\partial z}\right)(x,y,z^2) + 4z^2\left(\left(2\left(\frac{\partial v}{\partial z}\right)^2+2v\frac{\partial^2 v}{\partial z^2}+\varepsilon\frac{\partial^2 w}{\partial z^2}\right)(x,y,z^2)\right),
\end{aligned}$$

and similarly,

$$\begin{aligned}
\frac{\partial F_1}{\partial x}(x,y,z) &= \frac{\partial u}{\partial x}(x,y,z^2), \\
\frac{\partial F_1}{\partial y}(x,y,z) &= \frac{\partial u}{\partial y}(x,y,z^2), \\
\frac{\partial F_1}{\partial z}(x,y,z) &= 2z\left(\frac{\partial u}{\partial z}(x,y,z^2)\right), \\
\frac{\partial F_2}{\partial x}(x,y,z) &= \left(2v\frac{\partial v}{\partial x}+\varepsilon\frac{\partial w}{\partial x}\right)(x,y,z^2), \\
\frac{\partial F_2}{\partial y}(x,y,z) &= \left(2v\frac{\partial v}{\partial y}+\varepsilon\frac{\partial w}{\partial y}\right)(x,y,z^2), \\
\frac{\partial^2 F_2}{\partial x^2}(x,y,z) &= \left(2\left(\frac{\partial v}{\partial x}\right)^2+2v\frac{\partial^2 v}{\partial x^2}+\varepsilon\frac{\partial^2 w}{\partial x^2}\right)(x,y,z^2), \\
\frac{\partial^2 F_2}{\partial y^2}(x,y,z) &= \left(2\left(\frac{\partial v}{\partial y}\right)^2+2v\frac{\partial^2 v}{\partial y^2}+\varepsilon\frac{\partial^2 w}{\partial y^2}\right)(x,y,z^2), \\
\frac{\partial^2 F_2}{\partial x\partial y}(x,y,z) &= \left(2\frac{\partial v}{\partial x}\frac{\partial v}{\partial x}+2v\frac{\partial^2 v}{\partial x\partial y}+\varepsilon\frac{\partial^2 w}{\partial x\partial y}\right)(z,y,z^2), \\
\frac{\partial^2 F_2}{\partial x\partial z}(x,y,z) &= 2z\left(\left(2\frac{\partial v}{\partial x}\frac{\partial v}{\partial z}+2v\frac{\partial^2 v}{\partial x\partial z}+\varepsilon\frac{\partial^2 w}{\partial x\partial z}\right)(x,y,z^2)\right), \\
\frac{\partial^2 F_2}{\partial y\partial z}(x,y,z) &= 2z\left(\left(2\frac{\partial v}{\partial y}\frac{\partial v}{\partial z}+2v\frac{\partial^2 v}{\partial y\partial z}+\varepsilon\frac{\partial^2 w}{\partial y\partial z}\right)(x,y,z^2)\right).
\end{aligned}$$

Since $p=(0,0,0)$ with respect to both (x,y,z) and (u,v,w), for example,

$$\begin{aligned}
\left(\frac{\partial^2 F_2}{\partial x^2}\right)_p &= \left(2\left(\frac{\partial v}{\partial x}\right)^2+2v\frac{\partial^2 v}{\partial x^2}+\varepsilon\frac{\partial^2 w}{\partial x^2}\right)(0,0,0^2) \\
&= 2\left(\frac{\partial v}{\partial x}\right)_p^2 + 2\cdot 0\left(\frac{\partial^2 v}{\partial x^2}\right)_p + \varepsilon\left(\frac{\partial^2 w}{\partial x^2}\right)_p = 2\left(\frac{\partial v}{\partial x}\right)_p^2,
\end{aligned}$$

and similarly,

$$\left(\frac{\partial F_1}{\partial x}\right)_p = \left(\frac{\partial u}{\partial x}\right)_p,$$
$$\left(\frac{\partial F_1}{\partial y}\right)_p = \left(\frac{\partial u}{\partial y}\right)_p,$$
$$\left(\frac{\partial^2 F_2}{\partial y^2}\right)_p = 2\left(\frac{\partial v}{\partial y}\right)_p^2,$$
$$\left(\frac{\partial^2 F_2}{\partial z^2}\right)_p = 2\varepsilon\left(\frac{\partial w}{\partial z}\right)_p,$$
$$\left(\frac{\partial^2 F_2}{\partial x\partial y}\right)_p = 2\left(\frac{\partial v}{\partial x}\right)_p\left(\frac{\partial v}{\partial y}\right)_p,$$
$$\left(\frac{\partial F_1}{\partial z}\right)_p = \left(\frac{\partial F_2}{\partial x}\right)_p = \left(\frac{\partial F_2}{\partial y}\right)_p = \left(\frac{\partial F_2}{\partial z}\right)_p = \left(\frac{\partial^2 F_2}{\partial x\partial z}\right)_p = \left(\frac{\partial^2 F_2}{\partial y\partial z}\right)_p = 0.$$

We choose a pair of vector fields and calculate derivatives with respect to them as follows. Let η_1 and η_2 be C^∞ vector fields on U as

$$\eta_1 = \left(\frac{\partial u}{\partial y}\right)_p \frac{\partial}{\partial x} - \left(\frac{\partial u}{\partial x}\right)_p \frac{\partial}{\partial y},$$
$$\eta_2 = \frac{\partial}{\partial z}.$$

Noting that the coefficients of η_1 are constants,

$$\eta_1 F_2 = \left(\frac{\partial u}{\partial y}\right)_p \frac{\partial F_2}{\partial x} - \left(\frac{\partial u}{\partial x}\right)_p \frac{\partial F_2}{\partial y},$$
$$\begin{aligned}\eta_1\eta_1 F_2 &= \left(\left(\frac{\partial u}{\partial y}\right)_p \frac{\partial}{\partial x} - \left(\frac{\partial u}{\partial x}\right)_p \frac{\partial}{\partial y}\right)\left(\left(\frac{\partial u}{\partial y}\right)_p \frac{\partial F_2}{\partial x} - \left(\frac{\partial u}{\partial x}\right)_p \frac{\partial F_2}{\partial y}\right)\\ &= \left(\frac{\partial u}{\partial y}\right)_p^2 \frac{\partial^2 F_2}{\partial x^2} - 2\left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial u}{\partial y}\right)_p \frac{\partial^2 F_2}{\partial x\partial y} + \left(\frac{\partial u}{\partial x}\right)_p^2 \frac{\partial^2 F_2}{\partial y^2}.\end{aligned}$$

By the results of the previous paragraph,

$$(\eta_1 F_2)_p = \left(\frac{\partial u}{\partial y}\right)_p\left(\frac{\partial F_2}{\partial x}\right)_p - \left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial F_2}{\partial y}\right)_p = 0,$$
$$\begin{aligned}(\eta_1\eta_1 F_2)_p &= \left(\frac{\partial u}{\partial y}\right)_p^2\left(\frac{\partial^2 F_2}{\partial x^2}\right)_p - 2\left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial u}{\partial y}\right)_p\left(\frac{\partial^2 F_2}{\partial x\partial y}\right)_p + \left(\frac{\partial u}{\partial x}\right)_p^2\left(\frac{\partial^2 F_2}{\partial y^2}\right)_p\\ &= 2\left(\frac{\partial u}{\partial y}\right)_p^2\left(\frac{\partial v}{\partial x}\right)_p^2 - 4\left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial u}{\partial y}\right)_p\left(\frac{\partial v}{\partial x}\right)_p\left(\frac{\partial v}{\partial y}\right)_p + 2\left(\frac{\partial u}{\partial x}\right)_p^2\left(\frac{\partial v}{\partial y}\right)_p^2\\ &= 2\left(\left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial v}{\partial y}\right)_p - \left(\frac{\partial u}{\partial y}\right)_p\left(\frac{\partial v}{\partial x}\right)_p\right)^2 > 0.\end{aligned}$$

Similarly, we can obtain that $(\eta_1 F_1)_p = (\eta_2 F_1)_p = (\eta_2 F_2)_p = (\eta_1\eta_2 F_2)_p = (\eta_2\eta_1 F_2)_p = 0$ and

$$(\eta_2\eta_2 F_2)_p = \left(\frac{\partial^2 F_2}{\partial z^2}\right)_p = 2\varepsilon\left(\frac{\partial w}{\partial z}\right)_p \neq 0.$$

We are now ready to complete the proofs. By the above,

$$(dF_1)_p = \left(\left(\frac{\partial F_1}{\partial x}\right)_p, \left(\frac{\partial F_1}{\partial y}\right)_p, \left(\frac{\partial F_1}{\partial z}\right)_p\right) = \left(\left(\frac{\partial u}{\partial x}\right)_p, \left(\frac{\partial u}{\partial y}\right)_p, 0\right) \neq (0,0,0),$$

$$(dF_2)_p = \left(\left(\frac{\partial F_2}{\partial x}\right)_p, \left(\frac{\partial F_2}{\partial y}\right)_p, \left(\frac{\partial F_2}{\partial z}\right)_p\right) = (0,0,0).$$

Since $(\eta_1)_p$ and $(\eta_2)_p$ are linearly independent, and $(\eta_1 F_1)_p = (\eta_2 F_1)_p = (\eta_1 F_2)_p = (\eta_2 F_2)_p = 0$, we obtain the condition that $\ker(dF)_p = \left\langle (\eta_1)_p, (\eta_2)_p \right\rangle$. The matrix

$$\begin{pmatrix} (\eta_1\eta_1 F_2)_p & (\eta_1\eta_2 F_2)_p \\ (\eta_2\eta_1 F_2)_p & (\eta_2\eta_2 F_2)_p \end{pmatrix},$$

denoted by $(\mathbf{H}_{\eta_1,\eta_2} F_2)_p$, is equal to

$$\begin{pmatrix} 2\left(\left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial v}{\partial y}\right)_p - \left(\frac{\partial u}{\partial y}\right)_p\left(\frac{\partial v}{\partial x}\right)_p\right)^2 & 0 \\ 0 & 2\varepsilon\left(\frac{\partial w}{\partial z}\right)_p \end{pmatrix},$$

which shows that $\ker(\mathbf{H}_{\eta_1,\eta_2} F_2)_p = \{\mathbf{0}\}$. By Theorem 2, the point p is a fold point of F. Moreover, the fold point p of F is definite (resp. indefinite) if $\varepsilon > 0$ (resp. $\varepsilon < 0$), that is to say, the boundary fold point p of f is definite (resp. indefinite).

3.3 Cusp case

In this subsection, we give a proof of the fourth assertion of the proposition. Suppose that p is a boundary cusp point of f. Let $S(F)$ denote the singular set of F, let U be a sufficiently small neighborhood of p in $\mathbb{R}^3$, and let q be any point in $S(F) \cap U$.

The original map and the doubled map have local forms as follows. On one hand, there are local coordinate systems (u,v,w) and (s,t) of $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively, with respect to which $p = (0,0,0)$, $f(p) = (0,0)$, $\mathbb{R}^3_{\geq 0} = \{w \geq 0\}$ and $f(u,v,w) = (u, v^3 + uv + w)$. On the other hand, $F(x,y,z) = f(x,y,z^2)$ with respect to the coordinate system (x,y,z) of $\mathbb{R}^3$. We may suppose that $p = (0,0,0)$ with respect to (x,y,z). Suppose that F has a local form: $F(x,y,z) = (F_1(x,y,z), F_2(x,y,z))$ with respect to (x,y,z) and (s,t).

We detect the singular set of the doubled map as follows. Recall that the original map f has no singular points in U. The doubled map F inherits the regularity of f in $U \setminus \partial\mathbb{R}^3_{\geq 0}$. Recall also that f has only boundary regular points in $\partial\mathbb{R}^3_{\geq 0} \setminus \{3v^2 + u = w = 0\}$, and only boundary fold points in $\{3v^2 + u = w = 0\} \setminus \{p\}$. By the results of the previous subsections, $S(F)\cap U$ is either $\{3v^2+u=w=0\}\setminus\{p\}$ or $\{3v^2+u=w=0\}$. Since the singular set is a closed set in general, we conclude that $S(F)\cap U=\{3v^2+u=w=0\}$. Hence q is possibly p.

The relevant coordinate systems are related as follows. There is a coordinate transformation: $(x,y,z)\mapsto(u(x,y,z),v(x,y,z),w(x,y,z))$. Since $\{z\geq 0\}=\{w\geq 0\}$ and $q\in\{z=0\}=\{w=0\}$, the transformation satisfies the conditions that

$$\left(\frac{\partial u}{\partial x}\right)_q\left(\frac{\partial v}{\partial y}\right)_q-\left(\frac{\partial u}{\partial y}\right)_q\left(\frac{\partial v}{\partial x}\right)_q\neq 0,$$

$$\left(\frac{\partial w}{\partial x}\right)_q=\left(\frac{\partial w}{\partial y}\right)_q=\left(\frac{\partial^2 w}{\partial x^2}\right)_q=\left(\frac{\partial^2 w}{\partial y^2}\right)_q=\left(\frac{\partial^2 w}{\partial x\partial y}\right)_q=0,$$

$$\left(\frac{\partial w}{\partial z}\right)_q>0.$$

In particular, the top inequality implies that

$$\left(\left(\frac{\partial u}{\partial x}\right)_q,\left(\frac{\partial u}{\partial y}\right)_q\right)\neq(0,0).$$

We calculate partial derivatives with respect to the coordinates similarly to those in the previous subsection. We can obtain that

$$\frac{\partial F_1}{\partial x}(x,y,z)=\frac{\partial u}{\partial x}\left(x,y,z^2\right),$$

$$\frac{\partial F_1}{\partial y}(x,y,z)=\frac{\partial u}{\partial y}\left(x,y,z^2\right),$$

$$\frac{\partial F_1}{\partial z}(x,y,z)=2z\left(\frac{\partial u}{\partial z}\left(x,y,z^2\right)\right),$$

$$\frac{\partial F_2}{\partial x}(x,y,z)=\left(v\frac{\partial u}{\partial x}+\left(3v^2+u\right)\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\right)\left(x,y,z^2\right),$$

$$\frac{\partial F_2}{\partial y}(x,y,z)=\left(v\frac{\partial u}{\partial y}+\left(3v^2+u\right)\frac{\partial v}{\partial y}+\frac{\partial w}{\partial y}\right)\left(x,y,z^2\right),$$

$$\frac{\partial F_2}{\partial z}(x,y,z)=2z\left(\left(v\frac{\partial u}{\partial z}+\left(3v^2+u\right)\frac{\partial v}{\partial z}+\frac{\partial w}{\partial z}\right)\left(x,y,z^2\right)\right),$$

$$\frac{\partial^2 F_2}{\partial x^2}(x,y,z)=\left(2\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+v\frac{\partial^2 u}{\partial x^2}+6v\left(\frac{\partial v}{\partial x}\right)^2+\left(3v^2+u\right)\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 w}{\partial x^2}\right)\left(x,y,z^2\right),$$

$$\frac{\partial^2 F_2}{\partial y^2}(x,y,z)=\left(2\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}+v\frac{\partial^2 u}{\partial y^2}+6v\left(\frac{\partial v}{\partial y}\right)^2+\left(3v^2+u\right)\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial y^2}\right)\left(x,y,z^2\right),$$

$$\begin{aligned}\frac{\partial^2 F_2}{\partial z^2}(x,y,z)=&\,2\left(v\frac{\partial u}{\partial z}+(3v^2+u)\frac{\partial v}{\partial z}+\frac{\partial w}{\partial z}\right)\left(x,y,z^2\right)\\&+4z^2\left(\left(\frac{\partial v}{\partial z}\frac{\partial u}{\partial z}+v\frac{\partial^2 u}{\partial z^2}+\left(6v\frac{\partial v}{\partial z}+\frac{\partial u}{\partial z}\right)\frac{\partial v}{\partial z}\right.\right.\\&\left.\left.+(3v^2+u)\frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 w}{\partial z^2}\right)\left(x,y,z^2\right)\right),\end{aligned}$$

$$\frac{\partial^2 F_2}{\partial x \partial y}(x,y,z) = \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + v\frac{\partial^2 u}{\partial x \partial y} + 6v\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + (3v^2+u)\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial y}\right)(x,y,z^2),$$

$$\frac{\partial^2 F_2}{\partial x \partial z}(x,y,z) = 2z\left(\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial z} + v\frac{\partial^2 u}{\partial x \partial z} + \left(6v\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\right)\frac{\partial v}{\partial z} + (3v^2+u)\frac{\partial^2 v}{\partial x \partial z} + \frac{\partial^2 w}{\partial x \partial z}\right)(u,v,w^2)\right),$$

$$\frac{\partial^2 F_2}{\partial y \partial z}(x,y,z) = 2z\left(\left(\frac{\partial v}{\partial y}\frac{\partial u}{\partial z} + v\frac{\partial^2 u}{\partial y \partial z} + \left(6v\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\right)\frac{\partial v}{\partial z} + (3v^2+u)\frac{\partial^2 v}{\partial y \partial z} + \frac{\partial^2 w}{\partial y \partial z}\right)(u,v,w^2)\right),$$

$$\frac{\partial^3 F_2}{\partial x^3} = \left(3\frac{\partial u}{\partial x}\frac{\partial^2 v}{\partial x^2} + 6\left(\frac{\partial v}{\partial x}\right)^3 + 3\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial x^2} + 18v\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2} + v\frac{\partial^3 u}{\partial x^3} + (3v^2+u)\frac{\partial^3 v}{\partial x^3} + \frac{\partial^3 w}{\partial x^3}\right)(x,y,z^2),$$

$$\frac{\partial^3 F_2}{\partial x^2 \partial y} = \left(2\frac{\partial u}{\partial x}\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial x^2} + 6\left(\frac{\partial v}{\partial x}\right)^2\frac{\partial v}{\partial y} + 2\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial x \partial y} + 12v\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial v}{\partial y}\frac{\partial^2 u}{\partial x^2} + 6v\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial x^2} + v\frac{\partial^3 u}{\partial x^2 \partial y} + (3v^2+u)\frac{\partial^3 v}{\partial x^2 \partial y} + \frac{\partial^3 w}{\partial x^2 \partial y}\right)(x,y,z^2),$$

$$\frac{\partial^3 F_2}{\partial x \partial y^2} = \left(\frac{\partial u}{\partial x}\frac{\partial^2 v}{\partial y^2} + 2\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial x \partial y} + 6\frac{\partial v}{\partial x}\left(\frac{\partial v}{\partial y}\right)^2 + \frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial y^2} + 6v\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial y^2} + 2\frac{\partial v}{\partial y}\frac{\partial^2 u}{\partial x \partial y} + 12v\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial x \partial y} + v\frac{\partial^3 u}{\partial x \partial y^2} + (3v^2+u)\frac{\partial^3 v}{\partial x \partial y^2} + \frac{\partial^3 w}{\partial x \partial y^2}\right)(x,y,z^2),$$

$$\frac{\partial^3 F_2}{\partial y^3} = \left(3\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2} + 6\left(\frac{\partial v}{\partial y}\right)^3 + 3\frac{\partial v}{\partial y}\frac{\partial^2 u}{\partial y^2} + 18v\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y^2} + v\frac{\partial^3 u}{\partial y^3} + (3v^2+u)\frac{\partial^3 v}{\partial y^3} + \frac{\partial^3 w}{\partial y^3}\right)(x,y,z^2).$$

Let $(x_q, y_q, 0)$ and $(u_q, v_q, 0)$ be the coordinate representations of q with respect to the coordinate systems (x,y,z) and (u,v,w), respectively. Noting that $3v_q^2 + u_q = 0$, for example,

$$\left(\frac{\partial F_1}{\partial x}\right)_q = \frac{\partial u}{\partial x}(x_q, y_q, 0^2) = \left(\frac{\partial u}{\partial x}\right)_q,$$

$$\begin{aligned}\left(\frac{\partial F_2}{\partial x}\right)_q &= \left(v\frac{\partial u}{\partial x} + (3v^2+u)\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\right)(x_q, y_q, 0^2) \\ &= v_q\left(\frac{\partial u}{\partial x}\right)_q + (3v_q^2+u_q)\left(\frac{\partial v}{\partial x}\right)_q + \left(\frac{\partial w}{\partial y}\right)_q = v_q\left(\frac{\partial u}{\partial x}\right)_q,\end{aligned}$$

and similarly,

$$\left(\frac{\partial F_1}{\partial y}\right)_q = \left(\frac{\partial u}{\partial y}\right)_q,$$

$$\left(\frac{\partial F_2}{\partial y}\right)_q = v_q \left(\frac{\partial u}{\partial y}\right)_q.$$

Noting that $v_p = 0$,

$$\left(\frac{\partial F_2}{\partial x}\right)_p = \left(\frac{\partial F_2}{\partial y}\right)_p = 0.$$

Similarly, we can obtain that

$$\left(\frac{\partial^2 F_2}{\partial x^2}\right)_p = 2\left(\frac{\partial u}{\partial x}\right)_p \left(\frac{\partial v}{\partial x}\right)_p,$$

$$\left(\frac{\partial^2 F_2}{\partial y^2}\right)_p = 2\left(\frac{\partial u}{\partial y}\right)_p \left(\frac{\partial v}{\partial y}\right)_p,$$

$$\left(\frac{\partial^2 F_2}{\partial z^2}\right)_p = 2\left(\frac{\partial w}{\partial z}\right)_p,$$

$$\left(\frac{\partial^2 F_2}{\partial x \partial y}\right)_p = \left(\frac{\partial u}{\partial x}\right)_p \left(\frac{\partial v}{\partial y}\right)_p + \left(\frac{\partial u}{\partial y}\right)_p \left(\frac{\partial v}{\partial x}\right)_p,$$

$$\left(\frac{\partial^3 F_2}{\partial x^3}\right)_p = 3\left(\frac{\partial u}{\partial x}\right)_p \left(\frac{\partial^2 v}{\partial x^2}\right)_p + 6\left(\frac{\partial v}{\partial x}\right)_p^3 + 3\left(\frac{\partial v}{\partial x}\right)_p \left(\frac{\partial^2 u}{\partial x^2}\right)_p,$$

$$\begin{aligned}\left(\frac{\partial^3 F_2}{\partial x^2 \partial y}\right)_p = {} & 2\left(\frac{\partial u}{\partial x}\right)_p \left(\frac{\partial^2 v}{\partial x \partial y}\right)_p + \left(\frac{\partial u}{\partial y}\right)_p \left(\frac{\partial^2 v}{\partial x^2}\right)_p + 6\left(\frac{\partial v}{\partial x}\right)_p^2 \left(\frac{\partial v}{\partial y}\right)_p \\ & + 2\left(\frac{\partial v}{\partial x}\right)_p \left(\frac{\partial^2 u}{\partial x \partial y}\right)_p + \left(\frac{\partial v}{\partial y}\right)_p \left(\frac{\partial^2 u}{\partial x^2}\right)_p,\end{aligned}$$

$$\begin{aligned}\left(\frac{\partial^3 F_2}{\partial x \partial y^2}\right)_p = {} & \left(\frac{\partial u}{\partial x}\right)_p \left(\frac{\partial^2 v}{\partial y^2}\right)_p + 2\left(\frac{\partial u}{\partial y}\right)_p \left(\frac{\partial^2 v}{\partial x \partial y}\right)_p + 6\left(\frac{\partial v}{\partial x}\right)_p \left(\frac{\partial v}{\partial y}\right)_p^2 \\ & + \left(\frac{\partial v}{\partial x}\right)_p \left(\frac{\partial^2 u}{\partial y^2}\right)_p + 2\left(\frac{\partial v}{\partial y}\right)_p \left(\frac{\partial^2 u}{\partial x \partial y}\right)_p,\end{aligned}$$

$$\left(\frac{\partial^3 F_2}{\partial y^3}\right)_p = 3\left(\frac{\partial u}{\partial y}\right)_p \left(\frac{\partial^2 v}{\partial y^2}\right)_p + 6\left(\frac{\partial v}{\partial y}\right)_p^3 + 3\left(\frac{\partial v}{\partial y}\right)_p \left(\frac{\partial^2 u}{\partial y^2}\right)_p,$$

$$\left(\frac{\partial F_1}{\partial z}\right)_p = \left(\frac{\partial F_2}{\partial z}\right)_p = \left(\frac{\partial^2 F_2}{\partial x \partial z}\right)_p = \left(\frac{\partial^2 F_2}{\partial y \partial z}\right)_p = 0.$$

We choose a pair of vector fields and calculate derivatives with respect to them as follows. Let η_1 and η_2 be C^∞ vector fields on U as

$$\eta_1 = \frac{\partial u}{\partial y}\frac{\partial}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial}{\partial y},$$

$$\eta_2 = \frac{\partial}{\partial z}.$$

Noting that the coefficients of η_1 are derived functions,

$$\eta_1 F_2 = \frac{\partial u}{\partial y}\frac{\partial F_2}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial F_2}{\partial y},$$

$$\begin{aligned}
\frac{\partial}{\partial x}\eta_1 F_2 &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\frac{\partial F_2}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial F_2}{\partial y}\right) \\
&= \frac{\partial^2 u}{\partial x \partial y}\frac{\partial F_2}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial^2 F_2}{\partial x^2} - \frac{\partial^2 u}{\partial x^2}\frac{\partial F_2}{\partial y} - \frac{\partial u}{\partial x}\frac{\partial^2 F_2}{\partial x \partial y}, \\
\frac{\partial}{\partial y}\eta_1 F_2 &= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\frac{\partial F_2}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial F_2}{\partial y}\right) \\
&= \frac{\partial^2 u}{\partial y^2}\frac{\partial F_2}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial^2 F_2}{\partial x \partial y} - \frac{\partial^2 u}{\partial x \partial y}\frac{\partial F_2}{\partial y} - \frac{\partial u}{\partial x}\frac{\partial^2 F_2}{\partial y^2},
\end{aligned}$$

$$\begin{aligned}
\eta_1\eta_1 F_2 &= \left(\frac{\partial u}{\partial y}\frac{\partial}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial}{\partial y}\right)\eta_1 F_2 \\
&= \frac{\partial u}{\partial y}\left(\frac{\partial}{\partial x}\eta_1 F_2\right) - \frac{\partial u}{\partial x}\left(\frac{\partial}{\partial y}\eta_1 F_2\right) \\
&= \frac{\partial u}{\partial y}\left(\frac{\partial^2 u}{\partial x \partial y}\frac{\partial F_2}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial^2 F_2}{\partial x^2} - \frac{\partial^2 u}{\partial x^2}\frac{\partial F_2}{\partial y} - \frac{\partial u}{\partial x}\frac{\partial^2 F_2}{\partial x \partial y}\right) \\
&\quad - \frac{\partial u}{\partial x}\left(\frac{\partial^2 u}{\partial y^2}\frac{\partial F_2}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial^2 F_2}{\partial x \partial y} - \frac{\partial^2 u}{\partial x \partial y}\frac{\partial F_2}{\partial y} - \frac{\partial u}{\partial x}\frac{\partial^2 F_2}{\partial y^2}\right) \\
&= \left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}\right)\frac{\partial F_2}{\partial x} - \left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial y}\right)\frac{\partial F_2}{\partial y} \\
&\quad + \left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 F_2}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2 F_2}{\partial y^2} - 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2 F_2}{\partial x \partial y},
\end{aligned}$$

$$
\begin{aligned}
&\eta_1\eta_1\eta_1 F_2 \\
&= \left(\frac{\partial u}{\partial y}\frac{\partial}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial}{\partial y}\right)\left(\left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y} - \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}\right)\frac{\partial F_2}{\partial x} - \left(\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial y}\right)\frac{\partial F_2}{\partial y}\right. \\
&\quad \left. + \left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 F_2}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2 F_2}{\partial y^2} - 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2 F_2}{\partial x\partial y}\right) \\
&= \left(\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^3 u}{\partial y^3} - 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^3 u}{\partial x\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^3 u}{\partial x^2\partial y} - \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y}\left(\frac{\partial^2 u}{\partial x\partial y}\right)^2\right)\frac{\partial F_2}{\partial x} \\
&\quad - \left(\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^3 u}{\partial x\partial y^2} - 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^3 u}{\partial x^2\partial y} - \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x}\left(\frac{\partial^2 u}{\partial x\partial y}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^3 u}{\partial x^3}\right)\frac{\partial F_2}{\partial y} \\
&\quad + 3\left(-\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial x\partial y}\right)\frac{\partial^2 F_2}{\partial x^2} + 3\left(-\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x^2}\right)\frac{\partial^2 F_2}{\partial y^2} \\
&\quad + 3\left(\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial x^2}\right)\frac{\partial^2 F_2}{\partial x\partial y} + \left(\frac{\partial u}{\partial y}\right)^3\frac{\partial^3 F_2}{\partial x^3} - \left(\frac{\partial u}{\partial x}\right)^3\frac{\partial^3 F_2}{\partial y^3} \\
&\quad - 3\frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^3 F_2}{\partial x^2\partial y} + 3\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial u}{\partial y}\frac{\partial^3 F_2}{\partial x\partial y^2}.
\end{aligned}
$$

By the results of the previous paragraph,

$$
\begin{aligned}
(\eta_1 F_2)_q &= \left(\frac{\partial u}{\partial y}\right)_q\left(\frac{\partial F_2}{\partial x}\right)_q - \left(\frac{\partial u}{\partial x}\right)_q\left(\frac{\partial F_2}{\partial y}\right)_q \\
&= \left(\frac{\partial u}{\partial y}\right)_q v_q\left(\frac{\partial u}{\partial x}\right)_q - \left(\frac{\partial u}{\partial x}\right)_q v_q\left(\frac{\partial u}{\partial y}\right)_q = 0,
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{\partial}{\partial x}\eta_1 F_2\right)_p &= \left(\frac{\partial^2 u}{\partial x\partial y}\right)_p\left(\frac{\partial F_2}{\partial x}\right)_p + \left(\frac{\partial u}{\partial y}\right)_p\left(\frac{\partial^2 F_2}{\partial x^2}\right)_p \\
&\quad - \left(\frac{\partial^2 u}{\partial x^2}\right)_p\left(\frac{\partial F_2}{\partial y}\right)_p - \left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial^2 F_2}{\partial x\partial y}\right)_p \\
&= 2\left(\frac{\partial u}{\partial y}\right)_p\left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial v}{\partial x}\right)_p - \left(\frac{\partial u}{\partial x}\right)_p\left(\left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial v}{\partial y}\right)_p + \left(\frac{\partial u}{\partial y}\right)_p\left(\frac{\partial v}{\partial x}\right)_p\right) \\
&= -\left(\frac{\partial u}{\partial x}\right)_p\left(\left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial v}{\partial y}\right)_p - \left(\frac{\partial u}{\partial y}\right)_p\left(\frac{\partial v}{\partial x}\right)_p\right),
\end{aligned}
$$

$$
\begin{aligned}
(\eta_1\eta_1F_2)_p &= \left(\left(\frac{\partial u}{\partial y}\right)_p\left(\frac{\partial^2 u}{\partial x\partial y}\right)_p - \left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial^2 u}{\partial y^2}\right)_p\right)\left(\frac{\partial F_2}{\partial x}\right)_p \\
&\quad - \left(\left(\frac{\partial u}{\partial y}\right)_p\left(\frac{\partial^2 u}{\partial x^2}\right)_p - \left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial^2 u}{\partial x\partial y}\right)_p\right)\left(\frac{\partial F_2}{\partial y}\right)_p \\
&\quad + \left(\frac{\partial u}{\partial y}\right)_p^2\left(\frac{\partial^2 F_2}{\partial x^2}\right)_p - 2\left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial u}{\partial y}\right)_p\left(\frac{\partial^2 F_2}{\partial x\partial y}\right)_p + \left(\frac{\partial u}{\partial x}\right)_p^2\left(\frac{\partial^2 F_2}{\partial y^2}\right)_p \\
&= 2\left(\frac{\partial u}{\partial y}\right)_p^2\left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial v}{\partial x}\right)_p + 2\left(\frac{\partial u}{\partial x}\right)_p^2\left(\frac{\partial u}{\partial y}\right)_p\left(\frac{\partial v}{\partial y}\right)_p \\
&\quad - 2\left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial u}{\partial y}\right)_p\left(\left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial v}{\partial y}\right)_p + \left(\frac{\partial u}{\partial y}\right)_p\left(\frac{\partial v}{\partial x}\right)_p\right) \\
&= 0,
\end{aligned}
$$

and similarly,

$$
\left(\frac{\partial}{\partial y}\eta_1F_2\right)_p = -\left(\frac{\partial u}{\partial y}\right)_p\left(\left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial v}{\partial y}\right)_p - \left(\frac{\partial u}{\partial y}\right)_p\left(\frac{\partial v}{\partial x}\right)_p\right),
$$

$$
(\eta_1\eta_1\eta_1F_2)_p = -6\left(\left(\frac{\partial u}{\partial x}\right)_p\left(\frac{\partial v}{\partial y}\right)_p - \left(\frac{\partial u}{\partial y}\right)_p\left(\frac{\partial v}{\partial x}\right)_p\right)^3 \neq 0.
$$

Similarly, we can obtain that $(\eta_1F_1)_q = (\eta_2F_1)_p = (\eta_2F_2)_p = (\eta_1\eta_2F_2)_p = (\eta_2\eta_1F_2)_p = 0$ and

$$
(\eta_2\eta_2F_2)_p = \left(\frac{\partial^2 F_2}{\partial z^2}\right)_p = 2\left(\frac{\partial w}{\partial z}\right)_p > 0.
$$

We are now ready to complete the proof. By the above, $(dF_1)_p \neq (0,0,0)$ and $(dF_2)_p = (0,0,0)$. Since $(\eta_1)_p$ and $(\eta_2)_p$ are linearly independent, and $(\eta_1F_1)_p = (\eta_2F_1)_p = (\eta_1F_2)_p = (\eta_2F_2)_p = 0$, we obtain the condition that $\ker(dF)_p = \left\langle(\eta_1)_p, (\eta_2)_p\right\rangle$. Since $(\eta_1F_1)_q = (\eta_1F_2)_q = 0$, we obtain the condition that $(\eta_1)_q \in \ker(dF)_q$ for $q \in S(F)\cap U$. The matrix

$$
\begin{pmatrix} (\eta_1\eta_1F_2)_p & (\eta_1\eta_2F_2)_p \\ (\eta_2\eta_1F_2)_p & (\eta_2\eta_2F_2)_p \end{pmatrix},
$$

denoted by $(\mathbf{H}_{\eta_1,\eta_2}F_2)_p$, is equal to

$$
\begin{pmatrix} 0 & 0 \\ 0 & 2\left(\frac{\partial w}{\partial z}\right)_p \end{pmatrix},
$$

which shows that $\ker(\mathbf{H}_{\eta_1,\eta_2}F_2)_p = \left\langle(\eta_1)_p\right\rangle$. Since

$$
\left(\left(\frac{\partial}{\partial x}\eta_1F_2\right)_p, \left(\frac{\partial}{\partial y}\eta_1F_2\right)_p\right)
$$

$$= -\left(\left(\frac{\partial u}{\partial x}\right)_p \left(\frac{\partial v}{\partial y}\right)_p - \left(\frac{\partial u}{\partial y}\right)_p \left(\frac{\partial v}{\partial x}\right)_p\right) \left(\left(\frac{\partial u}{\partial x}\right)_p, \left(\frac{\partial u}{\partial y}\right)_p\right) \neq (0,0),$$

we obtain the condition that $(d(\eta_1 F_2))_p \neq (0,0,0)$, as well as $(\eta_1\eta_1\eta_1 F_2)_p \neq 0$. By Theorem 3, the point p is a cusp point of F.

3.4 $\Sigma_{1,0}^{2,0}$ case

The last assertion of the proposition can be proved by a simple observation as follows. Suppose that p is a $\Sigma_{1,0}^{2,0}$ point of f. Let $S(f)$ and $S(F)$ denote the singular sets of f and F, respectively, $S\left(f|_{\partial\mathbb{R}^3_{\geq 0}}\right)$ denote the boundary singular set of f, and U be a sufficiently small neighborhood of p in $\mathbb{R}^3$. Recall that $\left(S(f) \cup S\left(f|_{\partial\mathbb{R}^3_{\geq 0}}\right)\right) \cap U$ is a figure $\perp$ consisting only of fold points, boundary fold points and the $\Sigma_{1,0}^{2,0}$ point p. By the results of the previous subsections, $S(F) \cap U$ is a figure $+$ where the crossing point is p. This shows that p is neither a regular point, a fold point nor a cusp point of F.

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