ENTROPY FOR AUTOMORPHISMS
OF THE CROSSED PRODUCTS

Abstract. Let $G \subset \text{Aut}(A)$ be a discrete group which is exact, that is, admits an amenable action on some compact space. Then the entropy of an automorphism of the algebra $A$ does not change by the canonical extension to the crossed product $A \times G$. This is shown for the topological entropy of an exact $C^*$-algebra $A$ and for the dynamical entropy of an AFD von Neumann algebra $A$. These have applications to the case of transformations on Lebesgue spaces.

1. INTRODUCTION

The notion of the Kolmogoroff-Sinai entropy in the ergodic theory was brought into the theory of finite von Neumann algebras by Connes-Størmer ([10]), as a non-commutative extension. Replacing the finite trace to a state $\phi$, it was extended to general von Neumann algebras and to $C^*$-algebras by Connes-Narnhofer-Thirring ([9]). In this paper we call the Connes-Størmer entropy the CS-entropy and the Connes-Narnhofer-Thirring entropy the CNT-entropy. We denote by $H(\cdot)$ the CS-entropy and the by $h_\phi(\cdot)$ the CNT-entropy.

In the ergodic theory, we are given a probability space $(X, \mu)$ together with a measure preserving nonsingular transformation $T$ of $X$. Then we have the abelian von Neumann algebra $L^\infty(X, \mu)$ with the trace $\tau_\mu$ induced by $\mu$ and the automorphism $\alpha_T$ of $L^\infty(X, \mu)$ induced by $T$. In this setting, the Connes-Størmer entropy $H(\alpha_T)$ with respect to the trace $\tau_\mu$ is nothing but the Kolmogoroff-Sinai entropy $h(T)$.
The noncommutative algebra $M$ is given from this dynamical system $(X, \mu, T)$ by taking the crossed product $M = L^\infty(X, \mu) \times_\alpha \mathbb{Z}$. The automorphism $\alpha_T$ is extended naturally to the automorphism $\overline{\alpha_T}$ of $M$, and it preserves the natural extension $\tau_{\mu}$ of $\tau_{\mu}$. As a logical consequence, the following question was suggested by Stormer in ([17]) : Do we have $H(\overline{\alpha_T}) = h(T)$ ?

The first positive answer is due to Voiculescu. He showed that

$$H(\overline{\alpha_T}) = h(T) = \log n$$

for the ergodic measure preserving Bernoulli transformation $T$ on the space $(X, \mu)$, where $X$ is the product space $\{1, \cdots, n\}^\mathbb{Z}$ and the measure $\mu$ is the product measure $\mu_n^\otimes \mathbb{Z}$. Here $\mu_n$ is the equal weights probability measure on the set $\{1, \cdots, n\}$.

It was an application of the result on his topological entropy $ht(\cdot)$ introduced in the paper ([20]) for automorphisms of nuclear $C^*$-algebras. After then, Brown [3] extended the notion to automorphisms of more large class of $C^*$-algebras, that is, exact $C^*$-algebras.

Let us replace the integer group $\mathbb{Z}$ to a discrete group $G$, and let us replace the abelian von Neumann algebra $L^\infty(X, \mu)$ to a general von Neumann algebra $M$ with a state $\mu$, or a $C^*$-algebra $A$. Then we have the von Neumann crossed product $M \times_\alpha G$ with respect to an action $\alpha$ of $G$ on $M$ with

$$\mu \circ \alpha_g = \mu, \quad \text{for all } g \in G$$

and also we have the $C^*$-crossed product $A \times_\alpha G$ with respect to an action $\alpha$ of $G$ on $A$.

In the case of von Neumann algebras, the state $\mu$ has the natural extension $\bar{\mu}$ to $M \times_\alpha G$ which is $\bar{\theta}$-invariant. If an automorphism $\theta$ of $M$ satisfies that

$$\alpha_g \theta = \theta \alpha_g, \quad \text{for all } g \in G$$

and

$$\mu \circ \theta = \mu,$$

then $\theta$ can be canonically extended to the automorphism $\bar{\theta}$ of $M \times_\alpha G$, and the following problem naturally arises :  

$$h_\mu(\theta) = h_{\bar{\mu}}(\bar{\theta})$$
Similarly in the case of $C^{*}$-algebras, if an automorphism $\theta$ of $A$ satisfies $\alpha_{g}\theta = \theta \alpha_{g}$ for all $g \in G$, then it is canonically extended to the automorphism $\bar{\theta}$ of $A \times_{\alpha} G$, and the following problem naturally arises:

$$ht(\theta) = ht(\bar{\theta})$$

When $G$ is amenable, known results for these two problems are as follows:

**Theorem.** [6, 11, 13]. Assume that $G$ is an amenable discrete countable group.

1. [6, 11]. Let $A$ be a unital exact $C^{*}$-algebra and $\alpha$ an action of $G$ on $A$. If $\theta$ is an automorphism of $A$ such that $\alpha_{g}\theta = \theta \alpha_{g}$ for all $g \in G$, then

$$h_{\phi}(\theta) = h_{\bar{\phi}}(\bar{\theta})$$

2. [13]. Let $M$ be an approximately finite-dimensional von Neumann algebra with a normal state $\phi$ and $\alpha$ an action of $G$ on $M$ with $\phi \cdot \alpha_{g} = \phi$ for all $g \in G$.

If $\theta$ is an automorphism of $M$ such that $\phi \circ \theta = \phi$ and $\alpha_{g}\theta = \theta \alpha_{g}$ for all $g \in G$, then

$$h_{\phi}(\theta) = h_{\bar{\phi}}(\bar{\theta})$$

where $\bar{\phi}$ is the canonical extension of $\phi$ to $M \times_{\alpha} G$.

There are a large class of interesting non amenable discrete groups such as free groups $F_{n}, n \geq 2$ and discrete subgroups of connected Lie groups, etc. However each of these nonamenable groups has an amenable action on some compact space ([1, 2, 15]).

A discrete group $G$ has an amenable action on some compact space if and only if $G$ is exact in the sense of Kirchberg and Wassermann ([14]), that is, its reduced group $C^{*}$-algebra $C_{r}^{*}(G)$ is exact. This is first proved by Ozawa in [15].

Here, we report our results which show that the amenability of $G$ is not always necessary and it is replaced to more large class of groups, that is, exact groups.

2. **BASIC NOTATIONS AND TERMINOLOGIES.**

Proofs of the main results are given using partly some methods in [4, 5, 6, 7, 13]. Here we only denote some basic notations and terminologies.
2.1. Approximation property and exactness. Here our $C^*$-algebras are all separable. Let $M$ be a von Neumann algebra (resp. unital $C^*$-algebra). Then $M$ is called \textit{approximately finite dimensional} if there exists an increasing sequence $(N_k)_k$ of finite dimensional subalgebras such that $\cup_k N_k$ is weakly (resp. norm) dense in $M$.

This approximation property is extended in the case of $C^*$-algebras in $[14, 21]$ as exactness.

A $C^*$-algebra is \textit{exact} if there exists a representation $\pi$ of $A$ on a Hilbert space $H$ and triplets $(\varphi_i, B_i, \psi_i)_i$ of finite dimensional algebras $B_i$, completely positive maps $\varphi_i : A \to B_i$, $\psi_i : B_i \to B(H)$ such that
\[
\| \pi(a) - \psi_i \cdot \varphi_i(a) \| \to 0
\]
for all $a \in A$.

A discrete group $G$ is called \textit{exact} if the $C^*$-algebra $C^*_r(G)$ generated by the left regular representation is exact.

2.2. Entropy. Topological entropy $ht(\theta)$ is defined for an automorphism $\theta$ of an exact $C^*$-algebra. CS-entropy $H(\alpha)$ is defined for an automorphism $\alpha$ of a finite von Neumann algebra $M$ with a finite trace $\tau$ such that $\tau \cdot \alpha = \tau$ and CNT-entropy $h_\phi(\theta)$ is defined for an automorphism $\theta$ of a unital $C^*$-algebra $A$ with a state $\phi$ such that $\phi \cdot \theta = \phi$. They are both called dynamical entropies. CS-entropy $H(\alpha)$ depends on a finite trace $\tau$ such that $\tau \cdot \alpha = \tau$ and CNT-entropy $h_\phi(\theta)$ also depends on a state $\phi$ such that $\phi \cdot \theta = \phi$. Let $M$ be the von Neumann algebra generated by the GNS representation $\pi_\phi(A)$. Then such a $\theta$ as $\phi \cdot \theta = \phi$ is extended to the automorphism $\bar{\theta}$ of $M$ naturally. If $\phi$ is a tracial state, then the natural extension $\bar{\phi}$ is a trace of a finite von Neumann algebra $M$ and
\[
H(\bar{\theta}) = h_\phi(\theta).
\]
If $\phi \cdot \theta = \phi$, then topological entropy $ht(\theta)$ and CNT-entropy $h_\phi(\theta)$ has the following relation:
\[
h_\phi(\theta) \leq ht(\theta).
\]
We refer to these [3, 10, 9, 20]

2.2.1. **Topological entropy.** We refer [3, 20] for definitions and notations about the topological entropy.

2.2.2. **Dynamical entropy.** We refer [10, 9] for definitions and notations about the topological entropy.

2.3. **Crossed product.** Let $A$ be a unital $C^*$-algebra (resp. von Neumann algebra), $G$ a discrete countable group and $\alpha$ be an action of $G$ on $A$, that is a homomorphism from $G$ to the automorphism group Aut($A$) of $A$. We may assume that $A$ is acting on a Hilbert space $H$ faithfully. The crossed product $A \times_\alpha G$ is the $C^*$-subalgebra (resp. von Neumann subalgebra) of

$$B(l^2(G, H)) \cong B(l^2(G)) \otimes B(H)$$

generated by $\pi_\alpha(A)$ and $\lambda_G$, where

$$\pi_\alpha(a)\xi(g) = \alpha_{g^{-1}}(a)\xi(g), \quad (a \in A, g \in G, \xi \in l^2(G, H))$$

and

$$\lambda_g\xi(h) = \xi(g^{-1}h), \quad (a \in A, g \in G, \xi \in l^2(G, H)).$$

Essentially, we use the following representation as in [3, 4, 6, 7, 13, 19]:

$$\pi_\alpha(a)\lambda_g = \sum_{t \in G} e_{t, g^{-1}t} \otimes \alpha_t^{-1}(a), \quad (a \in A, g \in G),$$

where $\{e_{s,t}\}_{s,t \in G}$ is the standard matrix units in $B(l^2(G))$.

Since $G$ is discrete, there exists always the conditional expectation $E$ of $A \times_\alpha G$ onto $\pi_\alpha(A)$ such that

$$E(\lambda_g) = 0$$

for all $g \in G$ except the unit. If $\phi$ is a state of $A$ with $\phi \circ \alpha_g = \phi$ for all $g \in G$, we denote the state $\phi \circ E$ by $\bar{\phi}$ and call it the canonical extension of $\phi$ to $A \times_\alpha G$.

If $\theta \in$ Aut($A$) commutes with $\alpha_g$ for all $g \in G$, then there exists always an automorphism $\bar{\theta} \in$ Aut($A \times_\alpha G$) such that

$$\bar{\theta}(\pi_\alpha(a)\lambda_g) = \pi_\alpha(\theta(a))\lambda_g, \quad (a \in A, g \in G).$$

We call the $\bar{\theta}$ the canonical extension of $\theta$. 
2.4. Amenability. The notion of amenability for groups is generalized to amenability of actions of groups, that is, a group admits an amenable action on some compact space (cf. [1, 2, 5, 14]).

For example, the descriptions in [1] and [5] are as follows:

2.4.1. Amenable action. ([5]):

Let $G$ be a countable discrete group, and let $\alpha^G$ be the action $G \to \text{Aut}(l^\infty(G))$ given by

$$\alpha^G_g(x)(h) = x(g^{-1}h), \quad (x \in l^\infty(G), \; g, h \in G).$$

Let $l^1(G, l^\infty(G))$ be the closure of the linear space of finitely supported functions $T : G \to l^\infty(G)$ with respect to the norm

$$||T||_1 = \sum_g |T(g)| \; ||l^\infty(G)||.$$

Let us put

$$s.T(g) = \alpha^G_g(T(s^{-1}g)), \quad (s, g \in G).$$

The action $\alpha^G$ is amenable if there exist functions $T_n \in l^1(G, l^\infty(G))$ such that

1. $T_n$ is nonnegative (i.e. $T_n(g) \geq 0, (g \in G)$),
2. finitely supported,
3. $\sum_g T_n(g) = 1_{l^\infty(G)}$ and
4. $||s.T_n - T_n||_1 \to 0$ for all $s \in G$.

2.4.2. Amenable at infinity. ([1]):

A group $G$ is amenable at infinity if and only if there exists a sequence $(g_n)_{n \geq 1}$ of nonnegative functions on $G \times G$ with support in a tube such that

a) for each $n$ and each $s$,

$$\sum_t g_n(s, t) = 1,$$

b) uniformly on tubes,

$$\lim_n \sum_{u \in G} |g_n(s, u) - g_n(t, u)| = 0.$$

Here, a tube means the set $\{(s, t) : s^{-1}t \in F\}$ for some finite subset $F$ of $G$. 
2.4.3. **Equivalence.** These two notions of 3.4.1 and 3.4.2 are equivalent. In fact, let
\[(T_n(t))(s) = g_n(s^{-1}t, s^{-1}t)\]
for all \(s, t \in G\), then conditions in one side are implied by the other side.

A group \(G\) is exact if \(G\) admits an amenable action on a compact space ([15]) which is equivalent to that \(G\) is amenable at infinity ([1]) and also it is equivalent to that \(\alpha^G\) is amenable.

2.4.4. **Typical examples of exact groups.**
(1) Amenable groups.
(2) Free groups.
(3) Discrete subgroups of connected Lie groups.
(4) Subgroups, extensions, free products of the above groups.
(5) Quotients by classical amenable groups

3. **MAIN RESULTS**

Our results are followings:

3.1. **Theorem.** Let \(A\) be a unital exact \(C^*\)-algebra, \(G\) an exact discrete countable group, and \(\alpha\) an action of \(G\) on \(A\).

If \(\beta \in \text{Aut}(A \times_{\alpha} G)\) satisfies \(\beta(\lambda_g) = \lambda_g\) for all \(g \in G\) and \(\beta(\pi_{\alpha}(A)) = \pi_{\alpha}(A)\), then
\[ht(\beta) = ht(\beta|_{\pi_{\alpha}(A)}).\]

Here \(\pi_{\alpha}\) is the representation of \(A\) and \(\lambda\) is the unitary representation of \(G\) such that \(A \times_{\alpha} G\) is generated by \(\{\pi(A), \lambda_G\}\).

3.2. **Remark.** We have more general result on the topological entropy. In fact, by replacing the condition that
\[\beta(\lambda_g) = \lambda_g\quad\text{for all } g \in G\]
to the condition that
\[\beta(\lambda_G) = \lambda_G\]
we have a similar result in [7]. This gives an application to the proof of the main theorem in [12].
3.3. **Theorem.** Let $M$ be an approximately finite-dimensional von Neumann algebra with a normal state $\phi$, $G$ an exact discrete countable group, and $\alpha$ an action of $G$ on $M$ with

$$\phi \cdot \alpha_g = \phi \quad \text{for all } g \in G.$$ 

If $\theta$ is an automorphism of $M$ such that $\phi \circ \theta = \phi$ and 

$$\alpha_g \theta = \theta \alpha_g \quad \text{for all } g \in G,$$

then

$$h_\phi(\theta) = h_{\bar{\phi}}(\bar{\theta}),$$

where $\bar{\phi}$ is the canonical extension of $\phi$ to $M \times_\alpha G$.

3.4. **Proof.** Proofs for these results are in [8]. In [8], we adopt as exactness of the group $G$ the amenability of the canonical action $\alpha^G$ in [5].

**References**


