Displacement Operator and Generalization of Cameron–Martin–Girsanov Theorem

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Dedicated to Professor Nobuaki Obata on the occasion of his sixtieth birthday

Abstract

We study the displacement operators within the framework of quantum white noise calculus. The displacement operators are characterized by implementation problems which are equivalent to linear differential equations associated with the quantum white noise derivatives for white noise operators. Then the displacement operators are applied to study a generalization of the Cameron–Martin–Girsanov theorem. More precisely, we prove that the affine transform, with an isometric dilation and a regular drift, of a Brownian motion is again a Brownian motion with respect to a new probability measure which is derived explicitly in terms of the displacement operators.

Keywords: Boson filed, canonical commutation relation, Fock space, implementation problem, displacement operator, Girsanov transform

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1 Introduction

Let $C_0([0,1],\mathbb{R})$ be the (standard) Wiener space with the (standard) Wiener measure P and H_0 be the Cameron–Martin space, i.e., the Hilbert space consisting of all absolutely continuous functions on $[0,1]$ such that their derivatives are square integrable. Then famous Cameron-Martin translation theorem [2] states that the measure P is quasi-invariant under the transformation

$$C_0([0,1],\mathbb{R}) \ni \omega \longmapsto T_{x_0}(\omega) = \omega + x_0 \in C_0([0,1],\mathbb{R}), \tag{1.1}$$

where $x_0 \in \mathcal{H}$ with $x_0(0) = 0$. Furthermore, the Radon-Nikodym derivative is given by

$$\frac{dP(T_{x_0}(\omega))}{dP(\omega)} = \exp\left\{-\int_0^1 x_0'(s)d\omega(s) - \frac{1}{2}\int_0^1 (x_0'(s))^2 ds\right\}. \tag{1.2}$$

For each $t \in [0,1]$, consider the random variable $B_t : C_0([0,1],\mathbb{R}) \to \mathbb{R}$ defined as the evaluation map, i.e., $B_t(\omega) = \omega(t)$ for any $\omega \in C_0([0,1],\mathbb{R})$. Then the stochastic process $\{B_t\}_{t\in[0,1]}$ is called a (standard) *Wiener process* or *Brownian motion* satisfying that

(B1) $P(\{\omega; B_0(\omega) = 0\}) = 1$;

(B2) for each $0 \le s < t \le 1$, $B_t - B_s$ is a Gaussian random variable with mean 0 and variance $t - s$;

(B3) $\{B_t\}_{t\in[0,1]}$ has independent increments, i.e., for any $0 \le t_1 < t_2 < \cdots < t_n \le 1$, the random variables B_{t_1}, $B_{t_2} - B_{t_1}$, $\cdots$, $B_{t_n} - B_{t_{n-1}}$ are independent;

(B4) almost all sample paths of $\{B_t\}_{t\in[0,1]}$ are continuous.

In general, a stochastic process $\{B_t\}_{t\in[0,1]}$ satisfying the properties (B1)–(B4) is called a *Brownian motion*, see [17]. In fact, the condition (B4) can be proved from the condition (B2) by applying the Kolmogorov continuity theorem.

The Cameron-Martin translation theorem has been extended by Girsanov [4] to the shifts of a Brownian motion and then Girsanov proved that a Brownian motion with a regular drift is again a Brownian motion with respect a new probability measure which is called the *Girsanov transform* (see also [21]).

On the other hand, the displacement operator in the quantum field theory plays an important role in the study of coherent and squeezed states. Since the meaning of the displacement operator is realized by its quadrature representation, we can find some relations between the Cameron-Martin translation theorem and the quadrature representation of the displacement operator. In this paper, motivated from this observation, we study the Cameron–Martin–Girsanov theorem in terms of the displacement operators.

Main purpose of this paper is to study an affine transform of a Brownian motion and then we prove that the affine transform, with an isometric dilation and a regular drift, of a Brownian motion is again a Brownian motion with respect to a new probability measure which is explicitly described.

For our purpose, we basically accept the idea used in [14]. Based on the quantum white noise calculus, we first study the displacement operators which is a slight generalization of the typical displacement operators in the quantum field theory. Then the displacement operators are characterized by implementation problems which are equivalent to linear differential equations associated with the quantum white noise derivatives for white noise operators. From the implementation problems we induce an implementation problem for a Brownian motion and its affine transform. Then the (generalized) displacement operators as the solution of the implementation problem are applied to study a generalization of the Cameron–Martin–Girsanov theorem to the affine transform of a Brownian motion. More precisely, we prove that the affine transform, with an isometric dilation and a regular drift, of a Brownian motion is again a Brownian motion with respect to a new probability measure which is derived explicitly in terms of the displacement operators.

This paper is organized as follows. In Section 2, we briefly review the basic notions in quantum white noise calculus. In Section 3, we recall the quantum white noise derivatives, the Wick derivations and differential equations of Wick type. In Section 4, we study the displacement operators as solutions of implementation problems and their properties. In Section 5, we establish a generalization of the Cameron–Martin–Girsanov theorem to the affine transform of a Brownian motion.

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2 White Noise Operators

2.1 Gaussian Spaces

Let $H = L^2(I, dt)$ be the complex Hilbert space of square-integrable functions on an interval I, where $I = \mathbb{R}^n$ or $I = [0, T]^n$ for $T > 0$ and $n \in \mathbb{N}$. The canonical $\mathbb{C}$-bilinear form on H and the norm are defined by

$$\langle \xi, \eta \rangle = \int_I \xi(t)\eta(t)\, dt, \qquad |\xi|_0^2 = \langle \bar{\xi}, \xi \rangle = \int_I |\xi(t)|^2\, dt,$$

respectively.

Let A be a positive, selfadjoint operator densely defined in H with Hilbert-Schmidt inverse, and assume that A is real, i.e., if ξ is a $\mathbb{R}$-valued function, so is $A\xi$. For each $p \geq 0$, the power A^p becomes canonically a selfadjoint operator with a dense domain $\mathrm{Dom}\,(A^p) \subset H$. Then, the domain $E_p := \mathrm{Dom}\,(A^p)$ itself becomes a Hilbert space equipped with the norm

$$|\xi|_p = |A^p \xi|_0, \qquad \xi \in \mathrm{Dom}\,(A^p).$$

For a positive $p > 0$, we define E_{-p} to be the Hilbert space by taking the completion of H with respect to the norm:

$$|\xi|_{-p} = |A^{-p}\xi|_0, \qquad \xi \in H.$$

Then we come to a chain of Hilbert spaces and their limit spaces:

$$E \subset \cdots \subset E_p \subset \cdots \subset H \subset \cdots \subset E_{-p} \subset \cdots \subset E^*, \tag{2.1}$$

where

$$E = \operatorname*{proj\,lim}_{p\to\infty} E_p, \qquad E^* = \operatorname*{ind\,lim}_{p\to\infty} E_{-p}\,.$$

Since the natural injection $E_{p+1} \to E_p$ is of Hilbert-Schmidt type by assumption on A, we know that E is a nuclear space. Hence we construct a Gelfand triple: $E \subset H \subset E^*$ of which the real Gelfand triple is denoted by $E_{\mathbb{R}} \subset H_{\mathbb{R}} \subset E_{\mathbb{R}}^*$.

By the Bochner–Minlos–Yamasaki theorem there exists a unique probability measure μ on $E^*_{\mathbb{R}}$ such that

$$\exp\left(-\frac{1}{2}|\,\xi\,|_0^2\right) = \int_{E^*_{\mathbb{R}}} e^{i\langle x,\,\xi\rangle}\mu(dx), \qquad \xi \in E_{\mathbb{R}}.$$

This μ is referred to as the *(standard) Gaussian measure* and the probability space $(E^*_{\mathbb{R}}, \mu)$ as a *white noise space* or a *Gaussian space.* In general, a (generalized) function on the white noise space is called a *white noise function.*

2.2 Hida–Kubo–Takenaka Space

The *(Boson) Fock space* over the Hilbert space H is defined by

$$\Gamma(H) = \left\{\phi = (f_n)\,;\, f_n \in H^{\hat{\otimes} n}\,,\ \|\phi\|^2 = \sum_{n=0}^{\infty} n!|f_n|_0^2 < \infty\right\},$$

where $|f_n|_0$ is the usual norm of the n-fold symmetric tensor power $H^{\hat{\otimes} n} = L^2_{\mathrm{sym}}(I^n)$. Constructing Fock spaces over the chain of Hilbert spaces (2.1), we obtain a chain of Fock spaces and their limit spaces:

$$(E) \subset \cdots \subset \Gamma(E_p) \subset \cdots \subset \Gamma(H) \subset \cdots \subset \Gamma(E_{-p}) \subset \cdots \subset (E)^*, \tag{2.2}$$

where

$$(E) = \operatorname*{proj\,lim}_{p\to\infty} \Gamma(E_p), \qquad (E)^* = \operatorname*{ind\,lim}_{p\to\infty} \Gamma(E_{-p}).$$

In particular, we come to the Gelfand triple:

$$(E) \subset \Gamma(H) \subset (E)^*, \tag{2.3}$$

which is referred to as the *Hida–Kubo–Takenaka space* [15] (see also [16, 20]). By construction, (E) is a countable Hilbert nuclear space whose topology is defined by the norms

$$\|\,\phi\,\|_p^2 = \sum_{n=0}^{\infty} n!\,|f_n|_p^2, \qquad \phi = (f_n) \in (E), \qquad p \in \mathbb{R},$$

and $(E)^*$ is the strong dual space of (E). The canonical $\mathbb{C}$-bilinear form $\langle\!\langle \cdot,\, \cdot\rangle\!\rangle$ on $(E)^* \times (E)$ takes the form:

$$\langle\!\langle \Phi,\, \phi\rangle\!\rangle = \sum_{n=0}^{\infty} n!\,\langle F_n,\, f_n\rangle\,, \qquad \Phi = (F_n) \in (E)^*, \quad \phi = (f_n) \in (E),$$

where $\langle F_n,\, f_n\rangle$ is the canonical $\mathbb{C}$-bilinear form on $(E^{\otimes n})^* \times E^{\otimes n}$.

There is a canonical isomorphism, referred to as the *Wiener–Itô–Segal isomorphism,* between $L^2(E^*_{\mathbb{R}}, \mu)$ and $\Gamma(H)$ determined uniquely by the correspondence

$$\phi_\xi(x) \equiv \exp\left(\langle x,\, \xi\rangle - \frac{1}{2}\langle \xi,\, \xi\rangle\right) \quad \longleftrightarrow \quad \left(1, \xi, \frac{\xi^{\otimes 2}}{2!}, \cdots, \frac{\xi^{\otimes n}}{n!}, \cdots\right), \qquad \xi \in E.$$

The above ϕ_ξ is called an *exponential vector* or a *coherent vector.* If $\phi \in L^2(E^*_{\mathbb{R}}, \mu)$ and $(f_n) \in \Gamma(H)$ are related under the Wiener–Itô–Segal isomorphism, we write $\phi = (f_n)$ for simplicity. In that case it holds that

$$\|\,\phi\,\|_0^2 \equiv \int_{E^*_{\mathbb{R}}} |\phi(x)|^2\mu(dx) = \sum_{n=0}^{\infty} n!\,|\,f_n\,|_0^2\,. \tag{2.4}$$

2.3 White Noise Operators

A continuous linear operator from (E) into $(E)^*$ is called a *white noise operator*. The space of all white noise operators from (E) into $(E)^*$ is denoted by $\mathcal{L}((E),(E)^*)$ and is equipped with the bounded convergence topology.

For each $\Xi \in \mathcal{L}((E),(E)^*)$ we denote by $\Xi^* \in \mathcal{L}((E),(E)^*)$ the adjoint operator with respect to the canonical bilinear form, i.e.,

$$\langle\langle \Xi\phi,\, \psi \rangle\rangle = \langle\langle \Xi^*\psi,\, \phi \rangle\rangle, \qquad \phi, \psi \in (E).$$

The hermitian inner product of $\Gamma(H)$ is denoted by

$$\langle\langle \phi | \psi \rangle\rangle = \langle\langle \bar{\phi}, \psi \rangle\rangle = \sum_{n=0}^{\infty} n! \langle \bar{f}_n, g_n \rangle, \qquad \phi = (f_n), \quad \psi = (g_n) \in \Gamma(H),$$

and then the hermitian adjoint $\Xi^\dagger$ of $\Xi \in \mathcal{L}(\Gamma(H),\Gamma(H))$ satisfies that

$$\langle\langle \Xi\phi \,|\, \psi \rangle\rangle = \langle\langle \phi \,|\, \Xi^\dagger \psi \rangle\rangle, \qquad \phi, \psi \in \Gamma(H).$$

We have a simple relation:

$$\Xi^\dagger \phi = \overline{\Xi^* \overline{\phi}}, \quad \phi \in \Gamma(H).$$

With each $x \in E^*$ we associate a white noise operator, called the *annihilation operator*, uniquely specified by

$$a(x) : (0, \dots, 0, \xi^{\otimes n}, 0, \dots) \mapsto (0, \dots, 0, n \langle x,\, \xi \rangle \, \xi^{\otimes (n-1)}, 0, \dots), \qquad \xi \in E,$$

and the adjoint $a^*(x)$, called the *creation operator*, is uniquely specified by

$$a^*(x) : (0, \dots, 0, \xi^{\otimes n}, 0, \dots) \mapsto (0, \dots, 0, x \hat{\otimes} \xi^{\otimes n}, 0, \dots), \qquad \xi \in E.$$

These are unbounded operators in Fock space $\Gamma(H)$, but become white noise operators.

Lemma 2.1. *Let $x \in E^*$ be given.*

(1) *We have $a(x) \in \mathcal{L}((E),(E))$ and $a^*(x) \in \mathcal{L}((E)^*,(E)^*)$.*

(2) *If $x \in E$, then both $a(x)$ and $a^*(x)$ belong to $\mathcal{L}((E),(E)) \cap \mathcal{L}((E)^*,(E)^*)$.*

Proof. The proof is by routine application of Schwarz inequality, for relevant argument see [20]. □

Note also that the canonical commutation relation (CCR) takes the form:

$$[a(\eta), a(\zeta)] = [a^*(\eta), a^*(\zeta)] = 0, \qquad [a(\eta), a^*(\zeta)] = \langle \eta,\, \zeta \rangle\,, \tag{2.5}$$

where η and ζ are members of E or may be taken from E^* whenever the commutators are well-defined according to Lemma 2.1.

The *annihilation and creation operators* at a point $t \in I$ are defined by

$$a_t = a(\delta_t), \qquad a_t^* = a^*(\delta_t),$$

respectively. We often refer to $\{a_t, a_t^* \,;\, t \in I\}$ as the *quantum white noise* over I. For each $\kappa \in (E^{\otimes(l+m)})^*$, a white noise operator $\Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}((E),(E)^*)$ is defined by

$$\Xi_{l,m}(\kappa_{l,m}) = \int_{I^{l+m}} \kappa_{l,m}(s_1, \cdots, s_l, t_1, \cdots, t_m) \\ \times a_{s_1}^* \cdots a_{s_l}^* a_{t_1} \cdots a_{t_m} ds_1 \cdots ds_l dt_1 \cdots dt_m$$

and is called an *integral kernel operator*. The above integral expression is instructive but formal, for the precise definition see [6, 20]. Similar expressions have been used in common literatures along different formulations [1, 5, 18]. By definition, for each $x \in E^*$, we have

$$a(x) = \Xi_{0,1}(x) = \int_I x(t) a_t \, dt, \qquad a^*(x) = \Xi_{1,0}(x) = \int_I x(t) a_t^* \, dt. \tag{2.6}$$

By the nuclear kernel theorem we have $\mathcal{L}(E, E^*) \cong (E \otimes E)^*$, where the correspondence is given by

$$\langle \tau_S, \, \eta \otimes \xi \rangle = \langle S\xi, \, \eta \rangle \,, \qquad \xi, \eta \in E,$$

or even formally,

$$S\xi(s) = \int_I \tau_S(s,t)\xi(t) \, dt.$$

With each $S \in \mathcal{L}(E, E^*)$, we associate the integral kernel operator defined by

$$\Lambda(S) = \Xi_{1,1}(\tau_S) = \int_{I \times I} \tau_S(s,t) a_s^* a_t \, ds dt,$$

which is called a *conservation operator*. It is known that $\Lambda(S) \in \mathcal{L}((E),(E)^*)$ and that $\Lambda(S) \in \mathcal{L}((E),(E))$ if and only if $S \in \mathcal{L}(E, E)$, or equivalently $\tau_S \in E \otimes E^*$. For a more detailed study of white noise operators, we refer to [6, 13, 19, 20].

3 Wick Derivations and Associated Equations

3.1 Creation and Annihilation Derivatives

For $\zeta \in E$ and $\Xi \in \mathcal{L}((E),(E)^*)$, we define

$$D_\zeta^+ \Xi = [a(\zeta), \Xi] = a(\zeta)\Xi - \Xi a(\zeta), \tag{3.1}$$

$$D_\zeta^- \Xi = -[a^*(\zeta), \Xi] = \Xi a^*(\zeta) - a^*(\zeta)\Xi, \tag{3.2}$$

where the composition of white noise operators in the right-hand sides are well-defined by (2) in Lemma 2.1. Then $D_\zeta^+ \Xi$ and $D_\zeta^- \Xi$ are called the *creation derivative* and *annihilation derivative* of Ξ, respectively, and both together is called the *quantum white noise derivatives* (see [8, 9, 10]). By definition,

$$(D_\zeta^+ \Xi)^* = D_\zeta^- \Xi^*, \qquad (D_\zeta^- \Xi)^* = D_\zeta^+ \Xi^*. \tag{3.3}$$

Moreover, it is proved [11] that $(\zeta, \Xi) \mapsto D_\zeta^\pm \Xi$ is a continuous bilinear map from $E \times \mathcal{L}((E),(E)^*)$ into $\mathcal{L}((E),(E)^*)$.

In view of the definition (3.1) the quantum white noise derivative D_x^+ is defined for $x \in E^*$ if the action is restricted to $\mathcal{L}((E),(E))$. In fact, $(x, \Xi) \mapsto D_x^+\Xi$ becomes a continuous bilinear map from $E^* \times \mathcal{L}((E),(E))$ into $\mathcal{L}((E),(E))$. Similarly, $(x, \Xi) \mapsto D_x^-\Xi$ becomes a continuous bilinear map from $E^* \times \mathcal{L}((E)^*,(E)^*)$ into $\mathcal{L}((E)^*,(E)^*)$. These assertions are shown by simple application of Lemma 2.1. For $x = \delta_t$ we come to a pointwisely defined derivatives $D_t^\pm\Xi$, in this connection see [9, 10].

Example 3.1. Let $x \in E^*$ and $S \in \mathcal{L}(E, E^*)$. Then for each $\zeta \in E$, it holds that

$$\begin{aligned} D_\zeta^- a(x) &= \langle x,\, \zeta\rangle\,, & D_\zeta^+ a(x) &= 0, \\ D_\zeta^- a^*(x) &= 0, & D_\zeta^+ a^*(x) &= \langle x,\, \zeta\rangle\,, \\ D_\zeta^- \Lambda(S) &= a^*(S\zeta), & D_\zeta^+ \Lambda(S) &= a(S^*\zeta). \end{aligned} \tag{3.4}$$

3.2 Wick Derivations

It is known that $\{\phi_\xi\,;\, \xi \in E\}$ is linearly independent and spans a dense subspace of (E). Hence every white noise operator $\Xi \in \mathcal{L}((E),(E)^*)$ is uniquely determined by its *symbol* defined by

$$\widehat{\Xi}(\xi, \eta) = \langle\!\langle \Xi\phi_\xi,\, \phi_\eta \rangle\!\rangle\,, \qquad \xi, \eta \in E.$$

More precisely, the operator symbols of white noise operators are characterized by certain analytic and growth conditions, so called the analytic characterization [19, 20]. Similar results have been obtained for various classes of white noise operators, see e.g., [7] and references cited therein.

For each $\Xi_1, \Xi_2 \in \mathcal{L}((E),(E)^*)$, by applying the analytic characterization of symbols, we see that there exists a unique white noise operator $\Xi \in \mathcal{L}((E),(E)^*)$ such that

$$\widehat{\Xi}(\xi, \eta) = \widehat{\Xi}_1(\xi, \eta)\widehat{\Xi}_2(\xi, \eta)e^{-\langle \xi, \eta\rangle}, \qquad \xi, \eta \in E.$$

The above Ξ is called the *Wick product* or *normal-ordered product* and is denoted by

$$\Xi = \Xi_1 \diamond \Xi_2\,.$$

Equipped with the Wick product, $\mathcal{L}((E),(E)^*)$ becomes a commutative $*$-algebra. For more discussion, see [3].

A continuous linear map $\mathcal{D} : \mathcal{L}((E),(E)^*) \to \mathcal{L}((E),(E)^*)$ is called a *Wick derivation* if

$$\mathcal{D}(\Xi_1 \diamond \Xi_2) = (\mathcal{D}\Xi_1) \diamond \Xi_2 + \Xi_1 \diamond (\mathcal{D}\Xi_2), \qquad \Xi_1, \Xi_2 \in \mathcal{L}((E),(E)^*).$$

The following result is important to our approach.

Theorem 3.2 ([12]). *For any $\zeta \in E$, the creation and annihilation derivatives $D_\zeta^\pm$ are Wick derivations from $\mathcal{L}((E),(E)^*)$ into itself.*

Remark 3.3. Roughly speaking, every Wick derivation is a linear combination of the quantum white noise derivatives. For more details, see [12, Theorem 3.7].

3.3 Differential Equations Associated with Wick Derivations

Given a Wick derivation $\mathcal{D} : \mathcal{L}((E),(E)^*) \to \mathcal{L}((E),(E)^*)$ and a white noise operator $G \in \mathcal{L}((E),(E)^*)$, we consider a differential equation of the form:

$$\mathcal{D}\Xi = G \diamond \Xi. \tag{3.5}$$

As in the case of ordinary differential equations, the solution is described by a type of the exponential function. For a white noise operator Y, the *Wick exponential* is defined by

$$\operatorname{wexp} Y = \sum_{n=0}^{\infty} \frac{1}{n!} Y^{\diamond n}$$

whenever the series converges in $\mathcal{L}((E),(E)^*)$, see [3, 12].

Theorem 3.4 ([12]). *Let $\mathcal{D} : \mathcal{L}((E),(E)^*) \to \mathcal{L}((E),(E)^*)$ be a Wick derivation and let $G \in \mathcal{L}((E),(E)^*)$ be a white noise operator. Assume that there exists an operator $Y \in \mathcal{L}((E),(E)^*)$ such that $\mathcal{D}Y = G$ and* $\operatorname{wexp} Y$ *is defined as an operator in $\mathcal{L}((E),(E)^*)$. Then every solution to* (3.5) *is given by*

$$\Xi = F \diamond \operatorname{wexp} Y, \tag{3.6}$$

where $F \in \mathcal{L}((E),(E)^*)$ satisfying $\mathcal{D}F = 0$.

Example 3.5. Let $\Xi \in \mathcal{L}((E),(E)^*)$ be given. Then $D_\zeta^+\Xi = D_\zeta^-\Xi = 0$ for all $\zeta \in E$ if and only if Ξ is a scalar operator (see Lemma 3.9 in [9]).

For each $S \in \mathcal{L}(E, E^*)$, the second quantized operator $\Gamma(S) \in \mathcal{L}((E),(E)^*)$ is defined by

$$\Gamma(A)\phi = (A^{\otimes n} f_n), \qquad \phi = (f_n) \in (E).$$

It is noted that $\Gamma(S)$ is expressible in terms of Wick exponential as follows.

Lemma 3.6. *For $S \in \mathcal{L}(E, E^*)$, we have*

$$\Gamma(S) = \operatorname{wexp} \Lambda(S - 1).$$

4 Displacement Operators

For each $S \in \mathcal{L}(E, E)$, $x \in E^*$ and $\xi \in E$, put

$$b_{S,x}(\xi) = a(S\xi) - \langle x,\, \xi \rangle, \qquad b^*_{S,x}(\xi) = a^*(S\xi) - \langle x,\, \xi \rangle .$$

We note that

$$\begin{aligned}(b_{S,,x}(\xi))^\dagger &= \overline{(a(S\xi) - \langle x,\, \xi \rangle)^*} = a^*(\overline{S}\,\overline{\xi}) - \langle \overline{x},\, \overline{\xi} \rangle \\ &= b^*_{\overline{S},\overline{x}}(\overline{\xi}).\end{aligned}$$

In particular, the operators $b_{I,x}(\xi)$ and $b^*_{I,x}(\xi)$ are denoted by $b_x(\xi)$ and $b^*_x(\xi)$, respectively.

Lemma 4.1. $\{b_{S,x}(\zeta), b^*_{\overline{S},x}(\zeta)\,;\, S \in \mathcal{L}(E,E),\ x \in E^*,\ \zeta \in E\}$ *satisfies the canonical commutation relation if and only if*

$$S^\dagger S = I. \tag{4.1}$$

Proof. For each $S \in \mathcal{L}(E,E)$, $x \in E^*$ and $\eta, \zeta \in E$, we obtain that

$$[b_{S,x}(\eta), b^*_{\overline{S},x}(\zeta)] = [a(S\eta), a^*(\overline{S}\zeta)] = \left\langle S\eta,\, \overline{S}\zeta \right\rangle,$$

which implies the assertion. □

Let $S \in \mathcal{L}(E,E)$ and $x \in E^*$. We now consider the problem to find an operator $D_{S,x} \in \mathcal{L}((E),(E)^*)$ satisfying the intertwining properties:

$$D_{S,x}a(\xi) = b_{S,x}(\xi)D_{S,x}, \tag{4.2}$$
$$D_{S,x}a^*(\xi) = b^*_{\overline{S},\overline{x}}(\xi)D_{S,x} \tag{4.3}$$

for any $\xi \in E$. In particular, for simple notation, $D_{I,x}$ is denoted by D_x and it is called the *displacement operator* associated with x. Therefore, the operator $D_{S,x}$ as a solution of the implementation problems (4.2) and (4.3) can be considered as a generalization of the displacement operator. The displacement operator D_x satisfies the intertwining properties:

$$D_x a(\xi) = b_x(\xi)D_x, \qquad D_x a^*(\xi) = b^*_{\overline{x}}(\xi)D_x \tag{4.4}$$

for any $\xi \in E$.

Let $S \in \mathcal{L}(E,E)$ such that $S^* \in \mathcal{L}(E,E)$. Then S can be extended to E^* as a continuous linear operator acting on E^* by

$$\langle Sw,\, \xi\rangle = \langle w,\, S^*\xi\rangle$$

for any $w \in E^*$ and $\xi \in E$. For the extension, we used the same symbol S.

Theorem 4.2. *Let $S \in \mathcal{L}(E,E)$ such that $S^* \in \mathcal{L}(E,E)$ and $x \in E^*$. Suppose that S is an isometry. Then $D_{S,x} \in \mathcal{L}((E),(E)^*)$ is a solution of the implementation problem given as in* (4.2) *and* (4.3) *if and only if it is of the form*

$$D_{S,x} = H \diamond e^{a^*(\overline{S}x)}\Gamma(\overline{S})e^{-a(\overline{x})} \tag{4.5}$$

for some $H \in \mathcal{L}((E),(E)^*)$ such that $D^+_{S\xi}H = 0$ and $D^-_{\xi}H = 0$. In particular, for any constant $c \in \mathbb{C}$, the white noise operator $D_{S,x} \in \mathcal{L}((E),(E)^*)$ given by

$$D_{S,x} = ce^{a^*(\overline{S}x)}\Gamma(\overline{S})e^{-a(\overline{x})} \tag{4.6}$$

is a solution of the implementation problem given as in (4.2) *and* (4.3).

Proof. Let $\Xi \in \mathcal{L}((E),(E)^*)$ be a solution of the implementation problem given as in (4.2) and (4.3). Then Ξ satisfies the following equations:

$$\Xi a(\xi) = (a(S\xi) - \langle x,\, \xi\rangle)\,\Xi,$$
$$\Xi a^*(\xi) = \left(a^*(\overline{S}\xi) - \langle \overline{x},\, \xi\rangle\right)\Xi$$

for any $\xi \in E$, which is equivalent to the differential equations associated with the quantum white noise derivatives:

$$D_{S\xi}^{+}\Xi = (a((I-S)\xi) + \langle x,\, \xi\rangle) \diamond \Xi, \tag{4.7}$$

$$D_{\xi}^{-}\Xi = \left(a^*((\overline{S}-I)\xi) - \langle \overline{x},\, \xi\rangle\right) \diamond \Xi \tag{4.8}$$

for any $\xi \in E$. On the other hand, by applying Example 3.1 we obtain that

$$D_{S\xi}^{+}\left(\Lambda(\overline{S}-I) + a^*(\overline{S}x)\right) = a((I-S)\xi) + \left\langle \overline{S}x,\, S\xi\right\rangle = a((I-S)\xi) + \langle x,\, \xi\rangle\,,$$

from which, by applying Theorem 3.4, the solution Ξ of (4.7) is of the form

$$\Xi = F \diamond \mathrm{wexp}\left\{\Lambda(\overline{S}-I) + a^*(\overline{S}x)\right\} \tag{4.9}$$

for some $F \in \mathcal{L}((E),(E)^*)$ such that $D_{S\xi}^{+}F = 0$. Also, by applying Example 3.1 again we obtain that

$$D_{\xi}^{-}\left(\Lambda(\overline{S}-I) - a(\overline{x})\right) = a^*((\overline{S}-I)\xi) - \langle \overline{x},\, \xi\rangle\,,$$

which implies that the solution Ξ of (4.8) is of the form

$$\Xi = G \diamond \mathrm{wexp}\left\{\Lambda(\overline{S}-I) - a(\overline{x})\right\} \tag{4.10}$$

for some $G \in \mathcal{L}((E),(E)^*)$ such that $D_{\xi}^{-}G = 0$. Therefore, from (4.9) and (4.10), a solution Ξ of (4.7) and (4.8) is of the form

$$\Xi = H \diamond \mathrm{wexp}\left\{a^*(\overline{S}x) + \Lambda(\overline{S}-I) - a(\overline{x})\right\}$$

for some $H \in \mathcal{L}((E),(E)^*)$ such that $D_{S\xi}^{+}H = 0$ and $D_{\xi}^{-}H = 0$. Since $\mathrm{wexp}\,\Lambda(S-I) = \Gamma(S)$ from Lemma 3.6, we complete the proof. □

Corollary 4.3. *Let $x \in E^*$ be given. Then $D_x \in \mathcal{L}((E),(E)^*)$ is the displacement operator associated with x, i.e., it is a solution of the implementation problem given as in* (4.4) *if and only if it is of the form*

$$D_x = c e^{a^*(x)} e^{-a(\overline{x})} \tag{4.11}$$

for some constant $c \in \mathbb{C}$.

Proof. By Theorem 4.2, the solution of (4.4) is given as in (4.5) with $S = I$ and $H \in \mathcal{L}((E),(E)^*)$ such that $D_{\xi}^{+}H = D_{\xi}^{-}H = 0$, and then by Example 3.5, H is a scalar operator. Hence the proof is completed. □

Theorem 4.4. *Let $S \in \mathcal{L}(E,E)$ such that $S^* \in \mathcal{L}(E,E)$ and $x \in H$. Suppose that S is an isometry. Then the operator $D_{S,x} \in \mathcal{L}((E),(E)^*)$ given as in* (4.6) *is an isometry if and only if $c = ze^{-|x|_0^2/2}$ for some $z \in \mathbb{C}$ with $|z| = 1$.*

Proof. From (4.6), we have

$$D_{S,x}^{\dagger} = \overline{c}e^{-a^*(x)}\Gamma(S^*)e^{a(S\overline{x})} \tag{4.12}$$

Note that

$$\begin{aligned} e^{a(f)}e^{a^*(g)} &= e^{\langle f,g\rangle}e^{a^*(g)}e^{a(f)}, \\ \Gamma(S)e^{a^*(\zeta)} &= e^{a^*(S\zeta)}\Gamma(S), \\ e^{a(\zeta)}\Gamma(S) &= \Gamma(S)e^{a(S^*\zeta)} \end{aligned} \tag{4.13}$$

Therefore, since S is an isometry, i.e., $S^\dagger S = 1$, by applying (4.13), we obtain that

$$\begin{aligned} D_{S,x}^\dagger D_{S,x} &= |c|^2 e^{-a^*(x)}\Gamma(S^*)e^{a(S\overline{x})}e^{a^*(\overline{S}x)}\Gamma(\overline{S})e^{-a(\overline{x})} \\ &= |c|^2 e^{\left\langle S\overline{x}, \overline{S}x\right\rangle} e^{-a^*(x)}\Gamma(S^*)e^{a^*(\overline{S}x)}e^{a(S\overline{x})}\Gamma(\overline{S})e^{-a(\overline{x})} \\ &= |c|^2 e^{\langle \overline{x}, x\rangle} e^{-a^*(x)}e^{a^*(S^*\overline{S}x)}\Gamma(S^*)\Gamma(\overline{S})e^{a(S^\dagger S\overline{x})}e^{-a(\overline{x})} \\ &= |c|^2 e^{|x|_0^2}. \end{aligned}$$

Hence $D_{S,x}$ is an isometry if and only if $|c|^2 e^{|x|_0^2} = 1$ if and only if $c = ze^{-|x|_0^2/2}$ with $|z| = 1$. □

From now on, we consider the isometric operator $D_{S,x}$ given by

$$D_{S,x} = e^{-\frac{1}{2}|x|_0^2} e^{a^*(\overline{S}x)}\Gamma(\overline{S})e^{-a(\overline{x})}, \tag{4.14}$$

with isometry S and $x \in H$. If S is real, i.e., $\overline{S} = S$, then the isometric operator $D_{S,x}$ is given by

$$D_{S,x} = e^{-\frac{1}{2}|x|_0^2} e^{a^*(Sx)}\Gamma(S)e^{-a(x)}, \tag{4.15}$$

Corollary 4.5. *Let $S \in \mathcal{L}(E,E)$ such that $S^* \in \mathcal{L}(E,E)$ and $x \in H$. Suppose that S and x are real and S is an isometry. Then the isometric operator $D_{S,x}$ given as in* (4.15) *satisfies the intertwining properties:*

$$\begin{aligned} D_{S,x}a(\xi) &= b_{S,x}(\xi)D_{S,x}, \\ D_{S,x}a^*(\xi) &= b_{S,x}^*(\xi)D_{S,x} \end{aligned} \tag{4.16}$$

for any $\xi \in E$.

Proof. The proof is immediate from Theorems 4.2 and 4.4. □

Theorem 4.6. *Let $S \in \mathcal{L}(E,E)$ such that $S^* \in \mathcal{L}(E,E)$ and $x \in H$. Suppose that S is unitary. Then the operator $D_{S,x} \in \mathcal{L}((E),(E)^*)$ given as in* (4.14) *is unitary.*

Proof. By Theorem 4.4, the operator $D_{S,x}$ given as in (4.14) is an isometry. Also, by similar computations used in the proof of Theorem 4.4, for $c = e^{-|x|_0^2/2}$, we obtain that

$$\begin{aligned} D_{S,x}D_{S,x}^\dagger &= |c|^2 e^{a^*(\overline{S}x)}\Gamma(\overline{S})e^{-a(\overline{x})}e^{-a^*(x)}\Gamma(S^*)e^{a(S\overline{x})} \\ &= |c|^2 e^{\langle \overline{x}, x\rangle} e^{a^*(\overline{S}x)}\Gamma(\overline{S})e^{-a^*(x)}e^{-a(\overline{x})}\Gamma(S^*)e^{a(S\overline{x})} \\ &= |c|^2 e^{\langle \overline{x}, x\rangle} e^{a^*(\overline{S}x)}e^{-a^*(\overline{S}x)}\Gamma(\overline{S})\Gamma(S^*)e^{-a(S\overline{x})}e^{a(S\overline{x})} \\ &= I. \end{aligned}$$

Therefore, the operator $D_{S,x}$ is a coisometry and so it is unitary. □

Let $\mathcal{U}(H)$ be the family of all unitary operators in $\mathcal{L}(H,H)$. Then the set $\mathcal{U}(H) \times H$ becomes a group, denoted by $\mathcal{U}(H) \ltimes H$, with the group operation defined by

$$(S,x)\cdot(T,y) = (ST, T^*x + y), \qquad (S,x),(T,y) \in \mathcal{U}(H) \times H. \tag{4.17}$$

Also, the set $\mathbb{T} \times \mathcal{U}(H) \times H$ becomes a group, denoted by $\mathbb{T} \times \mathcal{U}(H) \ltimes H$, with the group operation defined by

$$(\alpha, S, x)\cdot(\beta, T, y) = (\alpha\beta e^{i\mathrm{Im}(\langle x, T\overline{y}\rangle)}, ST, T^*x + y) \tag{4.18}$$

for any $(\alpha, S, x), (\beta, T, y) \in \mathbb{T} \times \mathcal{U}(H) \ltimes H$, where $\mathbb{T} = \{z \in \mathbb{C}\,;\, |z| = 1\}$.

Theorem 4.7. *For any $(S,x),(T,y) \in \mathcal{U}(H) \times H$, it holds that*

$$D_{S,x}D_{T,y} = e^{i\mathrm{Im}(\langle x,\, T\overline{y}\rangle)} D_{ST,T^*x+y}. \tag{4.19}$$

$$D_{S,x}D_{S,y} = e^{i\mathrm{Im}(\langle x,\, S\overline{y}\rangle)} D_{S^2,S^*x+y}, \tag{4.20}$$

$$D_{S,x}^{-1} = D_{S,x}^{\dagger} = D_{S^\dagger,-\overline{S}x}. \tag{4.21}$$

Proof. By direct computation using the intertwining properties given as in (4.13), we obtain that

$$\begin{aligned}
D_{S,x}D_{T,y} &= e^{-\frac{1}{2}(|x|_0^2+|y|_0^2)} e^{a^*(\overline{S}x)} \Gamma(\overline{S}) e^{-a(\overline{x})} e^{a^*(\overline{T}y)} \Gamma(\overline{T}) e^{-a(\overline{y})} \\
&= e^{-\frac{1}{2}\left(|x|_0^2+|y|_0^2+2\langle \overline{x},\overline{T}y\rangle\right)} e^{a^*(\overline{S}x)} \Gamma(\overline{S}) e^{a^*(\overline{T}y)} e^{-a(\overline{x})} \Gamma(\overline{T}) e^{-a(\overline{y})} \\
&= e^{-\frac{1}{2}\left(|x|_0^2+|y|_0^2+2\langle \overline{x},\overline{T}y\rangle\right)} e^{a^*(\overline{S}x+\overline{S}\,\overline{T}y)} \Gamma(\overline{S}\,\overline{T}) e^{-a(T^\dagger\overline{x}+\overline{y})} \\
&= e^{-\frac{1}{2}\left(|x|_0^2+|y|_0^2+2\langle \overline{x},\overline{T}y\rangle-|T^*x+y|_0^2\right)} D_{ST,T^*x+y} \\
&= e^{-\frac{1}{2}\left(\langle \overline{x},\overline{T}y\rangle-\overline{\langle \overline{x},\overline{T}y\rangle}\right)} D_{ST,T^*x+y} \\
&= e^{i\mathrm{Im}(\langle x,\, T\overline{y}\rangle)} D_{ST,T^*x+y},
\end{aligned}$$

which gives the proof of (4.19). The proofs of (4.20) and (4.21) are straightforward from (4.19). □

Let $\mathcal{U}(H_{\mathbb{R}})$ be the family of all real and unitary operators in $\mathcal{L}(H_{\mathbb{R}}, H_{\mathbb{R}})$. Then $\mathcal{U}(H_{\mathbb{R}}) \ltimes H_{\mathbb{R}}$ and $\mathbb{T} \times \mathcal{U}(H_{\mathbb{R}}) \ltimes H_{\mathbb{R}}$ are subgroups of $\mathcal{U}(H) \ltimes H$ and $\mathbb{T} \times \mathcal{U}(H) \ltimes H$, respectively.

Corollary 4.8. *Let $(S,x) \in \mathcal{U}(H_{\mathbb{R}}) \ltimes H_{\mathbb{R}}$. Then the unitary operator $D_{S,x}$ on $\Gamma(H)$ given as in* (4.15) *satisfies the intertwining properties given as in* (4.16).

Proof. The proof is immediate from Corollary 4.5 and Theorem 4.6. □

5 Generalized Girsanov Transforms

For each $\zeta \in H$, there exists a sequence $\{\zeta_n\}_{n=1}^\infty \subset E$ such that $\{\zeta_n\}_{n=1}^\infty$ converges to ζ in H. On the other hand, for each $n \in \mathbb{N}$, $X_{\zeta_n} = \langle \cdot,\, \zeta_n\rangle$ is a Gaussian random variable defined on $E_{\mathbb{R}}^*$ and $\{X_{\zeta_n}\}_{n=1}^\infty$ is a Cauchy sequence in $L^2(E_{\mathbb{R}}^*, \mu)$. In fact, it holds that

$$\int_{E_{\mathbb{R}}^*} |X_{\zeta_n} - X_{\zeta_m}|^2 \, d\mu = \int_{E_{\mathbb{R}}^*} |X_{\zeta_n-\zeta_m}|^2 \, d\mu = |\zeta_n - \zeta_m|_0^2$$

for any $n, m \in \mathbb{N}$. Therefore, we can define the random variable $X_\zeta := \langle \cdot,\, \zeta\rangle$ on $E_{\mathbb{R}}^*$ by

$$X_\zeta = \lim_{n\to\infty} X_{\zeta_n} \quad \text{in } \; L^2(E_{\mathbb{R}}^*, \mu). \tag{5.1}$$

Then X_ζ is a Gaussian random variable with mean 0 and variance $|\zeta|_0^2$. Also, X_ζ can be considered as a multiplication operator in $\mathcal{L}((E),(E)^*)$ and then we obtain that

$$\begin{aligned}
\langle\langle X_\zeta \phi_\xi,\, \phi_\eta\rangle\rangle &= \langle\langle X_\zeta,\, \phi_\xi\phi_\eta\rangle\rangle = e^{\langle \xi,\, \eta\rangle} \langle\langle X_\zeta,\, \phi_{\xi+\eta}\rangle\rangle = \langle \zeta,\, \xi+\eta\rangle\, e^{\langle \xi,\, \eta\rangle} \\
&= \langle\langle (a(\zeta)+a^*(\zeta))\, \phi_\xi,\, \phi_\eta\rangle\rangle,
\end{aligned}$$

which implies that

$$X_\zeta = a(\zeta) + a^*(\zeta), \tag{5.2}$$

which is called the *quantum decomposition* of X_ζ (see [20]).

Let $\zeta \in H$. Then from (5.2), it is obvious that

$$\mathrm{Dom}(X_\zeta) = \mathrm{Dom}(a(\zeta)) \cap \mathrm{Dom}(a^*(\zeta)),$$

and so for any $\xi \in H$, $\phi_\xi \in \mathrm{Dom}(X_\zeta)$.

Lemma 5.1. *Let $S \in \mathcal{L}(E,E)$ such that $S^* \in \mathcal{L}(E,E)$ and $\zeta \in H$. Then for any $\xi, x \in H$, $D_{S,x}\phi_\xi \in \mathrm{Dom}(X_{S\zeta})$.*

Proof. Let $\xi, x \in$ be given. Then we obtain that

$$\begin{aligned} D_{S,x}\phi_\xi &= e^{-\frac{1}{2}|x|_0^2} e^{a^*(Sx)}\Gamma(S)e^{-a(x)}\phi_\xi = e^{-\frac{1}{2}|x|_0^2-\langle x,\,\xi\rangle} e^{a^*(Sx)}\phi_{S\xi} \\ &= e^{-\frac{1}{2}|x|_0^2-\langle x,\,\xi\rangle}\phi_{S(x+\xi)}. \end{aligned}$$

Therefore, $D_{S,x}\phi_\xi \in \mathrm{Dom}(X_{S\zeta})$. □

Theorem 5.2. *Let $S \in \mathcal{L}(E,E)$ such that $S^* \in \mathcal{L}(E,E)$ and $x \in H$. Suppose that S and x are real and S is an isometry. Then for each $\zeta \in H$, it holds that*

$$D_{S,x}X_\zeta = \left(X_{S\zeta} - 2\langle x,\,\zeta\rangle\right) D_{S,x} \tag{5.3}$$

on $\mathrm{LS}(\{\phi_\xi\,;\,\xi \in H\})$, *where $D_{S,x}$ is the isometric operator given as in* (4.15) *and* $\mathrm{LS}(Y)$ *is the linear span of $Y \subset \Gamma(H)$.*

Proof. There exists a sequence $\{\zeta_n\}_{n=1}^\infty \subset E$ such that $\{\zeta_n\}_{n=1}^\infty$ converges to ζ in H and $\{X_{\zeta_n}\}_{n=1}^\infty$ converges to X_ζ in $L^2(E_{\mathbb{R}}^*, \mu) \cong \Gamma(H)$. Then for any $n \in \mathbb{N}$, by applying Corollary 4.5, we see that the isometric operator $D_{S,x}$ given as in (4.15) satisfies the intertwining properties:

$$D_{S,x}a(\zeta_n) = b_{S,x}(\zeta_n)D_{S,x}, \tag{5.4}$$

$$D_{S,x}a^*(\zeta_n) = b_{S,x}^*(\zeta_n)D_{S,x}, \tag{5.5}$$

which implies that

$$D_{S,x}X_{\zeta_n} = \left(X_{S\zeta_n} - 2\langle x,\,\zeta_n\rangle\right) D_{S,x}.$$

Therefore, for any $\phi \in \mathrm{LS}(\{\phi_\xi\,;\,\xi \in H\})$, we obtain that

$$\begin{aligned} D_{S,x}X_\zeta\phi &= \lim_{n\to\infty} D_{S,x}X_{\zeta_n}\phi = \lim_{n\to\infty} \left(X_{S\zeta_n} - 2\langle x,\,\zeta_n\rangle\right) D_{S,x}\phi \\ &= \left(X_{S\zeta} - 2\langle x,\,\zeta\rangle\right) D_{S,x}\phi, \end{aligned}$$

which proves (5.3). □

Remark 5.3. The isometric operator $D_{S,x}$ given as in (4.15) as a solution of the implementation problem given as in (5.3) is independent of the choice of $\zeta \in H$.

Theorem 5.4. *Let $S \in \mathcal{L}(E,E)$ such that $S^* \in \mathcal{L}(E,E)$ and $x \in H$. Suppose that S and x are real and S is unitary. Then for each $\zeta \in H$, it holds that*

$$D_{S,x}X_\zeta = \left(X_{S\zeta} - 2\langle x,\,\zeta\rangle\right) D_{S,x} \tag{5.6}$$

on $\mathrm{LS}(\{\phi_\xi\,;\,\xi \in H\})$, *where $D_{S,x}$ is the unitary operator given as in* (4.15).

Proof. The proof is straightforward from Theorems 4.6 and 5.2. □

Remark 5.5. As a solution of the implementation problem given as in (5.3), the isometric operator $D_{S,x}$ is called a *quantum Girsanov transform*, see [14].

Keeping the same notations and assumptions as in Theorem 5.2, we see from (5.3) that

$$D_{S,x}X_\zeta^m = (X_{S\zeta} - 2\langle x,\, \zeta\rangle)^m D_{S,x}$$

for $m = 0, 1, 2, \ldots$. Then since $D_{S,x}$ is an isometry, we have

$$\begin{aligned}\langle\!\langle D_{S,x}\phi_0 \,|\, (X_{S\zeta} - 2\langle x,\, \zeta\rangle)^m D_{S,x}\phi_0 \rangle\!\rangle &= \langle\!\langle D_{S,x}\phi_0 \,\big|\, D_{S,x}X_\zeta^m\phi_0 \rangle\!\rangle \\ &= \langle\!\langle \phi_0 \,\big|\, X_\zeta^m\phi_0 \rangle\!\rangle,\end{aligned} \tag{5.7}$$

where $\langle\!\langle\cdot|\cdot\rangle\!\rangle$ is the hermitian inner product of $\Gamma(H)$. Therefore, following quantum probabilistic language, we see that the spectral distribution of $X_{S\zeta} - 2\langle x,\, \zeta\rangle$ in the transformed vacuum state $D_{S,x}\phi_0 = e^{-\frac{1}{2}|x|_0^2}\phi_{Sx}$ coincides with that of X_ζ in the vacuum state ϕ_0. Here $D_{S,x}\phi_0$ is given by

$$D_{S,x}\phi_0 = e^{-\frac{1}{2}|x|_0^2}\phi_{Sx} = e^{\langle\cdot,\, Sx\rangle - |x|_0^2}.$$

From (5.7) we obtain that

$$\begin{aligned}\int_{E_{\mathbb{R}}^*}\langle z,\zeta\rangle^m\mu(dz) &= \int_{E_{\mathbb{R}}^*}(\langle z, S\zeta\rangle - 2\langle x,\zeta\rangle)^m\left(D_{S,x}\phi_0(z)\right)^2\mu(dz) \\ &= \int_{E_{\mathbb{R}}^*}(\langle z, S\zeta\rangle - 2\langle x,\zeta\rangle)^m e^{2\langle z,Sx\rangle - 2\langle x,x\rangle}\mu(dz) \\ &= \int_{E_{\mathbb{R}}^*}\langle S^*z - 2x, \zeta\rangle^m\phi_{2Sx}(z)\mu(dz),\end{aligned}$$

which implies that

$$\int_{E_{\mathbb{R}}^*}\langle z,\zeta\rangle^m\mu(dz) = \int_{E_{\mathbb{R}}^*}\langle S^*z - 2x, \zeta\rangle^m\phi_{2Sx}(z)\mu(dz). \tag{5.8}$$

On the other hand, the Cameron–Martin theorem says that the Gaussian measure μ on $E_{\mathbb{R}}^*$ is quasi-invariant under the translation by $\eta \in E_{\mathbb{R}}$ (in fact, $\eta \in H_{\mathbb{R}}$ is necessary and sufficient) and the Radon–Nikodym derivative is given by

$$\frac{\mu(dz-\eta)}{\mu(dz)} = \phi_\eta(z), \qquad z \in E_{\mathbb{R}}^*.$$

We thus observe that (5.8) produces a generalization of the Cameron–Martin theorem.

For each $t \geq 0$, put

$$B_t = X_{\mathbf{1}_{[0,t]}} = \left\langle\cdot,\, \mathbf{1}_{[0,t]}\right\rangle.$$

Then $\{B_t\}_{t\geq 0}$ is called a *realization of Brownian motion*. Let $S \in \mathcal{L}(E, E)$ be real and an isometry. Suppose that $S^* \in \mathcal{L}(E, E)$. Then for each $t \geq 0$, put

$$B_t^S = \left\langle S^*\cdot,\, \mathbf{1}_{[0,t]}\right\rangle, \qquad \widetilde{B}_t^S := B_t^S - \int_0^t x(s)ds$$

for some $x \in H_{\mathbb{R}}$. Here if $I = [0, T]^n$, then $\mathbf{1}_{[0,t]}$ is the vector in $L^2([0, T]^n, dt)$, i.e.,

$$\mathbf{1}_{[0,t]} = \left(\mathbf{1}_{[0,t]}, \cdots, \mathbf{1}_{[0,t]}\right) \in L^2([0, T]^n, dt),$$

and so $\{B_t\}_{t\geq 0}$ is called a realization of n-dimensional Brownian motion.

Theorem 5.6. *Let $S \in \mathcal{L}(E, E)$ such that $S^* \in \mathcal{L}(E, E)$ and $x \in H$. Suppose that S and x are real and S is an isometry. Then for each $t \geq 0$, it holds that*

$$D_{S,x/2} B_t = \widetilde{B}_t^S D_{S,x/2} \tag{5.9}$$

on $\mathrm{LS}(\{\phi_\xi\,;\, \xi \in H\})$, *where $D_{S,x/2}$ is the isometric operator given by*

$$D_{S,x/2} = e^{-\frac{1}{8}|x|_0^2} e^{\frac{1}{2}a^*(Sx)} \Gamma(S) e^{-\frac{1}{2}a(x)}. \tag{5.10}$$

Proof. The proof is immediate from Theorem 5.2. Here from Remark 5.3, the isometric operator $D_{S,x/2}$ dose not depend on t. □

Theorem 5.7. *Notations and assumptions being as in Theorem 5.6, $\{\widetilde{B}_t^S\}_{t\geq 0}$ is a Brownian motion with respect to the probability measure $Q_{S,x}$ given by*

$$\frac{dQ_{S,x}(z)}{d\mu(z)} = \left[\left(D_{S,x/2}\phi_0\right)(z)\right]^2 = e^{\langle z,\, Sx\rangle - \frac{1}{2}|x|_0^2} = \phi_{Sx}(z).$$

Proof. The proof is straightforward from Theorem 5.6, i.e., we can prove that $\{\widetilde{B}_t^S\}_{t\geq 0}$ satisfies the properties (B1)–(B4) stated in Introduction. In fact, it is obvious that $\widetilde{B}_0^S = 0$. By applying Theorem 5.6, we see that

$$\int_{E_{\mathbb{R}}*} e^{i\lambda \widetilde{B}_t^S} dQ_{S,x} = e^{-\frac{1}{2}\lambda^2 t},$$

which implies that $\widetilde{B}_t^S$ is a Gaussian random variable with mean 0 and variance t. Also, for any $0 \leq s < t \leq u < v$, by direct computation, we have

$$\int_{E_{\mathbb{R}}^*} \left(\widetilde{B}_t^S - \widetilde{B}_s^S\right)\left(\widetilde{B}_v^S - \widetilde{B}_u^S\right) dQ_{s,x} = \left\langle S\mathbf{1}_{[s,t]},\, S\mathbf{1}_{[u,v]}\right\rangle = 0,$$

which implies that $\widetilde{B}_t^S - \widetilde{B}_s^S$ and $\widetilde{B}_v^S - \widetilde{B}_u^S$ are independent. Finally, by applying Kolmogorov's continuity theorem with Theorem 5.6, we see that almost all sample paths of $\{\widetilde{B}_t^S\}_{t\geq 0}$ are continuous. □

For the case of $S = I$ the identity operator, put $\widetilde{B}_t = \widetilde{B}_t^S$ for each $t \geq 0$.

Corollary 5.8 (Girsanov Theorem). *The stochastic process $\{\widetilde{B}_t\}_{t\geq 0}$ is a Brownian motion with respect to the probability measure Q_x given by*

$$\frac{dQ_x(z)}{d\mu(z)} = \left[\left(D_{I,x/2}\phi_0\right)(z)\right]^2 = e^{\langle z,\, x\rangle - \frac{1}{2}|x|_0^2} = \phi_x(z).$$

Proof. The proof is immediate from Theorem 5.7. □

Remark 5.9. In the literature, the statement of Corollary 5.8 is called the *Cameron–Martin–Girsanov theorem*. Also, since the measure Q_x given as in Corollary 5.8 is called a *Girsanov transform*, the measure $Q_{S,x}$ given as in Theorem 5.7 can be considered as a generalization of Girsanov transform.

Example 5.10. Let $E \equiv \mathcal{S}$ be the Schwartz space of rapidly decreasing C^∞-functions on $\mathbb{R}_+ = [0, \infty)$ and $E^* \equiv \mathcal{S}'$ the space of tempered distributions on $\mathbb{R}_+$.

(1) **(Time inversion)** We now consider the time inversion property of a Brownian motion which is a powerful tool to study path properties of a Brownian motion.

Define a linear operator $S : E \to E$ by

$$(S\eta)(s) = \int_0^{1/s} \eta(u)du - \frac{1}{s}\eta(1/s), \qquad \eta \in E. \tag{5.11}$$

Then we can easily see that S is a continuous linear operator from E into itself. Also, by applying the integration by parts formula and change of variable formula, for any $\xi, \eta \in E$, we can see that

$$\langle \eta, S^*\xi \rangle = \langle S\eta, \xi \rangle = \int_0^\infty \eta(t) \left(\int_0^{1/t} \xi(s)ds - \frac{1}{t}\xi(1/t) \right) dt,$$

which implies that

$$S^*\xi(t) = \int_0^{1/t} \xi(s)ds - \frac{1}{t}\xi(1/t). \tag{5.12}$$

Therefore, the operator S is self-adjoint (symmetric and real). Moreover, we obtain that

$$\begin{aligned} S^*(S\eta)(t) &= \int_0^{1/t} \left[\int_0^{1/s} \eta(u)du - \frac{1}{s}\eta(1/s) \right] ds - \frac{1}{t} \left[\int_0^t \eta(u)du - t\eta(t) \right] \\ &= \eta(t) + \int_0^{1/t} \left(\int_0^{1/s} \eta(u)du \right) ds - \int_0^{1/t} \frac{1}{s}\eta(1/s)ds - \frac{1}{t} \int_0^t \eta(u)du \\ &= \eta(t), \end{aligned}$$

which implies that S is an isometry. On the other hand, we see that

$$S\mathbf{1}_{[0,t]} = t\mathbf{1}_{[0,1/t]}$$

for any $t > 0$. Therefore, by Theorem 5.7, $\{\widetilde{B}_t^S\}_{t\geq 0}$ is a Brownian motion with respect to the probability measure $Q_{S,x}$ given by

$$\frac{dQ_{S,x}(z)}{d\mu(z)} = \left[\left(D_{S,x/2}\phi_0 \right)(z) \right]^2 = e^{\langle z,\, Sx \rangle - \frac{1}{2}|x|_0^2} = \phi_{Sx}(z),$$

where $x \in H$ and

$$\widetilde{B}_t^S := B_t^S - \int_0^t x(s)ds = tB_{\frac{1}{t}} - \int_0^t x(s)ds.$$

In particular, the time inversion $\{tB_{\frac{1}{t}}\}_{t\geq 0}$ of the Brownian motion $\{B_t\}_{t\geq 0}$ is also a Brownian motion with respect to the initial Gaussian measure μ.

(2) **(Scaling invariance)** Let $a \in \mathbb{R}$ with $a \neq 0$ be given. Define a linear operator $S : E \to E$ by

$$(S\eta)(s) = \frac{1}{a}\eta\left(\frac{t}{a^2}\right), \qquad \eta \in E. \tag{5.13}$$

Then it is obvious that S is continuous and

$$S^*\xi(t) = a\xi(a^2 t), \tag{5.14}$$

and S is an isometry. On the other hand, we have

$$S\mathbf{1}_{[0,t]} = \frac{1}{a}\mathbf{1}_{[0,a^2t]}, \qquad t \geq 0.$$

Therefore, by Theorem 5.7, $\{\widetilde{B}_t^S\}_{t\geq 0}$ is a Brownian motion with respect to the probability measure $Q_{S,x}$ given by

$$\frac{dQ_{S,x}(z)}{d\mu(z)} = \left[\left(D_{S,x/2}\phi_0\right)(z)\right]^2 = e^{\langle z,\, Sx\rangle - \frac{1}{2}|x|_0^2} = \phi_{Sx}(z),$$

where $x \in H$ and

$$\widetilde{B}_t^S := B_t^S - \int_0^t x(s)ds = \frac{1}{a}B_{a^2t} - \int_0^t x(s)ds.$$

In particular, the time scaling $\{\frac{1}{a}B_{a^2t}\}_{t\geq 0}$ of the Brownian motion $\{B_t\}_{t\geq 0}$ is also a Brownian motion with respect to the initial Gaussian measure μ.

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