<table>
<thead>
<tr>
<th>Title</th>
<th>FACTORIZATION AND HAAGERUP TYPE NORMS ON OPERATOR SPACES (Quantum Analysis in Operator Algebras)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Itoh, Takashi</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 2004, 1354: 20-28</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2004-01</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/25160">http://hdl.handle.net/2433/25160</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
FACTORIZATION AND HAAGERUP TYPE NORMS
ON OPERATOR SPACES

群馬大学・教育学部 伊藤 隆 (Takashi Itoh)
Dept. of Math., Fac. of Edu., Gunma University

This is joint work with M.Nagisa (Chiba Univ.). The problem of
the factorization through a Hilbert space for a bounded linear map
was considered in Banach space theory and its study was started by
Grothendieck [7]. Let $X$ and $Y$ be Banach spaces. It is called that
$T : X \rightarrow Y$ factors through a Hilbert space if there exist a Hilbert
space $\mathcal{H}$ and bounded linear maps $a : X \rightarrow \mathcal{H}$, $b : \mathcal{H} \rightarrow Y$ such that
$T = ba$.

We note that given $T : X \rightarrow Y$, if $T : X \rightarrow Y^{**}$ factors though
a Hilbert space $\mathcal{H}$ then $T$ itself factors through a Hilbert space which
is a closed subspace of $\mathcal{H}$. So it is essential to consider the problem in
case that $Y$ is a dual space.

Grothendieck introduced the norm $\| \|_H$ on the algebraic tensor
product $X \otimes Y$ in [7] by

$$
\| u \|_H = \inf \{ \sup \{ (\sum_{i=1}^{n} |f(x_i)|^2)^{\frac{1}{2}} (\sum_{i=1}^{n} |g(y_i)|^2)^{\frac{1}{2}} \} \}
$$

where the supremum is taken over all $f \in X^*$, $g \in Y^*$ with $\|f\|, \|g\| \leq 1$
and the infimum is taken over all representation $u = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$. In this note, we denote by $X \otimes_\alpha Y$ the completion of $X \otimes Y$ by the
norm $\| \|_\alpha$, and denote by $\| \|_{\alpha^*}$ the norm of the dual space $(X \otimes_\alpha Y)^*$. He showed that $T : X \rightarrow Y^{*}$ factors through a Hilbert space if and
only if $T \in (X \otimes_H Y)^*$ by the natural identification $\langle T(x), y \rangle = T(x \otimes y)$
for $x \in X, y \in Y$, moreover $\inf \{ \| b \| \| a \| \mid T = ba \} = \| T \|_{H^*}$.

In [15], Lindenstrauss and Pelczynski studied a bounded linear map
$T : X \rightarrow Y$ with the condition:
given any $n$ and $n \times n$ matrices $[a_{ij}] \in \mathcal{M}(\mathbb{C})$ with $\|[a_{ij}]\| \leq 1$, then
\[
\sum_{i=1}^{n} \left\| \sum_{j=1}^{n} a_{ij}T(x_{j}) \right\| \leq C \sum_{j=1}^{n} \|x_{j}\|^2 \text{ for any } x_{1}, \ldots, x_{n}.
\]

We consider $T \otimes \alpha : X \otimes \ell_{n}^{2} \rightarrow Y \otimes \ell_{n}^{2}$ for $T : X \rightarrow Y$ and define a norm $\| \sum_{i=1}^{n} x_{i} \otimes e_{i} \|^2 = \sum_{i=1}^{n} \|x_{i}\|^2$. Then the above condition is equivalent to $\|T \otimes \alpha\| \leq C\|\alpha\|$ for all $\alpha : \ell_{n}^{2} \rightarrow \ell_{n}^{2}$.

Their theorems are summarized for a bounded linear map $T : X \rightarrow Y^*$ as follows:

The following are equivalent:
1. $\|T \otimes \alpha\| \leq \|\alpha\|$ for all $\alpha : \ell_{n}^{2} \rightarrow \ell_{n}^{2}$ and $n \in \mathbb{N}$.
2. $\|T\|_{H^*} \leq 1$.
3. $T$ factors through a Hilbert space $\mathcal{K}$ by bounded linear maps $a : X \rightarrow \mathcal{K}$ and $b : \mathcal{K} \rightarrow Y^*$ such that
   \[ i.e., T = ba \text{ with } \|a\|\|b\| \leq 1. \]

In $C^*$-algebra theory and operator space theory, many important factorization theorems have been proved.

**Theorem 1.** (Haagerup, [8]) Suppose that $A$ and $B$ are $C^*$-algebras, and $T : A \rightarrow B^*$ is a bounded linear map. Then $T$ factors through a Hilbert space such that $T = ba$ with $\|T\| \leq 2\|b\|\|a\|$.

We recall the column (resp. row) Hilbert space $\mathcal{H}_{c}(\text{resp. } \mathcal{H}_{r})$ for a Hilbert space $\mathcal{H}$. If $\xi = [\xi_{ij}] \in M_{n}(\mathcal{H})$, then we define a map $C_{n}(\xi)$ by
\[
C_{n}(\xi) : \mathbb{C}^{n} \ni [\lambda_{1}, \ldots, \lambda_{n}] \mapsto \left[ \sum_{j=1}^{n} \lambda_{j} \xi_{ij} \right] i \in \mathcal{H}^{n}
\]
and denote the column matrix norm by $\|\xi\|_{c} = \|C_{n}(\xi)\|$. This operator space structure on $\mathcal{H}$ is called the column Hilbert space and denoted by $\mathcal{H}_{c}$.

To consider the row Hilbert space, let $\overline{\mathcal{H}}$ be the conjugate Hilbert space for $\mathcal{H}$. We define a map $R_{n}(\xi)$ by
\[
R_{n}(\xi) : \overline{\mathcal{H}}^{n} \ni [\overline{\eta}_{1}, \ldots, \overline{\eta}_{n}] \mapsto \left[ \sum_{j=1}^{n} (\xi_{ij}|\eta_{j}) \right] i \in \mathbb{C}^{n}
\]
and the row matrix norm by $\|\xi\|_{r} = \|R_{n}(\xi)\|$. This operator space structure on $\mathcal{H}$ is called the row Hilbert space and denoted by $\mathcal{H}_{r}$.
Let $A$ and $B$ be operator spaces. The Haagerup norm [4] on $A \otimes B$ is defined by
\[
\|u\|_h = \inf\{\|x_1, \ldots, x_n\|\|y_1, \ldots, y_n\|^t | u = \sum_{i=1}^n x_i \otimes y_i\},
\]
where $[x_1, \ldots, x_n] \in M_{1,n}(A)$ and $[y_1, \ldots, y_n]^t \in M_{n,1}(B)$.

**Theorem 2.** (Effros-Ruan, [5]) Suppose that $A$ and $B$ are operator spaces, and $T : A \to B^*$ is a completely bounded map. Then $T$ factors through a row Hilbert space $\mathcal{H}_r$ if and only if $T \in (A \otimes B)^*$ with $\|T\|_{h^*} = \inf\{\|b\|_{cb} \|a\|_{cb} | T = ba\}$.

**Theorem 3.** (Pisier-Shlyakhtenko, [21]) Suppose that $A$ and $B$ are $C^*$-algebras, and $T : A \to B^*$ is a completely bounded map. If one of the algebras $A, B$ is exact, then $T$ factors through $\mathcal{H}_c \oplus \mathcal{K}_r$ the direct sum of the column and row Hilbert spaces.

These factorizations form that

\[
\begin{array}{ccc}
A & \xrightarrow{T} & B^* \\
a & \downarrow & b \\
\mathcal{K} & \downarrow & \mathcal{K} \\
\ell^2 & \xrightarrow{a^t} & \ell^2^*
\end{array}
\]

On the other hand, in [12], it has been shown that the following factorization of a linear map $T$ from $\ell^1$ to $\ell^\infty$ in connection with a Schur multiplier:

\[
\begin{array}{ccc}
\ell^1 & \xrightarrow{T} & \ell^\infty \\
a & \downarrow & a^t \\
\ell^2 & \xrightarrow{b} & \ell^2^*
\end{array}
\]

where $a^t$ is the transposed map of $a$.

Motivated by this factorization, the aim of this note is to explain a square factorization theorem of a bounded linear map through a pair of Hilbert spaces $\mathcal{H}$ between an operator space and its dual space [13].

More precisely, let us suppose that $A$ and $B$ are operator spaces in $\mathcal{B}(\mathcal{H})$ and denote by $C^*(A)$ the $C^*$-algebra in $\mathcal{B}(\mathcal{H})$ generated by $A$. We define the **numerical radius Haagerup norm** of an element $u \in A \otimes B$ by
\[
\|u\|_{wh} = \inf\{\frac{1}{2}||[x_1, \ldots, x_n, y_1^*, \ldots, y_n^*]\|^2 | u = \sum_{i=1}^n x_i \otimes y_i\}.
\]
By the identity
\[ \inf_{\lambda>0} \frac{\lambda \alpha + \lambda^{-1} \beta}{2} = \sqrt{\alpha \beta} \]  
for positive real numbers \( \alpha, \beta \geq 0 \), the Haagerup norm can be rewritten as
\[ ||u||_h = \inf \left\{ \frac{1}{2} (||x_1, \ldots, x_n||^2 + ||y_1^*, \ldots, y_n^*||^2) \mid u = \sum_{i=1}^n x_i \otimes y_i \right\}. \]

Then it is easy to check that
\[ \frac{1}{2} ||u||_h \leq ||u||_{wh} \leq ||u||_h \]
and \( ||u||_{wh} \) is a norm.

We also define a norm of an element \( u \in C^*(A) \otimes C^*(A) \) by
\[ ||u||_{wh} = \inf \{ ||x_1, \ldots, x_n||^2 w(\alpha) \mid u = \sum x_i \alpha_{ij} \otimes x_j \}, \]
where \( w(\alpha) \) is the numerical radius norm of \( \alpha = [\alpha_{ij}] \) in \( M_n(\mathbb{C}) \).

\( A \otimes_{Wh} A \) is defined as the closure of \( A \otimes A \) in \( C^*(A) \otimes_{Wh} C^*(A) \).

**Theorem 4.** Let \( A \) be an operator space in \( \mathbb{B}(\mathcal{H}) \). Then \( A \otimes_{wh} A = A \otimes_{Wh} A \).

Let \( a : C^*(A) \rightarrow \mathcal{H}_c \) be a completely bounded map. We define a map \( d : C^*(A) \rightarrow \overline{\mathcal{H}} \) by \( d(x) = \overline{a(x^*)} \). It is not hard to check that \( d : C^*(A) \rightarrow \overline{\mathcal{H}} \) is completely bounded and \( \|a\|_{cb} = \|d\|_{cb} \) when we introduce the row Hilbert space structure to \( \overline{\mathcal{H}} \). In this paper, we define the adjoint map \( a^* \) of \( a \) by the transposed map of \( d \), that is, \( d^* : ((\mathcal{H})_r)^* = ((\mathcal{H}^*)_r)^* = (\mathcal{H}^{**})_c = \mathcal{H}_c \rightarrow C^*(A)^* \) (c.f. [5]). More precisely, we define
\[ \langle a^*(\eta), x \rangle = \langle \eta, d(x) \rangle = (\eta|a(x^*)) \quad \text{for} \quad \eta \in \mathcal{H}, x \in C^*(A). \]

Now we can state a square factorization theorem.

**Theorem 5.** Suppose that \( A \) is an operator space in \( \mathbb{B}(\mathcal{H}) \), and that \( T : A \times A \rightarrow \mathbb{C} \) is bilinear. Then the following are equivalent:

1. \( ||T||_{wh^*} \leq 1 \).
2. There exists a state \( p_0 \) on \( C^*(A) \) such
\[ |T(x,y)| \leq p_0(xx^*)^{1/2} p_0(y^*y)^{1/2} \quad \text{for} \quad x, y \in A. \]
(3) There exist a \(*\)-representation \(\pi : C^*(A) \to \mathcal{B}(\mathcal{K})\), a unit vector \(\xi \in \mathcal{K}\) and a contraction \(b \in \mathcal{B}(\mathcal{K})\) such that
\[
T(x, y) = (\pi(x)b\pi(y)\xi \mid \xi) \quad \text{for } x, y \in A.
\]

(4) There exist an extension \(T' : C^*(A) \to C^*(A)^*\) of \(T\) and completely bounded maps \(a : C^*(A) \to \mathcal{K}_c\), \(b : \mathcal{K}_c \to \mathcal{K}_c\) such that
\[
C^*(A) \xrightarrow{T'} C^*(A)^* \quad \text{i.e., } T' = a^*ba \text{ with } \|a\|_{cb}^2\|b\|_{cb} \leq 1.
\]

Remark 6. (i) If we replace the linear map \(\langle T(x), y \rangle = T(x, y)\) with \(\langle x, T(y) \rangle = T(x, y)\), then we have a factorization of \(T\) through a pair of the row Hilbert spaces \(\mathcal{H}_r\). More precisely, the following condition \((4)'\) is equivalent to the above conditions.

\((4)\)' There exist an extension \(T' : C^*(A) \to C^*(A)^*\) of \(T\) and completely bounded maps \(a : C^*(A) \to \mathcal{K}_r\), \(b : \mathcal{K}_r \to \mathcal{K}_r\) such that
\[
C^*(A) \xrightarrow{T'} C^*(A)^* \quad \text{i.e., } T' = a^*ba \text{ with } \|a\|_{cb}^2\|b\|_{cb} \leq 1.
\]

(ii) Let \(\ell^2_n\) be an \(n\)-dimensional Hilbert space with the canonical basis \(\{e_1, \ldots, e_n\}\). Given \(\alpha : \ell^2_n \to \ell^2_n\) with \(\alpha(e_j) = \sum_i \alpha_{ij} e_i\), we set the map \(\hat{\alpha} : \ell^2_n \to \ell^*_n\) by \(\hat{\alpha}(e_j) = \sum_i \alpha_{ij} \bar{e}_i\) where \(\{\bar{e}_i\}\) is the dual basis. For notational convenience, we shall also denote \(\hat{\alpha}\) by \(\alpha\). For \(\sum_{i=1}^n x_i \otimes e_i \in C^*(A) \otimes \ell^2_n\), we define a norm by \(\|\sum_{i=1}^n x_i \otimes e_i\| = \|[x_1, \ldots, x_n]^t\|\). Let \(T : C^*(A) \to C^*(A)^*\) be a bounded linear map. Consider \(T \otimes \alpha : C^*(A) \otimes \ell^2_n \to C^*(A)^* \otimes \ell^*_n\) with a numerical radius type norm \(w(\cdot)\) given by
\[
w(T \otimes \alpha) = \sup\{||\sum x_i^* \otimes e_i, T \otimes \alpha(\sum x_i \otimes e_i)|| \mid ||\sum x_i \otimes e_i|| \leq 1\}.
\]
Then we have
\[
\sup \left\{ \frac{w(T \otimes \alpha)}{w(\alpha)} \mid \alpha : \ell_n^2 \to \ell_n^2, \ n \in \mathbb{N} \right\} = \|T\|_{\text{wh}^*},
\]
since \( T(\sum x_i^* \alpha_{ij} \otimes x_j) = (\sum x_i^* \otimes e_i, T \otimes \alpha(\sum x_i \otimes e_i)) \).

(iii) Let \( u = \sum x_i \otimes y_i \in C^*(A) \otimes C^*(A) \). It is straightforward from Theorem 2.3 that
\[
\|u\|_{\text{wh}} = \sup u(\sum \varphi(x_i) b \varphi(y_i))
\]
where the supremum is taken over all \(*\)-preserving completely contractions \( \varphi \) and contractions \( b \).

We also define a variant of the numerical radius Haagerup norm of an element \( u \in A \otimes B \) by
\[
\|u\|_{\text{wh}'} = \inf \left\{ \frac{1}{2} \left\| [x_1, \ldots, x_n, y_1, \ldots, y_n]^t \right\|^2 \mid u = \sum_{i=1}^{n} x_i \otimes y_i \},
\]
where \([x_1, \ldots, x_n, y_1, \ldots, y_n]^t \in M_{2n,1}(A + B)\), and denote by \( A \otimes_{\text{wh}'} B \)
the completion of \( A \otimes B \) with the norm \( \| \cdot \|_{\text{wh}'} \).

We remark that \( \| \cdot \|_{\text{wh}} \) and \( \| \cdot \|_{\text{wh}'} \) are not equivalent, since \( \| \cdot \|_h \) in [10] is equivalent to \( \| \cdot \|_{wh'} \) and \( \| \cdot \|_h \) and \( \| \cdot \|_{\text{wh}'} \) are not equivalent [10], [14].

In the next theorem, we use the transposed map \( a^t : (\mathcal{K}_c)^* \to C^*(A)^* \) of \( a : C^*(A)^* \to \mathcal{K}_c \) instead of \( a^* : \mathcal{K}_c \to C^*(A)^* \). We note that \( (\mathcal{K}_c)^* = \overline{(\mathcal{K})}_r \) and the relation \( a \) and \( a^t \) is given by
\[
\langle a^t(\bar{\eta}), x \rangle = \langle \bar{\eta}, a(x) \rangle = (\bar{\eta}|a(x))_{\overline{\mathcal{K}}} \quad \text{for} \ \bar{\eta} \in \overline{\mathcal{K}}, \ x \in C^*(A).
\]

**Theorem 7.** Suppose that \( A \) is an operator space in \( \mathbb{B}(\mathcal{H}) \), and that \( T : A \times A \to \mathbb{C} \) is bilinear. Then the following are equivalent:

1. \( \|T\|_{\text{wh}^*} \leq 1 \).
2. There exists a state \( p_0 \) on \( C^*(A) \) such that
\[
|T(x, y)| \leq p_0(x^* x)^{1/2} p_0(y^* y)^{1/2} \quad \text{for} \ x, y \in A.
\]
3. There exist a \(*\)-representation \( \pi : C^*(A) \to \mathbb{B}(\mathcal{K}) \), a unit vector \( \xi \in \mathcal{K} \) and a contraction \( b : \mathcal{K} \to \overline{\mathcal{K}} \) such that
\[
T(x, y) = (b\pi(y)\xi \mid \pi(x)\xi)_{\overline{\mathcal{K}}} \quad \text{for} \ x, y \in A.
\]
(4) There exist a completely bounded map \( a : A \rightarrow \mathcal{K}_c \) and a bounded map \( b : \mathcal{K}_c \rightarrow (\mathcal{K}_c)^* \) such that

\[
\begin{array}{ccc}
A & \xrightarrow{T} & A^* \\
\downarrow a & & \uparrow a^t \\
\mathcal{K}_c & \xrightarrow{b} & (\mathcal{K}_c)^*
\end{array}
\]

i.e., \( T = a^t ba \) with \( \|a\|\|b\| \leq 1 \).

Now we can describe the above theorems in terms of Banach space theory.

Let \( X \) be a Banach space. Recall that the minimal quantization \( \text{Min}(X) \) of \( X \). Let \( \Omega_X \) be the unit ball of \( X^* \), that is, \( \Omega_X = \{ f \in X^* | \|f\| \leq 1 \} \). For \( [x_{ij}] \in M_n(X) \), \( \|[x_{ij}]\|_{\text{min}} \) is defined by

\[
\|[x_{ij}]\|_{\text{min}} = \sup\{ \|[f(x_{ij})]\| | f \in \Omega_X \}.
\]

Then \( \text{Min}(X) \) can be regarded as a subspace in the \( C^* \)-algebra \( C(\Omega_X) \) of all continuous functions on the compact Hausdorff space \( \Omega_X \). Here we define a norm of an element \( u \in X \otimes X \) by

\[
\|u\|_{wH} = \inf\{ \sup\{ (\sum_{i=1}^{n}|f(x_{i})|^2)^{\frac{1}{2}} (\sum_{i=1}^{n}|f(y_{i})|^2)^{\frac{1}{2}} \} | f \in \Omega_X \}.
\]

where the supremum is taken over all \( f \in X^* \) with \( \|f\| \leq 1 \) and the infimum is taken over all representation \( u = \sum_{i=1}^{n} x_i \otimes y_i \).

Let \( T : X \rightarrow X^* \) be a bounded linear map. We consider the map \( T \otimes \alpha : X \otimes \ell_n^2 \rightarrow X^* \otimes \ell_2^* \) and define a norm for \( \sum x_i \otimes e_i \in X \otimes \ell_n^2 \) by

\[
\| \sum x_i \otimes e_i \| = \sup\{ (\sum_{i=1}^{n}|f(x_{i})|^2)^{\frac{1}{2}} | f \in \Omega_X \}.
\]

We note that, given \( x \in X \), \( x^* \) is regarded as \( (x^*, f) = f(x) \) for \( f \in X^* \) in the definition of \( w(T \otimes \alpha) \), that is,

\[
w(T \otimes \alpha) = \sup\{ |\langle \sum x_i^* \otimes e_i, T \otimes \alpha(\sum x_i \otimes e_i) \rangle | | \sum x_i \otimes e_i \| \leq 1 \}.
\]

Finally we can state the following result which can be seen as a numerical radius norm version of Grothendieck, Lindenstrauss-Pelczynski's.
Corollary 8. Suppose that $X$ is a Banach space, and that $T : X \to X^*$ is a bounded linear map. Then the following are equivalent:

1. $w(T \otimes \alpha) \leq w(\alpha)$ for all $\alpha : \ell_2^n \to \ell_2^n$ and $n \in \mathbb{N}$.
2. $\|T\|_{wH^*} \leq 1$.
3. $T$ factors through a Hilbert space $\mathcal{K}$ and its dual space $\mathcal{K}^*$ by bounded linear maps $a : X \to \mathcal{K}$ and $b : \mathcal{K} \to \mathcal{K}^*$ as follows:

$$
\begin{array}{ccc}
X & \xrightarrow{T} & X^* \\
\downarrow & & \uparrow a^* \\
\mathcal{K} & \xrightarrow{b} & \mathcal{K}^*
\end{array}
$$

i.e., $T = a^*ba$ with $\|a\|^2\|b\| \leq 1$.
4. $T$ has an extension $T' : C(\Omega_X) \to C(\Omega_X)^*$ which factors through a pair of Hilbert spaces $\mathcal{K}$ by bounded linear maps $a : C(\Omega_X) \to \mathcal{K}$ and $b : \mathcal{K} \to \mathcal{K}$ as follows:

$$
\begin{array}{ccc}
C(\Omega_X) & \xrightarrow{T'} & C(\Omega_X)^* \\
\downarrow & & \uparrow a^* \\
\mathcal{K} & \xrightarrow{b} & \mathcal{K}
\end{array}
$$

i.e., $T' = a^*ba$ with $\|a\|^2\|b\| \leq 1$.

REFERENCES


