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Existence and non-existence of harmonic functions under integrable conditions

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Contents

1. Introduction

The purpose of this notes is to introduce a recent development of existence and non-existence of harmonic functions u under the integrability conditions $u \in L^p(M)$ for $p = 1, 2$ on a connected smooth Riemannian manifold M without boundary. [15, 13, 7]. We say that M enjoys $\mathcal{F}$-Liouville property if

$$\Delta u = 0,\ u \in \mathcal{F} \implies u \equiv \text{constant}$$

here Δ is the distributional Laplacian. Among various extensions, the most robust Liouville property is the L^2-Liouville property; namely,

Theorem 1 ([18, 16]). *Any complete Riemannian manifolds enjoys the L^2-Liouville property.*

This extends easily to $p \in (1, \infty)$, even for certain Dirichlet forms provided proper distance functions [17, 12, 9].

In contrast, there are counter examples of complete Riemannian manifolds for the L^1-Liouville property [2, 11, 10]. In this notes, we study first the L^2-Liouville property of incomplete manifolds and then next the L^1-Liouville property of manifolds with ends. More precisely, in Section 1 we will learn the L^2-Liouville property via it's relationship with the essential self-adjointness of the Laplacian, which plays an important role in the theory of quantum mechanics; and next, in Section 2 we introduce new classes of manifolds which guarantee the existence and non-existence of non-trivial L^1 harmonic functions, which is related to the mean exit time of Brownian motion of M to infinity.

2. L^2-Liouville property and the essential selfadjointness of the Lapalcian

The Laplacian is called essentially selfadjoint if it's restriction to the set $C_0^\infty(M)$ of smooth functions with compact support has the unique selfadjoint extension in L^2. This is equivalent to:

$$(\Delta + \lambda)u = 0,\ u \in L^2,\ \lambda < 0 \implies u \equiv 0, \tag{1}$$

here $\int u\Delta u \leq 0$ for any $u \in C_0^\infty(M)$.

The Laplacian of any complete manifold is essentially selfadjoint [1, 16]). Let us point out that Gaffney [3] proved the essential selfadjointness of the Laplacian Δ starting from a larger domain than $C_0^\infty(M)$. Both the L^2-Liouville property and the essential selfadjointness of complete manifolds is a direct consequence of the Caccioppoli type inequality (associated to the L^2-Liouville property): For any $0 < r_1 < r_2$

$$\int_{B_{r_1}} |du|^2 \leq \frac{C}{(r_2 - r_1)^2} \int_{B(r_2)\setminus B(r_1)} |u - \lambda|^2, \qquad \forall \lambda \in \mathbb{R},$$

where $B_r = \{x \in | \rho(x) < r\}$ and $\rho(x)$ is the distance from any fixed point $x_0 \in M$. The Caccioppoli inequality is a consequence of the existence of the sequence of cut-off functions:

$$\chi_{r_2,r_1}(x) = \left(\frac{\rho(x) - r_1}{r_2 - r_1} \wedge 1\right)_+$$

Note that χ_{r_2,r_1} solves

$$\begin{cases} |\nabla u(x)| = \frac{1}{r_2 - r_1}, & x \in B_{r_2} \setminus \overline{B_{r_1}} \\ u(x) = 1, & x \in B_{r_1} \\ u(x) = 0, & x \in M \setminus B_{r_2}. \end{cases} \tag{2}$$

This robust approach has been used to prove the same conclusion for certain Dirichlet forms [17, 12, 9].

The L^2-Liouville property and the essential selfadjointness of Δ are related as in

Theorem 2 ([13]). *For a general Riemannian manifold, the essential selfadjointness of Δ yields the L^2-Liouville property, and these two properties are equivalent if M has infinite volume and if M enjoys Poincaré's inequality: there exists $\lambda > 0$ such that*

$$\int u^2 \le \lambda \int |du|^2, \qquad \forall u \in W_0^1(M). \tag{3}$$

Let us take a closer look at this relationship in the case of model manifolds [5]:

Definition 1. We call $M_\sigma = (0, \infty) \times \mathbb{S}^{n-1}$ a *model manifold* if it's Riemannian metric has the form:

$$dr^2 + \sigma(r)^2 d\theta^2$$

where $\sigma(r) \in C^\infty([0,\infty))$ such that $\sigma(r) > 0$ for $r > 0$, $\sigma(0) = 0$, and $\sigma'(0) = 0$.

Note that M is incomplete[1]. By Weyl's criteria, Δ of a model manifold M_σ is essentially selfadjoint if and only if $n \ge 4$. We say (a general Riemannian manifold) M is stochastically complete if the heat kernel (minimal positive fundamental solution of the heat equation) k satisfies

$$\int k(t,x,y)\, \mu(dx) = 1, \qquad \forall t > 0,\ \forall y \in M.$$

If a model manifold M_σ is stochastically incomplete, then Friedrich's extension of Δ has discrete spectrum; hence, M_σ enjoys Poincaré's inequality (3). It is known that M_σ is stochastically complete if and only if

$$\int^\infty \frac{V(r)}{S(r)} dr = \infty,$$

where $S(r) = C\sigma^{n-1}$ and $V(r) = \int_0^r S(t)dt$. As a stochastically incomplete manifold needs to have infinite volume, we conclude that the L^2-Liouville property of a stochastically incomplete model manifold M_σ fails if and only if $n = 2, 3$.

Recall that the condition $n = 2, 3$ corresponds to the non-polarity of the Cauchy boundary $\partial_C M = \overline{M} \setminus M$, where $\overline{M}$ is the completion of M with respect to the Riemannian distance, associated with the $\mathrm{Cap}_{2.2}$ defined as

$$\mathrm{Cap}_{2.2}(\partial_C M) = \begin{cases} \inf_{u \in \mathcal{F}} \|u\|^2_{W^{2,2}}, & \mathcal{F} \ne \emptyset \\ \infty, & \mathcal{F} = \emptyset, \end{cases}$$

where $\mathcal{F} = \{u \in C^\infty(M) \mid u \ge 1 \text{ on a neighborhood of } \partial_C M\}$, $\|u\|^2_{W^{2,2}} = \|u\|^2 + \|du\|^2 + \|\Delta u\|^2$ and $\|\cdot\|$ is the L^2-norm. In contrast, by Bergman's result, M has the L^2-Liouville property if $\sigma(r) = r$ (the Euclidean case) for any $n \ge 2$. In summery, those observations, made in [13], suggest that in order to break the L^2-Liouville property, M needs to have both a not too small singularity (in the sense that $\partial_C M$ is not polar) and an *ample end* which we will define explicitly next.

The relationship between the $\mathrm{Cap}_{2.2}(\partial_C M)$ and the essential self-adjointness of the Laplacian should be compared with the following weaker but more complete relationship between $\mathrm{Cap}_{1.2}(\partial_C M)$ and the

[1] Usually, M_σ includes the pole, namely, $M_\sigma = [0,\infty) \times \mathbb{S}^{n-1}$.

Markov uniqueness of the Laplacian [2]. The capacity $\mathrm{Cap}_{1.2}$ of the Cauchy boundary $\partial_C M = \overline{M} \setminus M$ is defined as

$$\mathrm{Cap}_{1.2}(\partial_C M) = \begin{cases} \inf_{u \in \mathcal{F}} \|u\|^2_{W^{1,2}}, & \mathcal{F} \neq \emptyset \\ \infty, & \mathcal{F} = \emptyset. \end{cases}$$

Then

Theorem 3 ([6]). *For a general weighted Riemannian manifold M,*

$$Cap_{1.2}(\partial_C M) = 0 \implies \Delta \text{ is Markov unique} \implies M \text{ is stochastically complete.}$$

If $Cap_{1.2}(\partial_C M)$ is finite, then

$$Cap_{1.2}(\partial_C M) = 0 \iff \Delta \text{ is Markov unique.}$$

3. Existence and non-existence of non-trivial integrable harmonic functions

3.1. **Positive and negative results for the L^1-Liouville property.** Let us collect criteria which implies the L^1-Liouville property.

Theorem 4 ([10]). *Let M be complete and $x_0 \in M$. Let r denote the distance from x_0.*

$$Ric(x) \geq -C(1 + r^2(x)) \implies L^1\text{-Liouville property} \tag{4}$$

Note that the curvature condition (4) yields the stochastic completeness of M.

Theorem 5 ([14]). *Any model manifold has the L^1-Liouville property.*

The manifold in this theorem is allowed to be stochastically incomplete. Next example by Chung shows that the stochastic completeness does not yield the L^1-Liouville property:

Example 1 ([2]). Let $M = \mathbb{R} \times \mathbb{S}^1$ with parametrization (r, θ), $-\infty < r < \infty$ and $0 \leq \theta \leq 2\pi$ with the Riemannian metric $ds^2 = \sigma(r)^2(dr^2 + d\theta^2)$, where

$$\sigma(r) = \frac{1}{(r \log r)^2}, \quad |r| > 2$$

Then, $m(M) < \infty$ and M is complete since $\int_2^\infty \sqrt{\sigma} = \infty$. The function $H(r, \theta) = r$ is harmonic ($\Delta r = \sigma^{-2}(r)(\frac{\partial}{\partial r})^2 r = 0$) and is integrable since $\int_2^\infty \sigma(r) r \, dr < \infty$.

However, Grigoryan showed:

Theorem 6 ([4]). *If M is stochastically complete, then every positive super-harmonic function $u \in L^1$ is constant.*

3.2. **New results.** Inspired by the observation in the previous section, we study the existence and the non-existence of non-trivial integrable harmonic functions for manifolds with ends:

Definition 2 (Ends and Manifold with ends). An open set $E \subset M$ is called an end if it is connected, unbounded, and ∂E is compact. We assume ∂E is smooth. We call $E_\sigma = \{x \in M_\sigma \mid r(x) < 1\}$ a model end. A manifold with ends is a smooth connected manifold which is a disjoint union of finite number of end and a compact set K. If all ends of a manifold with ends M are model end then we call M a manifold with model ends.

Definition 3. Let $K \subset M$ be a compact non-polar set. A function h on M is called an Evans potential of K if

$$\begin{cases} \Delta h(x) = 0, & x \in M \setminus K \\ h(x) = 0, & x \in K \\ h(x) \to \infty, & x \to \infty. \end{cases}$$

The minimal and positive solution e to the following boundary value problem is called the equilibrium potential of K

$$\begin{cases} \Delta e(x) = 0, & x \in M \setminus K \\ e(x) = 1, & x \in K. \end{cases}$$

[2] Recall that a selfadjoint operator in L^2 is called Markovian if the associated L^2-semigroup satisfies the Markov property:

$$0 \leq u \leq 1, \; u \in L^2 \implies 0 \leq T_t u \leq 1, \quad \forall t > 0,$$

A symmetric operator is called Markov unique if it has a unique Markovian extension.

Definition 4 ([7]). We say that M is narrow or ample, respectively, if there is a compact non-polar set $K \subset M$ such that it's Evans potential h is in $L^1(M)$ or it's equilibrium potential e is in $L^1(M)$, respectively. For an end E, we take h or e, respectively, to be the Evans potential or equilibrium potential on $\overline{E}$ with $K = \partial E$. We say that M is moderate if it is not ample nor narrow.

The (minimal and positive) Green function G of M is defined as

$$G(x,y) = \int_0^\infty k(t,x,y)\,dt, \quad x,y \in M.$$

Note that it is allowed that $G \equiv \infty$ (for instance, $M = \mathbb{R}^n$ with $n = 1,2$), and if not, then

$$\Delta G(\cdot, x) = -\delta_x.$$

We also note

- the integrability of $e = e_K$ and $G = G(x,\cdot)$ are independent of the choice of $K \subset M$ and $x \in M$;
- e is integrable if and only if so is G.

The former is a consequence of the maximum principle and local Harnack inequality, and the latter follows from the fact that e and G are obtained as the limit of the equilibrium potentials e_n and the Green functions G_n of an exhaustion $\{\Omega_n\}$ of M with the Dirichlet boundary condition.

Proposition 1 ([7, 4]). *The following assertions are equivalent.*

(1) *M is ample.*
(2) $G(x,\cdot) \in L^1(M)$, $\exists/\forall x \in M$.
(3) $\tau_M(x) < \infty$, $\exists/\forall x \in M$.
(4) *There exists an integrable non-trivial super-harmonic function on M.*

A manifold M is called parabolic if $G \equiv \infty$. Hansen and Netuka [8] showed that M has an Evans potential if and only if it is parabolic. By Fubini's lemma, M is ample if and only if the mean exit time τ_M of Brownian motion on M starting from $x \in M$ to escape to ∞ is finite, that is,

$$\tau_M(x) = \int_M \int_0^\infty k(t,x,y)\,m(dy) < \infty.$$

Recall that the stochastic completeness means that the life time of Brownian motion on M is finite almost surely. Combining those facts together, we have the following implications:

(5) narrow $\Longrightarrow$ parabolic $\Longrightarrow$ stochastically complete $\Longrightarrow$ not ample

Hereafter, let M be a manifold with at least two ends otherwise stated excplicitly. Note that such M can be decomposed into a disjoint union of two ends as $M = E_1 \cup \overline{E}_2$.

Proposition 2 ([7]). *Let $M = E_1 \cup \overline{E}_2$.*

(1) *E_1 and E_2 are ample $\Longrightarrow$ M is ample.*
(2) *$m(E_1) < \infty$ and E_2 is ample $\Longrightarrow$ M is ample.*
(3) *E_1 is not ample and $m(E_1) = \infty$ $\Longrightarrow$ M is not ample.*

A model manifold is stochastically complete if and only if it is not ample [5]; however, it is not true in general if M is not a model manifold. Indeed, by Proposition 2,

Example 2 ([7]). Let $M = E_1 \cup \overline{E}_2$, where E_2 is not stochastically complete. Then

E_1 is not ample and $m(E_1) = \infty$ $\Longrightarrow$ M is not ample and not stochastically complete.

Recently, Pessoa, Pigola, and Setti [15] obtained the same conclusion under a different assumption:

Example 3 (Example 35 [15]). Let $M = E_1 \cup \overline{E}_2$, where E_2 is complete and not stochastically complete. Then

E_1 is complete and non-paraboilic and enjoys a parabolic Harnack inequality.

$\Longrightarrow$ M is not ample and not stochastically complete.

The idea of Example 3 is to get a lower bound of the Green function G via the parabolic Harnack inequality so that $G(x,\cdot)$ is not integrable for $x \in M$.

We state the main results in [7]:

Theorem 7. *Let $M = E_1 \cup \overline{E}_2$.*

(1) *If E_1 is narrow and E_2 is ample, then M admits a positive integrable harmonic function H such that* $\sup H = \infty$.
(2) *If E_1 and E_2 are both narrow, and if M enjoys Poincaré's inequality for functions with 0-mean, then M admits an integrable harmonic function H such that* $\inf H = -\infty$ *and* $\sup H = \infty$.

Theorem 8. *Let M be a manifold with model end(s), and let N be the number of the end(s). Then, M enjoys the L^1-Liouville property if one of the following conditions is satisfied.*
(1) $N = 1$.
(2) $N \geq 2$, *and each end is ample or moderate.*
(3) $N \geq 2$, *only one end is narrow, and the other ends are moderate.*

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