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FACTORIZATION AND HAAGERUP TYPE NORMS ON OPERATOR SPACES

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This is joint work with M.Nagisa (Chiba Univ.). The problem of the factorization through a Hilbert space for a bounded linear map was considered in Banach space theory and its study was started by Grothendieck [7]. Let X and Y be Banach spaces. It is called that $T : X \longrightarrow Y$ factors through a Hilbert space if there exist a Hilbert space $\mathcal{H}$ and bounded linear maps $a : X \longrightarrow \mathcal{H}$, $b : \mathcal{H} \longrightarrow Y$ such that $T = ba$.

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "a"'] & & Y \\ & \mathcal{K} \arrow[ur, "b"'] & \end{tikzcd}$$

We note that given $T : X \longrightarrow Y$, if $T : X \longrightarrow Y^{**}$ factors though a Hilbert space $\mathcal{H}$ then T itself factors through a Hilbert space which is a closed subspace of $\mathcal{H}$. So it is essential to consider the problem in case that Y is a dual space.

Grothendieck introduced the norm $\|\ \|_H$ on the algebraic tensor product $X \otimes Y$ in [7] by

$$\|u\|_H = \inf\{\sup\{(\sum_{i=1}^{n} |f(x_i)|^2)^{\frac{1}{2}} (\sum_{i=1}^{n} |g(y_i)|^2)^{\frac{1}{2}}\}\}$$

where the supremum is taken over all $f \in X^*, g \in Y^*$ with $\|f\|, \|g\| \leq 1$ and the infimum is taken over all representation $u = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$. In this note, we denote by $X \otimes_\alpha Y$ the completion of $X \otimes Y$ by the norm $\|\ \|_\alpha$, and denote by $\|\ \|_{\alpha^*}$ the norm of the dual space $(X \otimes_\alpha Y)^*$. He showed that $T : X \longrightarrow Y^*$ factors through a Hilbert space if and only if $T \in (X \otimes_H Y)^*$ by the natural identification $\langle T(x), y\rangle = T(x \otimes y)$ for $x \in X, y \in Y$, moreover $\inf\{\|b\|\|a\| \mid T = ba\} = \|T\|_{H*}$.

In [15], Lindenstrauss and Pelczynski studied a bounded linear map $T : X \longrightarrow Y$ with the condition:

given any n and $n \times n$ matrices $[a_{ij}] \in \mathbb{M}(\mathbb{C})$ with $\|[a_{ij}]\| \leq 1$, then

$$\sum_{i=1}^{n} \| \sum_{j=1}^{n} a_{ij} T(x_j) \| \leq C \sum_{j=1} \|x_j\|^2 \quad \text{for any } x_1, \cdots, x_n.$$

We consider $T \otimes \alpha : X \otimes \ell^2_n \longrightarrow Y \otimes \ell^2_n$ for $T : X \longrightarrow Y$ and define a norm $\| \sum_{i=1}^n x_i \otimes e_i \|^2 = \sum_{i=1}^n \|x_i\|^2$. Then the above condition is equivalent to $\|T \otimes \alpha\| \leq C\|\alpha\|$ for all $\alpha : \ell^2_n \longrightarrow \ell^2_n$.

Their theorems are summarized for a bounded linear map $T : X \longrightarrow Y^*$ as follows:

The following are equivalent:

(1) $\|T \otimes \alpha\| \leq \|\alpha\|$ for all $\alpha : \ell^2_n \longrightarrow \ell^2_n$ and $n \in \mathbb{N}$.
(2) $\|T\|_{H^*} \leq 1$.
(3) T factors through a Hilbert space $\mathcal{K}$ by bounded linear maps $a : X \longrightarrow \mathcal{K}$ and $b : \mathcal{K} \longrightarrow Y^*$ such that

$$\text{i.e.,} \quad T = ba \quad \text{with} \quad \|a\|\|b\| \leq 1.$$

In C^*-algebra theory and operator space theory, many important factorization theorems have been proved.

Theorem 1. **(Haagerup,** [8]) *Suppose that A and B are C^*-algebras, and $T : A \longrightarrow B^*$ is a bounded linear map. Then T factors through a Hilbert space such that $T = ba$ with $\|T\| \leq 2\|b\|\|a\|$.*

We recall the column (resp. row) Hilbert space $\mathcal{H}_c$(resp. $\mathcal{H}_r$) for a Hilbert space $\mathcal{H}$. If $\xi = [\xi_{ij}] \in M_n(\mathcal{H})$, then we define a map $C_n(\xi)$ by

$$C_n(\xi) : \mathbb{C}^n \ni [\lambda_1, \ldots, \lambda_n] \longmapsto [\sum_{j=1}^{n} \lambda_j \xi_{ij}]_i \in \mathcal{H}^n$$

and denote the column matrix norm by $\|\xi\|_c = \|C_n(\xi)\|$. This operator space structure on $\mathcal{H}$ is called the column Hilbert space and denoted by $\mathcal{H}_c$.

To consider the row Hilbert space, let $\overline{\mathcal{H}}$ be the conjugate Hilbert space for $\mathcal{H}$. We define a map $R_n(\xi)$ by

$$R_n(\xi) : \overline{\mathcal{H}}^n \ni [\overline{\eta}_1, \ldots, \overline{\eta}_n] \longmapsto [\sum_{j=1}^{n} (\xi_{ij}|\eta_j)]_i \in \mathbb{C}^n$$

and the row matrix norm by $\|\xi\|_r = \|R_n(\xi)\|$. This operator space structure on $\mathcal{H}$ is called the row Hilbert space and denoted by $\mathcal{H}_r$.

Let A and B be operator spaces. The Haagerup norm [4] on $A \otimes B$ is defined by

$$\|u\|_h = \inf\{\|[x_1, \dots, x_n]\| \|[y_1, \dots, y_n]^t\| \mid u = \sum_{i=1}^{n} x_i \otimes y_i\},$$

where $[x_1, \dots, x_n] \in M_{1,n}(A)$ and $[y_1, \dots, y_n]^t \in M_{n,1}(B)$.

Theorem 2. (**Effros-Ruan,** [5]) *Suppose that A and B are operator spaces, and $T : A \longrightarrow B^*$ is a completely bounded map. Then T factors through a row Hilbert space $\mathcal{H}_r$ if and only if $T \in (A \otimes_h B)^*$ with $\|T\|_{h^*} = \inf\{\|b\|_{cb}\|a\|_{cb} \mid T = ba\}$.*

Theorem 3. (**Pisier-Shlyakhtenko,** [21]) *Suppose that A and B are C^*-algebras, and $T : A \longrightarrow B^*$ is a completely bounded map. If one of the algebras A, B is exact, then T factors though $\mathcal{H}_c \oplus \mathcal{K}_r$ the direct sum of the column and row Hilbert spaces.*

These factorizations form that

$$\begin{tikzcd} A \arrow[rr, "T"] \arrow[dr, "a"'] & & B^* \\ & \mathcal{K} \arrow[ur, "b"'] & \end{tikzcd}$$

On the other hand, in [12], it has been shown that the following factorization of a linear map T from ℓ^1 to ℓ^∞ in connection with a Schur multiplier:

$$\begin{tikzcd} \ell^1 \arrow[r, "T"] \arrow[d, "a"'] & \ell^\infty \\ \ell^2 \arrow[r, "b"'] & \ell^{2*} \arrow[u, "a^t"'] \end{tikzcd}$$

where a^t is the transposed map of a.

Motivated by this factorization, the aim of this note is to explain a square factorization theorem of a bounded linear map through a pair of Hilbert spaces $\mathcal{H}$ between an operator space and its dual space [13].

More precisely, let us suppose that A and B are operator spaces in $\mathbb{B}(\mathcal{H})$ and denote by $C^*(A)$ the C^*-albebra in $\mathbb{B}(\mathcal{H})$ generated by A. We define the **numerical radius Haagerup norm** of an element $u \in A \otimes B$ by

$$\|u\|_{wh} = \inf\{\frac{1}{2}\|[x_1, \dots, x_n, y_1^*, \dots, y_n^*]\|^2 \mid u = \sum_{i=1}^{n} x_i \otimes y_i\}.$$

By the identity

$$\inf_{\lambda>0} \frac{\lambda\alpha + \lambda^{-1}\beta}{2} = \sqrt{\alpha\beta} \qquad (\star)$$

for positive real numbers $\alpha, \beta \geq 0$, the Haagerup norm can be rewritten as

$$\|u\|_h = \inf\{\frac{1}{2}(\|[x_1, \ldots, x_n]\|^2 + \|[y_1^*, \ldots, y_n^*]\|^2) \mid u = \sum_{i=1}^{n} x_i \otimes y_i\}.$$

Then it is easy to check that

$$\frac{1}{2}\|u\|_h \leq \|u\|_{wh} \leq \|u\|_h$$

and $\|u\|_{wh}$ is a norm.

We also define a norm of an element $u \in C^*(A) \otimes C^*(A)$ by

$$\|u\|_{Wh} = \inf\{\|[x_1, \ldots, x_n]^t\|^2 w(\alpha) \mid u = \sum x_i^* \alpha_{ij} \otimes x_j\},$$

where $w(\alpha)$ is the numerical radius norm of $\alpha = [\alpha_{ij}]$ in $M_n(\mathbb{C})$.

$A \otimes_{Wh} A$ is defined as the closure of $A \otimes A$ in $C^*(A) \otimes_{Wh} C^*(A)$.

Theorem 4. *Let A be an operator space in $\mathbb{B}(\mathcal{H})$. Then $A \otimes_{wh} A = A \otimes_{Wh} A$.*

Let $a : C^*(A) \longrightarrow \mathcal{H}_c$ be a completely bounded map. We define a map $d : C^*(A) \longrightarrow \overline{\mathcal{H}}$ by $d(x) = \overline{a(x^*)}$. It is not hard to check that $d : C^*(A) \longrightarrow \overline{\mathcal{H}}_r$ is completely bounded and $\|a\|_{cb} = \|d\|_{cb}$ when we introduce the row Hilbert space structure to $\overline{\mathcal{H}}$. In this paper, we define the adjoint map a^* of a by the transposed map of d, that is, $d^t : ((\overline{\mathcal{H}})_r)^* = ((\mathcal{H}^*)_r)^* = (\mathcal{H}^{**})_c = \mathcal{H}_c \longrightarrow C^*(A)^*$ (c.f. [5]). More precisely, we define

$$\langle a^*(\eta), x\rangle = \langle \eta, d(x)\rangle = (\eta | a(x^*)) \qquad \text{for} \qquad \eta \in \mathcal{H}, x \in C^*(A).$$

Now we can state a square factorization theorem.

Theorem 5. *Suppose that A is an operator space in $\mathbb{B}(\mathcal{H})$, and that $T : A \times A \longrightarrow \mathbb{C}$ is bilinear. Then the following are equivalent:*

(1) $\|T\|_{wh^*} \leq 1$.

(2) *There exists a state p_0 on $C^*(A)$ such*

$$|T(x, y)| \leq p_0(xx^*)^{\frac{1}{2}} p_0(y^*y)^{\frac{1}{2}} \qquad \textit{for } x, y \in A.$$

(3) *There exist a $*$-representation $\pi : C^*(A) \longrightarrow \mathbb{B}(\mathcal{K})$, a unit vector $\xi \in \mathcal{K}$ and a contraction $b \in \mathbb{B}(\mathcal{K})$ such that*

$$T(x, y) = (\pi(x) b \pi(y) \xi \mid \xi) \qquad \textit{for } x, y \in A.$$

(4) *There exist an extension $T' : C^*(A) \longrightarrow C^*(A)^*$ of T and completely bounded maps $a : C^*(A) \longrightarrow \mathcal{K}_c$, $b : \mathcal{K}_c \longrightarrow \mathcal{K}_c$ such that*

$$\begin{array}{ccc} C^*(A) & \xrightarrow{T'} & C^*(A)^* \\ {\scriptstyle a}\downarrow & & \uparrow{\scriptstyle a^*} \\ \mathcal{K}_c & \xrightarrow[b]{} & \mathcal{K}_c \end{array}$$

i.e., $T' = a^* b a$ with $\|a\|_{cb}^2 \|b\|_{cb} \leq 1$.

Remark 6. (i) If we replace the linear map $\langle T(x), y\rangle = T(x, y)$ with $\langle x, T(y)\rangle = T(x, y)$, then we have a factorization of T through a pair of the row Hilbert spaces $\mathcal{H}_r$. More precisely, the following condition (4)$'$ is equivalent to the above conditions.

(4)$'$ There exist an extension $T' : C^*(A) \longrightarrow C^*(A)^*$ of T and completely bounded maps $a : C^*(A) \longrightarrow \mathcal{K}_r$, $b : \mathcal{K}_r \longrightarrow \mathcal{K}_r$ such that

$$\begin{array}{ccc} C^*(A) & \xrightarrow{T'} & C^*(A)^* \\ {\scriptstyle a}\downarrow & & \uparrow{\scriptstyle a^*} \\ \mathcal{K}_r & \xrightarrow[b]{} & \mathcal{K}_r \end{array}$$

i.e., $T' = a^* b a$ with $\|a\|_{cb}^2 \|b\|_{cb} \leq 1$.

(ii) Let ℓ_n^2 be an n-dimensional Hilbert space with the canonical basis $\{e_1, \ldots, e_n\}$. Given $\alpha : \ell_n^2 \longrightarrow \ell_n^2$ with $\alpha(e_j) = \sum_i \alpha_{ij} e_i$, we set the map $\dot{\alpha} : \ell_n^2 \longrightarrow \ell_n^{2*}$ by $\dot{\alpha}(e_j) = \sum_i \alpha_{ij} \bar{e}_i$ where $\{\bar{e}_i\}$ is the dual basis. For notational convenience, we shall also denote $\dot{\alpha}$ by α. For $\sum_{i=1}^n x_i \otimes e_i \in C^*(A) \otimes \ell_n^2$, we define a norm by $\|\sum_{i=1}^n x_i \otimes e_i\| = \|[x_1, \ldots, x_n]^t\|$. Let $T : C^*(A) \longrightarrow C^*(A)^*$ be a bounded linear map. Consider $T \otimes \alpha : C^*(A) \otimes \ell_n^2 \longrightarrow C^*(A)^* \otimes \ell_n^{2*}$ with a numerical radius type norm $w(\cdot)$ given by

$$w(T \otimes \alpha) = \sup\{|\langle \sum x_i^* \otimes e_i, T \otimes \alpha(\sum x_i \otimes e_i)\rangle| \mid \|\sum x_i \otimes e_i\| \leq 1\}.$$

Then we have

$$\sup\{\frac{w(T\otimes\alpha)}{w(\alpha)} \mid \alpha:\ell_n^2\longrightarrow\ell_n^2,\ n\in\mathbb{N}\}=\|T\|_{wh^*},$$

since $T(\sum x_i^*\alpha_{ij}\otimes x_j)=\langle\sum x_i^*\otimes e_i, T\otimes\alpha(\sum x_i\otimes e_i)\rangle$.

(iii) Let $u=\sum x_i\otimes y_i\in C^*(A)\otimes C^*(A)$. It is straightfoward from Theorem 2.3 that

$$\|u\|_{wh}=\sup w(\sum\varphi(x_i)b\varphi(y_i))$$

where the supremum is taken over all $*$- preserving completely contractions φ and contractions b.

We also define a variant of the numerical radius Haagerup norm of an element $u\in A\otimes B$ by

$$\|u\|_{wh'}=\inf\{\frac{1}{2}\|[x_1,\ldots,x_n,y_1,\ldots,y_n]^t\|^2 \mid u=\sum_{i=1}^n x_i\otimes y_i\},$$

where $[x_1,\ldots,x_n,y_1,\ldots,y_n]^t\in M_{2n,1}(A+B)$, and denote by $A\otimes_{wh'}B$ the completion of $A\otimes B$ with the norm $\|\ \|_{wh'}$.

We remark that $\|\ \|_{wh}$ and $\|\ \|_{wh'}$ are not equivalent, since $\|\ \|_{'h}$ in [10] is equivalent to $\|\ \|_{wh'}$ and $\|\ \|_h$ and $\|\ \|_{'h}$ are not equivalent [10], [14].

In the next theorem, we use the transposed map $a^t:(\mathcal{K}_c)^*\longrightarrow C^*(A)^*$ of $a:C^*(A)^*\longrightarrow\mathcal{K}_c$ instead of $a^*:\mathcal{K}_c\longrightarrow C^*(A)^*$. We note that $(\mathcal{K}_c)^*=(\overline{\mathcal{K}})_r$ and the relation a and a^t is given by

$$\langle a^t(\bar\eta),x\rangle=\langle\bar\eta,a(x)\rangle=(\bar\eta|\overline{a(x)})_{\overline{\mathcal{K}}}\quad\text{for }\bar\eta\in\overline{\mathcal{K}},x\in C^*(A).$$

Theorem 7. *Suppose that A is an operator space in $\mathbb{B}(\mathcal{H})$, and that $T:A\times A\longrightarrow\mathbb{C}$ is bilinear. Then the following are quivalent:*

(1) $\|T\|_{wh'^*}\le 1$.

(2) *There exists a state p_0 on $C^*(A)$ such that*

$$|T(x,y)|\le p_0(x^*x)^{\frac{1}{2}}p_0(y^*y)^{\frac{1}{2}}\quad\textit{for } x,y\in A.$$

(3) *There exist a $*$-representation $\pi:C^*(A)\longrightarrow\mathbb{B}(\mathcal{K})$, a unit vector $\xi\in\mathcal{K}$ and a contraction $b:\mathcal{K}\longrightarrow\overline{\mathcal{K}}$ such that*

$$T(x,y)=(b\pi(y)\xi\mid\overline{\pi(x)\xi})_{\overline{\mathcal{K}}}\quad\textit{for } x,y\in A.$$

(4) *There exist a completely bounded map* $a : A \longrightarrow \mathcal{K}_c$ *and a bounded map* $b : \mathcal{K}_c \longrightarrow (\mathcal{K}_c)^*$ *such that*

$$\begin{array}{ccc} A & \xrightarrow{T} & A^* \\ {\scriptstyle a}\downarrow & & \uparrow{\scriptstyle a^t} \\ \mathcal{K}_c & \xrightarrow[b]{} & (\mathcal{K}_c)^* \end{array}$$

$$i.e., \quad T = a^t ba \quad with \quad \|a\|_{cb}^2 \|b\| \leq 1.$$

Now we can describe the above theorems in terms of Banach space theory.

Let X be a Banach space. Recall that the minimal quantization $\mathrm{Min}(X)$ of X. Let Ω_X be the unit ball of X^*, that is, $\Omega_X = \{f \in X^* | \ \|f\| \leq 1\}$. For $[x_{ij}] \in M_n(X)$, $\|[x_{ij}]\|_{\min}$ is defined by

$$\|[x_{ij}]\|_{\min} = \sup\{\|[f(x_{ij})]\| \mid f \in \Omega_X\}.$$

Then $\mathrm{Min}(X)$ can be regarded as a subspace in the C^*-algebra $C(\Omega_X)$ of all continuous functions on the compact Hausdorff space Ω_X. Here we define a norm of an element $u \in X \otimes X$ by

$$\|u\|_{wH} = \inf\{\sup\{(\sum_{i=1}^{n} |f(x_i)|^2)^{\frac{1}{2}} (\sum_{i=1}^{n} |f(y_i)|^2)^{\frac{1}{2}}\}\},$$

where the supremum is taken over all $f \in X^*$ with $\|f\| \leq 1$ and the infimum is taken over all representation $u = \sum_{i=1}^{n} x_i \otimes y_i$.

Let $T : X \longrightarrow X^*$ be a bounded linear map. We consider the map $T \otimes \alpha : X \otimes \ell_n^2 \longrightarrow X^* \otimes \ell_n^{2*}$ and define a norm for $\sum x_i \otimes e_i \in X \otimes \ell_n^2$ by

$$\|\sum x_i \otimes e_i\| = \sup\{(\sum |f(x_i)|^2)^{\frac{1}{2}} \mid f \in \Omega_X\}.$$

We note that, given $x \in X$, x^* is regarded as $\langle x^*, f\rangle = \overline{f(x)}$ for $f \in X^*$ in the definition of $w(T \otimes \alpha)$, that is,

$$w(T \otimes \alpha) = \sup\{|\langle \sum x_i^* \otimes e_i, T \otimes \alpha(\sum x_i \otimes e_i)\rangle| \mid \|\sum x_i \otimes e_i\| \leq 1\}.$$

Finally we can state the following result which can be seen as a numerical radius norm version of Grothendieck, Lindenstrauss-Pelczynski's.

Corollary 8. *Suppose that X is a Banach space, and that $T : X \longrightarrow X^*$ is a bounded linear map. Then the following are quivalent:*

(1) $w(T \otimes \alpha) \leq w(\alpha)$ *for all* $\alpha : \ell_n^2 \longrightarrow \ell_n^2$ *and* $n \in \mathbb{N}$.

(2) $\|T\|_{wH^*} \leq 1$.

(3) *T factors through a Hilbert space $\mathcal{K}$ and its dual space $\mathcal{K}^*$ by bounded linear maps $a : X \longrightarrow \mathcal{K}$ and $b : \mathcal{K} \longrightarrow \mathcal{K}^*$ as follows:*

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "a"'] & X^* \\ \mathcal{K} \arrow[r, "b"'] & \mathcal{K}^* \arrow[u, "a^t"'] \end{tikzcd}$$

$$\textit{i.e.,} \quad T = a^t ba \quad \textit{with} \quad \|a\|^2\|b\| \leq 1.$$

(4) *T has an extention $T' : C(\Omega_X) \longrightarrow C(\Omega_X)^*$ which factors through a pair of Hilbert spaces $\mathcal{K}$ by bounded linear maps $a : C(\Omega_X) \longrightarrow \mathcal{K}$ and $b : \mathcal{K} \longrightarrow \mathcal{K}$ as follows:*

$$\begin{tikzcd} C(\Omega_X) \arrow[r, "T'"] \arrow[d, "a"'] & C(\Omega_X)^* \\ \mathcal{K} \arrow[r, "b"'] & \mathcal{K} \arrow[u, "a^*"'] \end{tikzcd}$$

$$\textit{i.e.,} \quad T' = a^* ba \quad \textit{with} \quad \|a\|^2\|b\| \leq 1.$$

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