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# On Weighted Fock Spaces and Distributions

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**Abstract**

In this note, we shall consider a Gel'fand triple associated with weighted Fock spaces and revisit the characterization theorems for the $S$-transform and the operator symbol in terms of analytic and growth conditions. In addition, some results on higher order Bell numbers as a non-trivial example of weight sequences are summarized.

## 1 Preliminaries

### 1.1 Weighted Fock Spaces

Let $H$ be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $|\cdot|_0$. Let $A$ be a self-adjoint operator in $H$ with dense domain $\mathrm{Dom}(A)\subset H$ satisfying $\inf\mathrm{Spec}(A)\geq 1$. For each $p\geq 0$, a dense subspace of $H$, $\mathcal{D}_p:=\{\xi\in H;\ |\xi|_p:=|A^p\xi|_0<\infty\}$, is a Hilbert space. It is easy to see $\mathcal{D}_q\subset\mathcal{D}_p\subset H=\mathcal{D}_0$ for $0\leq p\leq q$. Then, consider $\mathcal{D}:=\mathrm{proj}\lim_{p\to\infty}\mathcal{D}_p$ and let $\mathcal{D}^*$ denote the dual space of $\mathcal{D}$. For each $p\geq 0$, let $\mathcal{D}_{-p}$ be the completion of $H$ with respect to the norm $|\xi|_{-p}:=|A^{-p}\xi|_0$. Then we get $H=\mathcal{D}_0\subset\mathcal{D}_{-p}\subset\mathcal{D}_{-q}$ for $0\leq p\leq q$, and $\mathcal{D}^*\cong\mathrm{ind}\lim_{p\to\infty}\mathcal{D}_{-p}$. As a result, with the identification $H\cong H^*$ by the Riesz representation theorem, we obtain a triple, $\mathcal{D}\subset H\subset\mathcal{D}^*$, where the bilinear form on $\mathcal{D}^*\times\mathcal{D}$ is also denoted by $\langle\cdot,\cdot\rangle$.

Let $\mathcal{F}_1(H)$ be a standard Boson Fock space over $H$ and $\alpha=\{\alpha(n)\}_{n=0}^{\infty}$ be a weight sequence of positive real numbers satisfying the condition,

$(A1)$ $\alpha(0)=1,\ \inf_{n\geq 0}\alpha(n)>0.$

Now we introduce a weighted Boson Fock space as follows. Let $\mathcal{F}_\alpha(\mathcal{D}_p)$ be a weighted Boson Fock space over $\mathcal{D}_p$ given by

$$\mathcal{F}_\alpha(\mathcal{D}_p):=\left\{\phi:=(f_n)_{n=0}^{\infty};\ f_n\in\mathcal{D}_p^{\hat{\otimes}n},\ \|\phi\|_{p,\alpha}^2:=\sum_{n=0}^{\infty}n!\alpha(n)|f_n|_p^2<\infty\right\}$$

where $\cdot^{\hat{\otimes}n}$ for the $n$-fold symmetric tensor product of $\cdot$ and $|f_n|_p:=|(A^p)^{\otimes n}f_n|_0$. The condition $\alpha(0)=1$ in $(A1)$ is simply to ensure that the norm on $\mathcal{D}_p^{\hat{\otimes}0}$ coincides with the absolute value on $\mathbb{C}$. The condition $\inf_{n\geq 0}\alpha(n)>0$ in $(A1)$ is required to have $\mathcal{F}_\alpha(H)\subset\mathcal{F}_1(H)$. By identifying $\mathcal{F}_1(H)$ with its dual space, we have a chain of weighted Fock spaces,

$$\cdots\subset\mathcal{F}_\alpha(\mathcal{D}_p)\subset\cdots\subset\mathcal{F}_\alpha(H)\subset\mathcal{F}_1(H)\subset\mathcal{F}_{1/\alpha}(H)\subset\cdots\subset\mathcal{F}_{1/\alpha}(\mathcal{D}_{-p})\subset\cdots,\ p\geq 0,$$

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where the norm on $\mathcal{F}_{1/\alpha}(\mathcal{D}_{-p})$ is given by $\|\cdot\|_{-p,1/\alpha} := \|(A^{-p})^{\otimes n}\cdot\|_{0,1/\alpha}$. Consider the space $\mathcal{F}_\alpha(\mathcal{D})$ of test functions defined by

$$\mathcal{F}_\alpha(\mathcal{D}) = \operatorname*{proj\,lim}_{p\to\infty} \mathcal{F}_\alpha(\mathcal{D}_p).$$

The dual space $\mathcal{F}_\alpha(\mathcal{D})^*$ of $\mathcal{F}_\alpha(\mathcal{D})$,

$$\mathcal{F}_\alpha(\mathcal{D})^* \cong \operatorname*{ind\,lim}_{p\to\infty} \mathcal{F}_{1/\alpha}(\mathcal{D}_{-p}),$$

is called the space of generalized functions. Then we get a triple,

$$\mathcal{F}_\alpha(\mathcal{D}) \subset \mathcal{F}_1(H) \subset \mathcal{F}_\alpha(\mathcal{D})^*.$$

We adopt the notation $\langle\langle\cdot,\cdot\rangle\rangle$ to denote the bilinear form on $\mathcal{F}_\alpha(\mathcal{D})^* \times \mathcal{F}_\alpha(\mathcal{D})$,

$$\langle\langle \Phi, \phi\rangle\rangle = \sum_{n=0}^{\infty} n!\langle F_n, f_n\rangle, \quad \Phi = (F_n) \in \mathcal{F}_\alpha(\mathcal{D})^*,\ \phi = (f_n) \in \mathcal{F}_\alpha(\mathcal{D}).$$

Due to the Cauchy-Schwartz inequality, we have $|\langle\langle \Phi, \phi\rangle\rangle| \leq \|\Phi\|_{-p,1/\alpha}\|\phi\|_{p,\alpha}$.

## 1.2 Growth Bound of the $S$-transform

Moreover, let us assume that

$(A2)$ $\lim_{n\to\infty} \left(\frac{\alpha(n)}{n!}\right)^{\frac{1}{n}} = 0.$

This condition implies that

$$G_\alpha(z) = \sum_{n=0}^{\infty} \frac{\alpha(n)}{n!} z^n$$

is an entire function. Then the exponential vector (coherent state) $e(\xi)$ given by

$$e(\xi) := \left(\frac{\xi^{\otimes n}}{n!}\right)_{n=0}^{\infty}, \quad \xi \in \mathcal{D}$$

belongs to $\mathcal{F}_\alpha(\mathcal{D})$ due to $\|e(\xi)\|_{p,\alpha}^2 = G_\alpha(|\xi|_p^2) < \infty$.

**Definition 1.1.** Assume $(A1)$ and $(A2)$. The $S$-transform $S\Phi$ of $\Phi = (F_n)_{n=0}^\infty \in \mathcal{F}_\alpha(\mathcal{D})^*$ is defined to be the function on $\mathcal{D}$ by

$$(S\Phi)(\xi) := \langle\langle \Phi, e(\xi)\rangle\rangle = \sum_{n=0}^{\infty} \langle F_n, \xi^{\otimes n}\rangle, \quad \xi \in \mathcal{D}.$$

The $S$-transform can be viewed as the generalization to distributions of the Segal-Bargmann transform.

**Lemma 1.2.** *Assume that conditions $(A1)(A2)$ hold. The $S$-transform $F = S\Phi$ of a generalized function $\Phi \in \mathcal{F}_\alpha(\mathcal{D})^*$ satisfies the growth condition*

$$|(S\Phi)(\xi)| \leq \|\Phi\|_{-p,1/\alpha} G_\alpha(|\xi|_p^2)^{1/2}, \quad \xi \in \mathcal{D}$$

*for some $p \geq 0$.*

Note that the condition $(A1)$ guarantees that $G_{1/\alpha}$ given by

$$G_{1/\alpha}(z) = \sum_{n=0}^{\infty} \frac{1}{n!\alpha(n)} z^n$$

is an entire function.

**Lemma 1.3.** *Assume that condition* $(A1)$ *holds. Then the* $S$*-transform* $F = S\varphi$ *of a test function* $\varphi \in \mathcal{F}_\alpha(\mathcal{D})$ *satisfies the growth condition*

$$|(S\varphi)(\xi)| \leq \|\varphi\|_{p,1/\alpha} G_{1/\alpha}(|\xi|_{-p}^2)^{1/2}, \quad \xi \in \mathcal{D},$$

*for any* $p \geq 0$.

Up to here, the nuclearity of $\mathcal{F}_\alpha(\mathcal{D})$ is not assumed.

## 2 Analytic Characterizations

The characterization of generalized functions in terms of analytic and growth conditions is called the analytic characterization, which was first discussed by Potthoff-Streit [24] for Hida distributions (Kuo *et al.* [20] for test functions). From the point of infinite dimensional analytic functions, equivalent results were obtained by Lee [21].

It is well-known that the nuclearity of $\mathcal{F}_\alpha(\mathcal{D})$ is a sufficient condition for the analytic characterization. It is recently proved [1] that the nuclearity of $\mathcal{F}_\alpha(\mathcal{D})$ is a necessary condition for it. In proof, the infinite dimensional Bargmann-Segal space [14][16], the space of square integrable analytic functions on infinite dimensional complex Gaussian space, plays important roles.

From now on, we suppose that the self-adjoint operator $A$ satisfies the condition,

$(H1)$ $\inf \mathrm{Spec}(A) > 1$ and $A^{-r}$ is of Hilbert-Schmidt type for some $r > 0$.

Then $\mathcal{D}$ becomes a nuclear space and so is $\mathcal{F}_\alpha(\mathcal{D})$. In such a case, we denote $\mathcal{D}$ and $\mathcal{F}_\alpha(\mathcal{D})$ by $\mathcal{E}$ and $\mathcal{F}_\alpha(\mathcal{E})$, respectively, and so a Gel'fand triple

$$\mathcal{F}_\alpha(\mathcal{E}) \subset \mathcal{F}_1(H) \subset \mathcal{F}_\alpha(\mathcal{E})^*$$

is referred to as a CKS-space where a condition $\inf_{n\geq 0} \alpha(n) > 0$ in $(A1)$ is assumed in [11]. However, a weaker condition,

$(A1)^*$ $\alpha(0) = 1,\ \inf_{n\geq 0} \alpha(n)\sigma^n > 0$ for some $\sigma \geq 1$,

is strong enough to assure that the nuclear space $\mathcal{F}_\alpha(\mathcal{E})$ is a subspace of $\mathcal{F}_1(H)$. This weaker condition was first introduced in [4]. Therefore, the condition $(A1)$ on Theorem 2.1 and Theorem 2.2 can be replaced by $(A1)^*$.

**Theorem 2.1** ([11]). *Assume that conditions* $(A1)^*(A2)$ *hold. The* $S$*-transform* $F = S\Phi$ *of a generalized function* $\Phi \in \mathcal{F}_\alpha(\mathcal{E})^*$ *satisfies the conditions:*

(a) *For any* $\xi, \eta \in \mathcal{D}$, *the function* $F(z\xi + \eta)$ *is an entire function of* $z \in \mathbb{C}$.

(b) *There exist constants* $K > 0, a > 0, p \geq 0$ *such that*

$$|F(\xi)| \leq K G_\alpha(a|\xi|_p^2)^{\frac{1}{2}}, \quad \xi \in \mathcal{E}.$$

*Conversely, assume that*

$(B1)$ $\limsup_{n\to\infty}\left(\frac{n!}{\alpha(n)}\inf_{r>0}\frac{G_\alpha(r)}{r^n}\right)^{\frac{1}{n}}<\infty$

*holds and let a* $\mathbb{C}$*-valued function* $F$ *on* $\mathcal{E}$ *satisfies the above two conditions* $(a)(b)$. *Then, there exists a unique* $\Phi\in\mathcal{F}_\alpha(\mathcal{E})^*$ *such that* $F=S\Phi$. *Moreover, for any* $q>p$ *with* $ae^2\|A^{-(q-p)}\|_{HS}^2<1$, *we have the norm estimate*

$$\|\Phi\|_{-q,1/\alpha}\le K(1-ae^2\|A^{-(q-p)}\|_{HS}^2)^{-\frac{1}{2}}.$$

For the space $\varphi\in\mathcal{F}_\alpha(\mathcal{E})$ of test functions, which was not studied in [11], we have

**Theorem 2.2** ([3]). *Assume that condition* $(A1)^*$ *holds. Then the* $S$*-transform* $F=S\varphi$ *of a test function* $\varphi\in\mathcal{F}_\alpha(\mathcal{E})$ *satisfies the conditions:*

(a) *For any* $\xi,\eta\in\mathcal{D}$, *the function* $F(z\xi+\eta)$ *is an entire function of* $z\in\mathbb{C}$.

(b) *For any* $p\ge 0, a>0$, *there exists a constant* $K>0$ *such that*

$$|F(\xi)|\le KG_{1/\alpha}(a|\xi|_{-p}^2)^{\frac{1}{2}},\quad \xi\in\mathcal{E}.$$

*Conversely, assume that*

$(\tilde{B}1)$ $\limsup_{n\to\infty}\left(n!\alpha(n)\inf_{r>0}\frac{G_{1/\alpha}(r)}{r^n}\right)^{\frac{1}{n}}<\infty$

*holds and let a* $\mathbb{C}$*-valued function* $F$ *on* $\mathcal{E}$ *satisfies the above two conditions* $(a)(b)$. *Then there exists a unique* $\varphi\in\mathcal{F}_\alpha(\mathcal{E})$ *such that* $F=S\varphi$. *Moreover, for any given* $a,p>0$, *choose* $q\in[0,p)$ *such that* $ae^2\|A^{-(p-q)}\|_{HS}^2<1$, *then we have the norm estimate*

$$\|\varphi\|_{q,\alpha}\le K(1-ae^2\|A^{-(p-q)}\|_{HS}^2)^{-\frac{1}{2}}.$$

*Remark* 2.3. (1) It will be seen that $(A3)$ and $(A4)$ given in Section 4 are necessary and sufficient conditions for $(B1)$ and $(\tilde{B}1)$, respectively.
(2) It was our starting point [5][6] to clarify minimal conditions on $\{\alpha(n)\}_{n=0}^\infty$ to carry out theories of generalized functions and operators associated with a CKS space,

$$\mathcal{F}_\alpha(\mathcal{E})\subset\mathcal{F}_1(H)\subset\mathcal{F}_\alpha(\mathcal{E})^*$$

such that Theorems 2.1 and 2.2 hold.

## 3 Examples and Log-concavity Criterion

**Example 3.1.** It is easy to see that the classical examples,

(1) $\alpha(n)=1$ for the Hida-Kubo-Takenaka space [19],

$$\mathcal{F}_1(\mathcal{E})\subset\mathcal{F}_1(H)\subset\mathcal{F}_1(\mathcal{E})^*,$$

and $\beta(n)=(n!)^\beta$ $(0\le\beta<1)$ for the Kondratiev-Streit space [17],

$$\mathcal{F}_\beta(\mathcal{E})\subset\mathcal{F}_1(H)\subset\mathcal{F}_\beta(\mathcal{E})^*,$$

satisfy $(A1)(A2)(B1)(\tilde{B}1)$, which can be checked by direct computations.

(3) Let $\exp_k(x)$ denotes the $k$-times iterated exponential function for an integer $k \geq 2$, that is,

$$\exp_k(x) = \exp(\exp \cdots (\exp(x))).$$

The $k$-th order Bell numbers $b_k(n)$ are defined by

$$\frac{\exp_k(x)}{\exp_k(0)} = \sum_{n=0}^{\infty} \frac{b_k(n)}{n!} x^n, \ k \geq 2,$$

where the numbers $b_2(n), n \geq 0$ are known as the (standard) Bell numbers. Then

$$\mathcal{F}_{b_k}(\mathcal{E}) \subset \mathcal{F}_{\beta}(\mathcal{E}) \subset \mathcal{F}_1(\mathcal{E}) \subset \mathcal{F}_1(H) \subset \mathcal{F}_1(\mathcal{E})^* \subset \mathcal{F}_{\beta}(\mathcal{E})^* \subset \mathcal{F}_{b_k}(\mathcal{E})^*.$$

It is not difficult to check $(A1)(A2)$ for $\{b_k(n)\}_{n=0}^{\infty}$. Cochran *et.al* [11] proved by direct computations that the condition $(B1)$ is satisfied, but they did not study $(\tilde{B}1)$. It seems impossible to check by direct computations whether or not $(\tilde{B}1)$ holds for the case of the $k$-th order bell numbers. Hence it is natural to seek an easy criterion for $(\tilde{B}1)$.

**Definition 3.2.** A sequence $\{\delta(n)\}_{n=0}^{\infty}$ is log-concave if $\delta(n)\delta(n+2) \leq \delta(n+1)^2$ and $\{\delta(n)\}_{n=0}^{\infty}$ is log-convex if $\delta(n+1)^2 \leq \delta(n)\delta(n+2)$.

In fact, the following criterion was mentioned in [11].

**Proposition 3.3.** *If $\{\alpha(n)/n!\}_{n=0}^{\infty}$ is log-concave, then $(B1)$ holds.*

Due to this proposition, it is easy to see the following.

**Corollary 3.4.** *If $\{1/n!\alpha(n)\}_{n=0}^{\infty}$ is log-concave, then $(\tilde{B}1)$ holds.*

However, it was not proved in [11] if the sequences $\{b_k(n)/n!\}_{n=0}^{\infty}$ and $\{1/n!b_k(n)\}_{n=0}^{\infty}$ are log-concave or not. We filled up these gaps [2].

**Theorem 3.5.** (1) *$\{b_k(n)/n!\}_{n=0}^{\infty}$ is log-concave.*

(2) *$\{b_k(n)\}_{n=0}^{\infty}$ is log-convex and hence $\{1/n!b_k(n)\}_{n=0}^{\infty}$ is log-concave.*

(3) *$\{b_k(n)\}_{n=0}^{\infty}$ satisfies $(A1)(A2)(B1)(\tilde{B}1)$.*

*Remark* 3.6. One can find a different way of proof by Engel [12] concerning the log-convexity of $\{b_2(n)\}_{n=0}^{\infty}$. Canfield [9] showed that the log-concavity of $\{b_2(n)/n!\}_{n=0}^{\infty}$ holds asymptotically.

In [18], the following "log-additivity" conditions were introduced in order to prove the continuity of various operators acting on $\mathcal{F}_{\alpha}(\mathcal{E})$ and $\mathcal{F}_{\alpha}(\mathcal{E})^*$:

$(C1)$ There exists a constant $c_1$ such that for any $n \leq m$,

$$\alpha(n) \leq c_1^m \alpha(m).$$

$(C2)$ There exists a constant $c_2$ such that for any $n, m$,

$$\alpha(n+m) \leq c_2^{n+m} \alpha(n)\alpha(m).$$

$(C3)$ There exists a constant $c_3$ such that for any $n, m$,

$$\alpha(n)\alpha(m) \leq c_3^{n+m} \alpha(n+m).$$

**Theorem 3.7.** *Let* $\{\alpha(n)\}_{n=0}^{\infty}$ *be a sequence of positive numbers with* $\alpha(0) = 1$.

(1) *If* $\{\alpha(n)\}_{n=0}^{\infty}$ *is log-convex, then*

$$\alpha(n)\alpha(m) \le \alpha(n+m), \quad n, m \ge 0.$$

(2) *If* $\{\alpha(n)/n!\}_{n=0}^{\infty}$ *is log-concave, then*

$$\alpha(n+m) \le 2^{n+m}\alpha(n)\alpha(m), \quad n, m \ge 0.$$

Due to Theorem 3.5 and Theorem 3.7, one has the following inequalities.

**Corollary 3.8.** $\{b_k(n)\}_{n=0}^{\infty}$ *satisfies* $(C1)(C2)(C3)$ *with* $c_1 = 1, c_2 = 2, c_3 = 1$, *that is,*

$$b_k(n)b_k(m) \le b_k(n+m) \le 2^{n+m}b_k(n)b_k(m), \quad n, m \ge 0.$$

*Remark* 3.9. In [18], it was proved that $(C3)$ implies $(C1)$ and the $k$-th order Bell numbers $\{b_k(n)\}_{n=0}^{\infty}$ satisfies $(C1)(C2)(C3)$ in an asymptotical consideration. Moreover, we proved in [2] that $c_1 = 1, c_3 = 1$ for any $k \ge 2$ and $c_2 = 2$ for $k = 2$ are best constants. It is not known if $c_2 = 2$ for $k \ge 3$ is the best constant.

## 4 Growth Functions

In this section, we shall recall key notions and results from [5][6]. Let $C_{+,\log}$ denote the collection of all positive continuous functions $u$ on $[0, \infty)$ satisfying

$$\lim_{r\to\infty} \frac{\log u(r)}{\log r} = \infty.$$

The Legendre transform $\ell_u$ of $u \in C_{+,\log}$ defined as the function,

$$\ell_u(t) := \inf_{r>0} \frac{u(r)}{r^t}, \quad t \in [0, \infty).$$

Let $C_{+,1/2}$ denotes the collection of all positive continuous functions $u$ on $[0, \infty)$ satisfying

$$\lim_{r\to\infty} \frac{\log u(r)}{\sqrt{r}} = \infty.$$

The dual Legendre transform $u^*$ of $u \in C_{+,1/2}$ is defined to be the function

$$u^*(r) = \sup_{s\ge 0} \frac{e^{2\sqrt{rs}}}{u(s)}, \quad r \in [0, \infty).$$

It can be proved that $u^* \in C_{+,1/2}$.

*Remark* 4.1. One can see that $\exp[\sqrt{r}] \in C_{+,\log}$, but $\notin C_{+,1/2}$. In addition, $\exp[2\sqrt{r}\log\sqrt{r}] \in C_{+,1/2}$.

**Definition 4.2.** We say that two sequences $\{a(n)\}$ and $\{b(n)\}$ are equivalent denoted by $\{a(n)\} \sim \{b(n)\}$ if there exist constants $K_1, K_2, c_1, c_2 > 0$ such that for all $n$,

$$K_1 c_1^n a(n) \le b(n) \le K_2 c_2^n a(n).$$

Now we state the weaker conditions for the sequence $\{\alpha(n)\}$:

$(A3)$ $\{\alpha(n)\}$ is equivalent to a positive sequence $\{\lambda(n)\}$ such that $\{\lambda(n)/n!\}$ is log-concave.

$(A4)$ $\{\alpha(n)\}$ is equivalent to a positive sequence $\{\lambda(n)\}$ such that $\{1/n!\lambda(n)\}$ is log-concave.

Then it is easy to see the following Lemma.

**Lemma 4.3.** (1) *$(B1)$ is equivalent to $(A3)$.*

(2) *$(\tilde{B}1)$ is equivalent to $(A4)$.*

For our discussion, the following conditions on $u$ play important roles:

$(U1)$ $\inf_{r\geq 0} u(r) = 1$.

$(U2)$ $\lim_{r\to\infty} \dfrac{\log u(r)}{r} < \infty$.

$(U3)$ $u(r^2)$ is a log-convex function on $[0.\infty)$.

For a given $u \in C_{+,\log}$, define a sequence $\{\alpha_u(n)\}_{n=0}^{\infty}$ given by

$$\alpha_u(n) := \frac{1}{\ell_u(n)n!},$$

which plays a role of a sequence $\{\alpha(n)\}_{n=0}^{\infty}$.

**Lemma 4.4.** (1) *If $u \in C_{+,\log}$ satisfies $(U1)(U2)$, then $\{\alpha_u(n)\}_{n=0}^{\infty}$ satisfies $(A1)^*$.*

(2) *If $u \in C_{+,1/2}$ satisfies $(U3)$, then $\{\alpha_u(n)\}_{n=0}^{\infty}$ satisfies $(A2)$.*

(3) *If $u \in C_{+,\log}$ satisfies $(U3)$, $\{\alpha_u(n)\}_{n=0}^{\infty}$ satisfies $(A3)$.*

(4) *If $u \in C_{+,\log}$, then $\{\alpha_u(n)\}_{n=0}^{\infty}$ satisfies $(A4)$.*

**Theorem 4.5.** *Suppose that $u \in C_{+1/2}$ satisfies $(U1)(U2)(U3)$. Then,*

(1) *a sequence $\{\alpha_u(n)\}_{n=0}^{\infty}$ satisfies conditions $(A1)^*(A2)(A3)(A4)$.*

(2) *$(A3)$ $(\Longleftrightarrow (B1))$ implies $(C2)$.*

(3) *$(A4)$ $(\Longleftrightarrow (\tilde{B}1))$ implies $(C3)$.*

(4) *$(C3)$ implies $(C1)$.*

**Definition 4.6.** Two positive functions $f$ and $g$ on $[0,\infty)$ are called equivalent, denoted by $f \sim g$, if there exists constants $c_1, c_2, a_1, a_2 > 0$ such that

$$c_1 f(a_1 r) \leq g(r) \leq c_2 f(a_2 r), \ \ r \in [0,\infty).$$

**Example 4.7.** (1) For $0 \leq \beta < 1$, one can see that

$$u_\beta(r) = \exp[(1+\beta)r^{\frac{1}{1+\beta}}] \in C_{+,1/2} \Longleftrightarrow u_\beta^*(r) = \exp[(1-\beta)r^{\frac{1}{1-\beta}}] \in C_{+,1/2}.$$

In fact, the series $G_\alpha$ and $G_{1/\alpha}$with $\alpha(n) = (n!)^\beta$ cannot have the closed forms unless $\beta = 0$, but we have the following estimates:

$$\begin{cases} \exp[(1-\beta)r^{\frac{1}{1-\beta}}] \leq G_\alpha(r) \leq 2^\beta \exp[(1-\beta)2^{\frac{\beta}{1-\beta}} r^{\frac{1}{1-\beta}}], \\ 2^{-\beta}\exp[(1+\beta)2^{-\frac{\beta}{1+\beta}} r^{\frac{1}{1+\beta}}] \leq G_{1/\alpha}(r) \leq \exp[(1+\beta)r^{\frac{1}{1+\beta}}]. \end{cases} \tag{4.1}$$

That is, $u_\beta(r) \sim \sum_{n=0}^{\infty} \frac{1}{(n!)^{1+\beta}} r^n$ and $u_\beta^*(r) \sim \sum_{n=0}^{\infty} \frac{1}{(n!)^{1-\beta}} r^n$.
(2) Let $\log_j(\cdot)$ denote the $j$-th iterated logarithmic function inductively defined by

$$\log_1(r) := \log(\max\{r, e\}), \ \log_j(r) := \log_1(\log_{j-1}(r)), \ j \geq 2.$$

Then we have

$$u_k^*(r) := \exp_k(r)/\exp_k(0) \in C_{+,1/2} \iff u_k(r) \sim w_k(r) = \exp[2\sqrt{r \log_{k-1} \sqrt{r}}] \in C_{+,1/2}$$

and $w_k(r) \sim \sum_{n=0}^{\infty} \frac{1}{n! b_k(n)} r^n$.

If one merges everything together with replacements of growth conditions in Theorem 2.1 and Theorem 2.2 respectively by

- $|F(\xi)| \leq K u^*(a|\xi|_p)^{\frac{1}{2}}$ for $\mathcal{F}_\alpha(\mathcal{E})^*$,
- $|F(\xi)| \leq K u(a|\xi|_{-p})^{\frac{1}{2}}$ for $\mathcal{F}_\alpha(\mathcal{E})$,

where $\alpha = \{\alpha_u(n)\}_{n=0}^{\infty}$, then we obtain

**Theorem 4.8.** *Suppose that $u \in C_{+1/2}$ satisfies $(U1)(U2)(U3)$. The $S$-transform $F = S\Phi$ of a generalized function $\Phi \in \mathcal{F}_\alpha(\mathcal{E})^*$ satisfies the conditions:*

*(a) For any $\xi, \eta \in \mathcal{E}$, the function $F(z\xi + \eta)$ is an entire function of $z \in \mathbb{C}$.*

*(b) There exist constants $K > 0, a > 0, p \geq 0$ such that*

$$|F(\xi)| \leq K u^*(a|\xi|_p^2)^{\frac{1}{2}}, \quad \xi \in \mathcal{E}.$$

*Conversely, let a $\mathbb{C}$-valued function $F$ on $\mathcal{E}$ satisfies the above two conditions $(a)(b)$. Then there exists a unique $\Phi \in \mathcal{F}_\alpha(\mathcal{E})^*$ such that $F = S\Phi$. Moreover, for any $q > p$ with $ae^2\|A^{-(q-p)}\|_{HS}^2 < 1$, we have the norm estimate*

$$\|\Phi\|_{-q,1/\alpha} \leq K(1 - ae^2\|A^{-(q-p)}\|_{HS}^2)^{-\frac{1}{2}}.$$

**Theorem 4.9.** *Suppose that $u \in C_{+1/2}$ satisfies $(U1)(U2)(U3)$. The $S$-transform $F = S\varphi$ of a test function $\varphi \in \mathcal{F}_\alpha(\mathcal{E})$ satisfies the conditions:*

*(a) For any $\xi, \eta \in \mathcal{E}$, the function $F(z\xi + \eta)$ is an entire function of $z \in \mathbb{C}$.*

*(b) For any $p \geq 0, a > 0$, there exists a constant $K > 0$ such that*

$$|F(\xi)| \leq K u(a|\xi|_{-p}^2)^{\frac{1}{2}}, \quad \xi \in \mathcal{E}.$$

*Conversely, let a $\mathbb{C}$-valued function $F$ on $\mathcal{E}$ satisfies the above two conditions $(a)(b)$. Then there exists a unique $\varphi \in \mathcal{F}_\alpha(\mathcal{E})$ such that $F = S\varphi$. Moreover, for any given $a, p > 0$, choose $q \in [0, p)$ such that $ae^2\|A^{-(p-q)}\|_{HS}^2 < 1$, then we have the norm estimate*

$$\|\varphi\|_{q,\alpha} \leq K(1 - ae^2\|A^{-(p-q)}\|_{HS}^2)^{-\frac{1}{2}}.$$

*Remark* 4.10. Consult our papers [6][7] to see connections Gannoun *et al.* [13].

## 5 Generalization of Obata's Theorem

Obata [22][23] characterized the operator symbol of $\Xi \in \mathcal{L}(\mathcal{F}_1(\mathcal{E}), \mathcal{F}_1(\mathcal{E})^*)$ and Chung *et al.* [10] presented a simplified proof.

**Definition 5.1.** For any $\Xi \in \mathcal{L}(\mathcal{F}_\alpha(\mathcal{E}), \mathcal{F}_\alpha(\mathcal{E})^*)$, the operator symbol $\widehat{\Xi}$ of $\Xi$ is defined by

$$\widehat{\Xi}(\xi, \eta) = \langle\langle \Xi e(\xi), e(\eta) \rangle\rangle, \quad \xi, \eta \in \mathcal{E}.$$

The operator symbol is an operator version of the $S$-transform. Therefore, one can generalize the characterization theorem for the operator symbol as follows.

**Theorem 5.2.** *Suppose that* $u \in C_{+1/2}$ *satisfies* $(U1)(U2)(U3)$. *The symbol* $G = \widehat{\Xi}$ *of* $\Xi \in \mathcal{L}(\mathcal{F}_\alpha(\mathcal{E}), \mathcal{F}_\alpha(\mathcal{E})^*)$ *satisfies the conditions:*

(a) *For any* $\xi_1, \xi_2, \eta_1, \eta_2 \in \mathcal{E}$, *the function* $G(z\xi_1 + \eta_1, w\xi_2 + \eta_2)$ *is an entire function of* $(z, w) \in \mathbb{C} \times \mathbb{C}$.

(b) *There exist constants* $K > 0, a > 0, p \geq 0$ *such that*

$$|G(\xi, \eta)| \leq K u^* \left( a(|\xi|_p^2 + |\eta|_p^2) \right)^{\frac{1}{2}}, \quad \xi, \eta \in \mathcal{E}.$$

*Conversely, suppose a* $\mathbb{C}$*-valued function* $G$ *on* $\mathcal{E} \times \mathcal{E}$ *satisfies the above two conditions* $(a)(b)$. *Then there exists a unique* $\Xi \in \mathcal{L}(\mathcal{F}_\alpha(\mathcal{E}), \mathcal{F}_\alpha(\mathcal{E})^*)$ *such that* $G = \widehat{\Xi}$.

*Proof.* Due to Theorem 4.5, there exist constants $c_1, c_2 > 0$ such that

$$u^*(s)u^*(t) \leq u^*(c_1(s+t)) \leq u^*(c_2 s)u^*(c_2 t), \quad s, t \geq 0. \tag{5.1}$$

Thanks to this inequality (5.1), one can apply the idea of Chung *et al.* [10]. It means that the proof can be done by applying Theorem 4.8 two times.

□

*Remark* 5.3. It is not difficult to generalize further and state Theorem 4.8, Theorem 4.9 and Theorem 5.2 in a unified manner as of Ji-Obata [15]. It is because essential properties what we need for norm estimates related with a CKS space can be derived from conditions $(A1)^*(A2)(A3)(A4)$ on $\{\alpha(n)\}$ and $(U1)(U2)(U3)$ on $u \in C_{+,1/2}$.

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