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# The second-order phase transition in the BCS-Bogoliubov model of superconductivity and its operator-theoretical proof

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## 1 Introduction and preliminaries

In this paper, we show that the transition to a superconducting state is a second-order phase transition in the BCS-Bogoliubov model of superconductivity without the magnetic fields from the viewpoint of operator theory. Here, the potential $U(\cdot,\ \cdot)$ of the BCS-Bogoliubov gap equation (1.2) below is not a constant but a function. Moreover we obtain the exact and explicit expression for the gap in the specific heat at constant volume at the transition temperature from the viewpoint of operator theory.

To this end we need to deal with the thermodynamic potential $\Omega$ in the BCS-Bogoliubov model without the magnetic fields:

$$\Omega=-\ln Z,$$

where $Z$ is the partition function. Throughout this paper we use the unit $k_B=1$. Here, $k_B$ denotes the Boltzmann constant. Generally speaking, the thermodynamic potential $\Omega$ is a function of the absolute temperature $T$, the chemical potential and the volume of our physical system under consideration. However we fix both the chemical potential and the volume of our physical system, and so we consider the thermodynamic potential $\Omega$ as a function of the temperature $T$ only. We have only to deal with the difference $\Psi$ between the thermodynamic potential corresponding to the superconducting state and that corresponding to the normal-conducting state. The difference $\Psi$ of the thermodynamic potential in the BCS-Bogoliubov model without the magnetic fields is given by

$$
\begin{aligned}
\Psi(T) \;=\; & -2N_0 \int_\varepsilon^{\hbar\omega_D} \left\{ \sqrt{\xi^2+u(T,\,\xi)^2} - \xi \right\} d\xi \\
& + N_0 \int_\varepsilon^{\hbar\omega_D} \frac{u(T,\,\xi)^2}{\sqrt{\xi^2+u(T,\,\xi)^2}} \tanh \frac{\sqrt{\xi^2+u(T,\,\xi)^2}}{2T}\, d\xi \\
& - 4N_0 T \int_\varepsilon^{\hbar\omega_D} \ln \frac{1+e^{-\sqrt{\xi^2+u(T,\,\xi)^2}/T}}{1+e^{-\xi/T}}\, d\xi, \quad T \in [\tau,\,T_c],
\end{aligned}
\tag{1.1}
$$

where $N_0 > 0$ and $\omega_D > 0$ stand for the density of states per unit energy at the Fermi surface and for the Debye angular frequency, respectively. and $u$ is the solution to the BCS-Bogoliubov gap equation (1.2) below. Here, $\tau > 0$ is introduced in the next section, and $T_c > 0$ is the transition temperature (see Definition 1.10 below) and satisfies $\tau < T_c$ . We introduce $\varepsilon > 0$, which is sufficiently small and fixed. We introduce a sufficiently small $\varepsilon > 0$ because of some mathematical reasons.

We consider the difference $\Psi$ defined mainly on the closed interval $[\tau,\, T_c]$ only. This is because we are interested in the phase transition at $T = T_c$ and we need to study some properties of $\Psi$ in the neighborhood of the transition temperature $T_c$ .

**Definition 1.1.** The transition to a superconducting state at the transition temperature $T_c$ is a second-order phase transition if the difference $\Psi$ of the thermodynamic potential satisfies the following:

(a) $\Psi \in C^2[\tau,\, T_c]$ and $\Psi(T_c) = 0$.

(b) $\dfrac{\partial \Psi}{\partial T}(T_c) = 0$.

(c) $\dfrac{\partial^2 \Psi}{\partial T^2}(T_c) \neq 0$.

*Remark* 1.2. Condition (a) of Definition 1.1 implies that the thermodynamic potential $\Omega$ is continuous at an arbitrary temperature $T$. Conditions (a) and (b) imply that the entropy $S = -(\partial\Omega/\partial T)$ is also continuous at an arbitrary temperature $T$ and that, as a result, no latent heat is observed at $T = T_c$ . Hence conditions (a) and (b) imply the transition to a superconducting state at $T_c$ is not a first-order phase transition. On the other hand, Conditions (a) and (c) imply that the specific heat at constant volume $C_V = -T\,(\partial^2\Omega/\partial T^2)$ is discontinuous at $T = T_c$ and that the gap $\Delta C_V$ in $C_V$ is observed at $T = T_c$ . Here, the gap $\Delta C_V$ at $T = T_c$ is given by

$$\Delta C_V = -T_c \frac{\partial^2 \Psi}{\partial T^2}(T_c).$$

For more details on the entropy and the specific heat at constant volume, see e.g. [2, Section III] or Niwa [11, Section 7.7.3].

In order to show that the transition to a superconducting state at the transition temperature is a second-order phase transition, we have to show that conditions (a) , (b) and (c) of Definition 1.1 are all fulfilled. To this end, we need to differentiate the difference $\Psi$ given by (1.1), and hence the solution $u$ to the BCS-Bogoliubov gap equation with respect to the temperature $T$ two times. We thus need to show that there is a unique (nonzero) solution to the BCS-Bogoliubov gap equation and that the solution is differentiable with respect to the temperature two times.

The following is the well-known BCS-Bogoliubov gap equation [2, 4] for superconductivity:

$$(1.2)\quad u(T,\,x)=\int_{\varepsilon}^{\hbar\omega_D}\frac{U(x,\,\xi)\,u(T,\,\xi)}{\sqrt{\xi^2+u(T,\,\xi)^2}}\tanh\frac{\sqrt{\xi^2+u(T,\,\xi)^2}}{2T}\,d\xi,\quad T\ge 0,\quad \varepsilon\le x\le \hbar\omega_D\,,$$

where the solution $u$ is a function of the absolute temperature $T$ and the energy $x$. The potential $U(\cdot,\,\cdot)$ satisfies $U(x,\,\xi)>0$ at all $(x,\,\xi)\in[\varepsilon,\,\hbar\omega_D]^2$. Here we again introduce $\varepsilon>0$, which is sufficiently small and fixed. In the original BCS-Bogoliubov gap equation, one sets $\varepsilon=0$. However we introduce a sufficiently small $\varepsilon>0$ because of some mathematical reasons.

In (1.2) we consider the solution $u$ as a function of the absolute temperature $T$ and the energy $x$. Accordingly, we deal with the integral with respect to the energy $\xi$ in (1.2). Sometimes one considers the solution $u$ as a function of the absolute temperature and the wave vector. Accordingly, instead of the integral in (1.2), one deals with the integral with respect to the wave vector over the three dimensional Euclidean space $\mathbb{R}^3$. Odeh [12], and Billard and Fano [3] established the existence and uniqueness of the solution to the BCS-Bogoliubov gap equation for $T=0$, and Vansevenant [13] for $T\ge 0$. Bach, Lieb and Solovej [1] studied the gap equation in the Hubbard model for a constant potential, and showed that its solution is strictly decreasing with respect to the temperature. Frank, Hainzl, Naboko and Seiringer [5] studied the asymptotic behavior of the transition temperature (the critical temperature) at weak coupling. Hainzl, Hamza, Seiringer and Solovej [6] proved that the existence of a positive solution to the BCS-Bogoliubov gap equation is equivalent to the existence of a negative eigenvalue of a certain linear operator, and showed the existence of a transition temperature. Hainzl and Seiringer [7] obtained upper and lower bounds on the transition temperature and the energy gap for the BCS-Bogoliubov gap equation. For interdisciplinary reviews of the BCS-Bogoliubov model of superconductivity, see Kuzemsky [8, 9, 10].

Let $U_1>0$ is a positive constant and set $U(x,\,\xi)=U_1$ at all $(x,\,\xi)\in[\varepsilon,\,\hbar\omega_D]^2$. Then the solution to the BCS-Bogoliubov gap equation becomes a function of the temperature $T$ only, and we denote the solution by $\Delta_1$. Accordingly, the BCS-Bogoliubov gap equation (1.2) is reduced to the simple gap equation [2]

$$(1.3)\qquad 1=U_1\int_{\varepsilon}^{\hbar\omega_D}\frac{1}{\sqrt{\xi^2+\Delta_1(T)^2}}\tanh\frac{\sqrt{\xi^2+\Delta_1(T)^2}}{2T}\,d\xi,\quad 0\le T\le\tau_1\,,$$

where the temperature $\tau_1>0$ is defined by (see [2])

$$(1.4)\qquad 1=U_1\int_{\varepsilon}^{\hbar\omega_D}\frac{1}{\xi}\tanh\frac{\xi}{2\tau_1}\,d\xi.$$

See also Niwa [11] and Ziman [18].

As is well known in the BCS-Bogoliubov model, physicists and engineers studying superconductivity always assume that there is a unique nonnegative solution $\Delta_1$ to the simple gap equation (1.3), that the solution $\Delta_1$ is continuous and strictly decreasing with respect to the temperature $T$, and that the solution $\Delta_1$ is of class $C^2$ with respect to the temperature $T$, and so on. But, as far as the present author knows, there is no mathematical proof for these assumptions of the BCS-Bogoliubov model. Applying the implicit function theorem to the simple gap equation (1.3), the present author obtained the following proposition that indeed gives a mathematical proof for these assumptions:

**Proposition 1.3** ([14, Proposition 1.2]). *Let $U_1>0$ is a positive constant and set $U(x,\,\xi)=U_1$ at all $(x,\,\xi)\in[\varepsilon,\,\hbar\omega_D]^2$. Then there is a unique nonnegative solution $\Delta_1:[0,\,\tau_1]\to[0,\,\infty)$ to* *the simple gap equation* (1.3) *such that the solution* $\Delta_1$ *is continuous, strictly decreasing with respect to the temperature* $T$ *on the closed interval* $[0,\ \tau_1]$, *and satisfies*

$$\Delta_1(0)=\frac{\sqrt{\left(\hbar\omega_D-\varepsilon\, e^{1/U_1}\right)\left(\hbar\omega_D-\varepsilon\, e^{-1/U_1}\right)}}{\sinh\frac{1}{U_1}}, \qquad \Delta_1(\tau_1)=0.$$

*Moreover, the solution* $\Delta_1$ *is of class* $C^2$ *with respect to the temperature* $T$ *on the interval* $[0,\ \tau_1)$ *and*

$$\Delta_1'(0)=\Delta_1''(0)=0, \qquad \lim_{T\uparrow\tau_1}\Delta_1'(T)=-\infty.$$

*Remark* 1.4. We set $\Delta_1(T)=0$ at $T>\tau_1$. See figure 1.

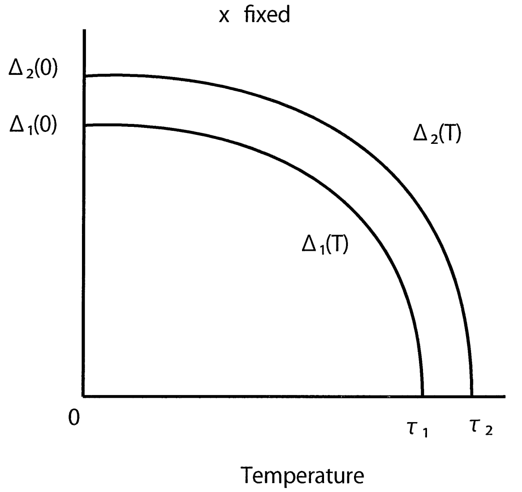


Figure 1: The graphs of the functions $\Delta_1$ and $\Delta_2$ with $x$ fixed.

We then introduce another positive constant $U_2>0$. Let $0<U_1<U_2$ and set $U(x,\ \xi)=U_2$ at all $(x,\ \xi)\in[\varepsilon,\ \hbar\omega_D]^2$. Then a similar discussion implies that for $U_2$, there is a unique nonnegative solution $\Delta_2:[0,\ \tau_2]\to[0,\ \infty)$ to the simple gap equation

$$1=U_2\int_\varepsilon^{\hbar\omega_D}\frac{1}{\sqrt{\xi^2+\Delta_2(T)^2}}\tanh\frac{\sqrt{\xi^2+\Delta_2(T)^2}}{2T}\,d\xi, \qquad 0\le T\le\tau_2. \tag{1.5}$$

Here, $\tau_2>0$ is defined by

$$1=U_2\int_\varepsilon^{\hbar\omega_D}\frac{1}{\xi}\tanh\frac{\xi}{2\tau_2}\,d\xi. \tag{1.6}$$

*Remark* 1.5. We again set $\Delta_2(T)=0$ at $T>\tau_2$.

**Lemma 1.6** ([14, Lemma 1.5]). (a) *The inequality* $\tau_1<\tau_2$ *holds.*
(b) *If* $0\le T<\tau_2$, *then* $\Delta_1(T)<\Delta_2(T)$. *If* $T\ge\tau_2$, *then* $\Delta_1(T)=\Delta_2(T)=0$.

See figure 1. The function $\Delta_2$ has properties similar to those of the function $\Delta_1$.

We define a nonlinear integral operator $A$ by

$$(1.7) \qquad Au(T,\, x) = \int_\varepsilon^{\hbar\omega_D} \frac{U(x,\, \xi)\, u(T,\, \xi)}{\sqrt{\xi^2 + u(T,\, \xi)^2}} \tanh \frac{\sqrt{\xi^2 + u(T,\, \xi)^2}}{2T}\, d\xi.$$

Here the right side of this equality is exactly the right side of the BCS-Bogoliubov gap equation (1.2). Since the solution to the BCS-Bogoliubov gap equation is a fixed point of our operator $A$, we apply fixed point theorems to our operator $A$.

Let us turn to the BCS-Bogoliubov gap equation (1.2). We assume the following condition on the potential $U(\cdot,\, \cdot)$:

$$(1.8) \qquad U(\cdot,\, \cdot) \in C([\varepsilon,\, \hbar\omega_D]^2), \qquad (0 <)\, U_1 \le U(x,\, \xi) \le U_2 \quad \text{at all} \quad (x,\, \xi) \in [\varepsilon,\, \hbar\omega_D]^2.$$

Let $0 \le T \le \tau_2$ and fix $T$. We now consider the Banach space $C[0,\, \hbar\omega_D]$ consisting of continuous functions of the energy $x$ only, and deal with the following temperature dependent subset $V_T$:

$$V_T = \{u(T,\, \cdot) \in C[\varepsilon,\, \hbar\omega_D] :\ \Delta_1(T) \le u(T,\, x) \le \Delta_2(T) \text{ at } x \in [\varepsilon,\, \hbar\omega_D]\}\,.$$

*Remark* 1.7. The set $V_T$ depends on the temperature $T$. See figures 1 and 2.

We define our nonlinear integral operator $A$ (1.7) on the set $V_T$ . The following gives another proof of the existence and uniqueness of the nonnegative solution to the BCS-Bogoliubov gap equation, and shows how the solution varies with the temperature.

**Theorem 1.8** ([14, Theorem 2.2]). *Assume* (1.8) *and let* $T \in [0,\, \tau_2]$ *be fixed. Then there is a unique nonnegative solution* $u_0(T,\, \cdot) \in V_T$ *to the BCS-Bogoliubov gap equation* (1.2)*:*

$$u_0(T,\, x) = \int_\varepsilon^{\hbar\omega_D} \frac{U(x,\, \xi)\, u_0(T,\, \xi)}{\sqrt{\xi^2 + u_0(T,\, \xi)^2}} \tanh \frac{\sqrt{\xi^2 + u_0(T,\, \xi)^2}}{2T}\, d\xi, \qquad x \in [\varepsilon,\, \hbar\omega_D].$$

*Consequently, the solution* $u_0(T,\, \cdot)$ *with* $T$ *fixed is continuous with respect to the energy* $x$ *and varies with the temperature as follows:*

$$\Delta_1(T) \le u_0(T,\, x) \le \Delta_2(T) \quad at \quad (T,\, x) \in [0,\, \tau_2] \times [\varepsilon,\, \hbar\omega_D].$$

See figure 2.

*Remark* 1.9. Let $u_0(T,\, \cdot)$ be the solution as in Theorem 1.8. If there is a point $x_1 \in [\varepsilon,\, \hbar\omega_D]$ satisfying $u_0(T,\, x_1) = 0$, then $u_0(T,\, x) = 0$ at all $x \in [\varepsilon,\, \hbar\omega_D]$. See [14, Proposition 2.4].

The existence and uniqueness of the transition temperature $T_c$ were pointed out in previous papers [5, 6, 7, 13]. In our case, we can define it as follows:

**Definition 1.10.** Let $u_0(T,\, \cdot)$ be the solution given by Theorem 1.8. Then the transition temperature $T_c$ is defined by

$$T_c = \inf\{T > 0 :\ u_0(T,\, x) = 0 \quad \text{at all} \quad x \in [\varepsilon,\, \hbar\omega_D]\}.$$

*Remark* 1.11. Let $u_0(T,\, \cdot)$ be the solution given by Theorem 1.8. At $T \ge T_c$ , we set $u_0(T,\, x) = 0$ at all $x \in [\varepsilon,\, \hbar\omega_D]$. The transition temperature $T_c$ is the critical temperature that divides normal conductivity and superconductivity, and satisfies $\tau_1 \le T_c \le \tau_2$ . See figure 2.

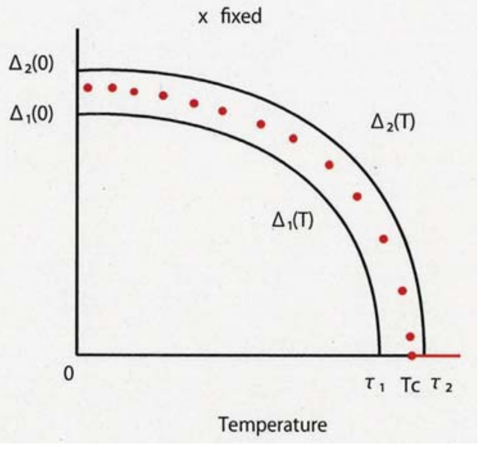


Figure 2: For each fixed $T$, the solution $u_0(T,\,x)$ is between $\Delta_1(T)$ and $\Delta_2(T)$.

But Theorem 1.8 tells us nothing about continuity and smoothness of the solution $u_0$ with respect to the temperature $T$. Applying the Banach fixed-point theorem, we then showed in [15, Theorem 1.2] that the solution $u_0$ is indeed continuous both with respect to the temperature $T$ and with respect to the energy $x$ under the restriction that the temperature $T$ is sufficiently small. See also [16].

Let us denote by $z_0 > 0$ a unique solution to the equation $\dfrac{2}{z} = \tanh z \quad (z > 0)$. Note that $z_0$ is nearly equal to 2.07. Let $\tau_0\,(> 0)$ satisfy

$$\Delta_1(\tau_0) = 2z_0\tau_0\,. \tag{1.9}$$

From (1.9) it follows immediately that $(0 <)\,\tau_0 < \tau_1$.

*Remark* 1.12. Observed values in many experiments by using superconductors imply the temperature $\tau_0$ is nearly equal to $T_c/2$.

Let $0 < \tau_3 < \tau_0$ and fix $\tau_3$. We then deal with the following subset $V$ of the Banach space $C([0,\,\tau_3] \times [\varepsilon,\,\hbar\omega_D])$:

$$\begin{aligned} V \;\; = \;\; & \big\{ u \in C([0,\,\tau_3] \times [\varepsilon,\,\hbar\omega_D]) : 0 \le u(T,\,x) - u(T',\,x) \le \gamma\,(T' - T) \;\; (T < T'), \\ & \Delta_1(T) \le u(T,\,x) \le \Delta_2(T),\; u \text{ is partially differentiable with respect to } T \text{ twice}, \\ & \frac{\partial u}{\partial T},\; \frac{\partial^2 u}{\partial T^2} \in C([0,\,\tau_3] \times [\varepsilon,\,\hbar\omega_D]) \Big\}\,. \end{aligned}$$

Here, see [17, (2.2)] for the positive constant $\gamma > 0$. We define our operator (1.7) on the subset $V$. We denote by $\overline{V}$ the closure of the subset $V$ with respect to the norm of the Banach space $C([0,\,\tau_3] \times [\varepsilon,\,\hbar\omega_D])$.

**Theorem 1.13** ([17, Theorem 1.10]). *Assume* (1.8). *Then the operator $A : \overline{V} \to \overline{V}$ has a unique fixed point $u_0 \in \overline{V}$, and so there is a unique nonnegative solution $u_0 \in \overline{V}$ to the BCS-Bogoliubov gap equation* (1.2)*:*

$$u_0(T,\,x) = \int_\varepsilon^{\hbar\omega_D} \frac{U(x,\,\xi)\,u_0(T,\,\xi)}{\sqrt{\xi^2 + u_0(T,\,\xi)^2}} \tanh \frac{\sqrt{\xi^2 + u_0(T,\,\xi)^2}}{2T}\,d\xi, \quad 0 \le T \le \tau_3\,, \quad \varepsilon \le x \le \hbar\omega_D\,.$$

*Consequently, the solution $u_0$ is continuous on $[0, \tau_3] \times [\varepsilon, \hbar\omega_D]$, i.e., the solution $u_0$ is continuous with respect to both the temperature $T$ and the energy $x$. Moreover, the solution $u_0$ is Lipschitz continuous and monotone decreasing with respect to the temperature $T$, and satisfies $\Delta_1(T) \leq u_0(T, x) \leq \Delta_2(T)$ at all $(T, x) \in [0, \tau_3] \times [\varepsilon, \hbar\omega_D]$. Furthermore, if $u_0 \in V$, then the solution $u_0$ is partially differentiable with respect to the temperature $T$ twice and the second-order partial derivative is continuous with respect to both the temperature $T$ and the energy $x$. On the other hand, if $u_0 \in \overline{V} \setminus V$, then the solution $u_0$ is approximated by such a smooth element of the subset $V$ with respect to the norm of the Banach space $C([0, \tau_3] \times [\varepsilon, \hbar\omega_D])$.*

See figure 3 for the graph of the solution $u_0$ with the energy $x$ fixed.

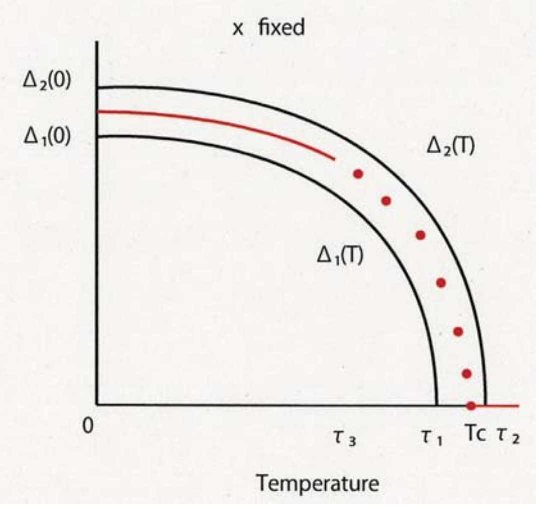


Figure 3: The solution $u_0$ belongs to the subset $\overline{V}$.

## 2 Main results

We choose an arbitrary $\tau > 0$ satisfying

$$\tau_1 < \tau < \tau_2 .$$

Here, $\tau_1 > 0$ (resp. $\tau_2 > 0$) is related to $U_1 > 0$ by (1.4) (resp. to $U_2 > 0$ by (1.6)). Let the potential $U(\cdot, \cdot)$ satisfy (1.8) and the following:

$$(0 <)\, a = \max_{x \in [\varepsilon, \hbar\omega_D]} \left( \int_{\varepsilon}^{\hbar\omega_D} \frac{U(x, \xi)}{\xi} \tanh \frac{\xi}{2\tau}\, d\xi \right) < 1. \tag{2.1}$$

Even when the potential $U(\cdot, \cdot)$ satisfies both (1.8) and (2.1), Theorem 1.8 again implies that there is a unique nonnegative solution $u_0(T, \cdot) \in V_T$ to the BCS-Bogoliubov gap equation (1.2).

By Definition 1.10, the transition temperature $T_c > 0$ is thus defined. Note that the transition temperature $T_c > 0$ is related to the solution $u_0(T, \cdot) \in V_T$. As for the relation between $\tau$ and $T_c$, we have $\tau < T_c$ or $\tau \geq T_c$ . So we choose the potential such that the relation $\tau < T_c$ holds true, and we then consider the solution $u_0(\cdot, \cdot)$ to the BCS-Bogoliubov gap equation defined on $[\tau, T_c] \times [\varepsilon, \hbar\omega_D]$.

Let us consider the following condition, which gives the behavior of the solution $u_0(\cdot, \cdot)$ to the BCS-Bogoliubov gap equation as $T \uparrow T_c$:

**Condition (C).** An element $u \in C([\tau, T_c] \times [\varepsilon, \hbar\omega_D])$ is partially differentiable with respect to the temperature $T \in [\tau, T_c)$ twice, and both $(\partial u/\partial T)$ and $(\partial^2 u/\partial T^2)$ belong to $C([\tau, T_c) \times [\varepsilon, \hbar\omega_D])$. Moreover, for the $u$ above, there are a unique $v \in C[\varepsilon, \hbar\omega_D]$ and a unique $w \in C[\varepsilon, \hbar\omega_D]$ satisfying the following:

(C1) $v(x) > 0$ at all $x \in [\varepsilon, \hbar\omega_D]$.

(C2) For an arbitrary $\varepsilon_1 > 0$, there is a $\delta > 0$ such that $|T_c - T| < \delta$ implies

$$\left| v(x) - \frac{u(T, x)^2}{T_c - T} \right| < T_c \varepsilon_1 \quad \text{and} \quad \left| v(x) + 2\, u(T, x)\, \frac{\partial u}{\partial T}(T, x) \right| < T_c \varepsilon_1 .$$

Here, the $\delta$ does not depend on $x \in [\varepsilon, \hbar\omega_D]$.

(C3) For an arbitrary $\varepsilon_1 > 0$, there is a $\delta > 0$ such that $|T_c - T| < \delta$ implies

$$\left| \frac{w(x)}{2} + \frac{u(T, x)^2 + (T_c - T)\, \frac{\partial}{\partial T} \left\{ u(T, x)^2 \right\}}{(T_c - T)^2} \right| < \varepsilon_1 \quad \text{and} \quad \left| w(x) - \frac{\partial^2}{\partial T^2} \left\{ u(T, x)^2 \right\} \right| < \varepsilon_1 .$$

Here, the $\delta$ does not depend on $x \in [\varepsilon, \hbar\omega_D]$.

*Remark* 2.1. Conditions (C2) and (C3) imply

$$\lim_{T \uparrow T_c} \frac{\partial\, u(T, x)^2}{\partial T} = -v(x) \quad \text{and} \quad \lim_{T \uparrow T_c} \frac{\partial^2\, u(T, x)^2}{\partial T^2} = w(x).$$

Each of them converges uniformly with respect to $x$.

Consider a subset $W$ of the Banach space $C([\tau, T_c] \times [\varepsilon, \hbar\omega_D])$:

$$\begin{aligned} W \;=\; & \left\{ u \in C([\tau, T_c] \times [\varepsilon, \hbar\omega_D]) : u(T, x) \geq u(T', x) \;\; (T < T'), \right. \\ & 0 = \Delta_1(T) \leq u(T, x) \leq \Delta_2(T) \;\text{ at }\; (T, x),\, (T', x) \in [\tau, T_c] \times [\varepsilon, \hbar\omega_D], \\ & \left. u \text{ satisfies Condition (C) above} \right\}. \end{aligned}$$

We then define a nonlinear integral operator $A$ (1.7) on the closure $\overline{W}$ of the subset $W$, and look for a fixed point in $\overline{W}$ of our operator $A$. Here, $\overline{W}$ denotes the closure of the subset $W$ with respect to the norm of the Banach space $C([\tau, T_c] \times [\varepsilon, \hbar\omega_D])$.

*Remark* 2.2. It follows directly from Condition (C2) that $u(T_c, x) = 0$ at all $x \in [\varepsilon, \hbar\omega_D]$ for $u \in W$.

One of our main results is the following:

**Theorem 2.3.** *Choose the potential* $U(\cdot, \cdot)$ *such that* $U(\cdot, \cdot)$ *satisfies* (1.8), (2.1) *and the relation* $\tau < T_c$ . *Then the operator* $A : \overline{W} \to \overline{W}$ *is contractive, and so there is a unique fixed point* $u_0 \in \overline{W}$ *of the operator* $A : \overline{W} \to \overline{W}$. *Consequently, there is a unique nonnegative solution* $u_0 \in \overline{W}$ *to the BCS-Bogoliubov gap equation* (1.2):

$$u_0(T, x) = \int_\varepsilon^{\hbar\omega_D} \frac{U(x, \xi)\, u_0(T, \xi)}{\sqrt{\xi^2 + u_0(T, \xi)^2}} \tanh \frac{\sqrt{\xi^2 + u_0(T, \xi)^2}}{2T}\, d\xi, \quad \tau \leq T \leq T_c, \quad \varepsilon \leq x \leq \hbar\omega_D .$$

*The solution $u_0$ is continuous on $[\tau, T_c] \times [\varepsilon, \hbar\omega_D]$ and is monotone decreasing with respect to the temperature $T$. Moreover, the solution $u_0$ satisfies that $0 = \Delta_1(T) \le u(T, x) \le \Delta_2(T)$ at all $(T, x) \in [\tau, T_c] \times [\varepsilon, \hbar\omega_D]$ and that $u_0(T_c, x) = 0$ at all $x \in [\varepsilon, \hbar\omega_D]$. If $u_0 \in W$, then the solution $u_0$ is smooth with respect to the temperature $T$ and satisfies Condition (C). Furthermore, if $u_0 \in \overline{W} \backslash W$, then the solution $u_0$ is approximated by a smooth element of the subset $W$ fulfilling Condition (C).*

See figure 4 for the graph of the solution $u_0$ near the transition temperature $T_c$ with the energy $x$ fixed.

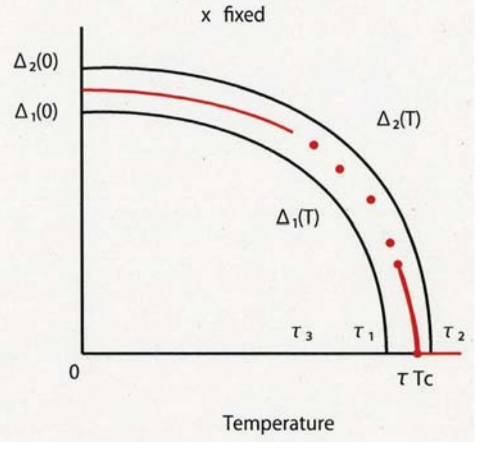


Figure 4: The graph of the solution $u_0$ near the transition temperature $T_c$.

**Approximation (A).** The function $u$ in $\Psi$ (1.1) is the solution $u_0 \in \overline{W}$ of Theorem 2.3. However, if the solution $u_0$ is in the set $\overline{W} \setminus W$, then the solution $u_0$ might not be differentiable with respect to the temperature $T$. This means that $\Psi$ (1.1) might not be differentiable with respect to the temperature $T$. We then approximate the solution $u_0 \in \overline{W} \setminus W$ by a suitably chosen element $u_1 \in W$, and we replace the function $u$ in (1.1) by this element $u_1 \in W$. In $\Psi$ (1.1) we thus use this element $u_1 \in W$ instead of the solution $u_0 \in \overline{W} \setminus W$. Accordingly, we consider the functions $v$ and $w$ in Condition (C) as those corresponding to this element $u_1 \in W$. In this way, we can differentiate this element $u_1 \in W$ with respect to the temperature $T$ twice, and hence $\Psi$ (1.1) with respect to the temperature $T$ twice. On the other hand, if the solution $u_0$ of Theorem 2.3 is in the subset $W$, then the solution $u_0$ is differentiable with respect to the temperature $T$ twice. Needless to say, in this case, we use the solution $u_0 \in W$ instead of this element $u_1 \in W$, and we need no approximation in this case.

We can show that all the conditions of Definition 1.1 hold true. We thus obtain the following:

**Theorem 2.4.** *Assume Approximation (A) if necessary. Then the transition to a superconducting state at the transition temperature $T_c$ is a second-order phase transition.*

Let $g : [0, \infty) \to \mathbb{R}$ be given by

$$g(\eta) = \begin{cases} \dfrac{1}{\eta^2 \cosh^2 \eta} - \dfrac{\tanh \eta}{\eta^3} & (\eta > 0), \\ -\dfrac{2}{3} & (\eta = 0). \end{cases}$$

Note that $g(\eta) < 0$.

We have one more main result:

**Theorem 2.5.** *Assume Approximation (A) if necessary. Let $v$ and $g$ be as above. Then the gap $\Delta C_V$ in the specific heat at constant volume at the transition temperature $T_c$ is given by*

$$\Delta C_V = -\frac{N_0}{8\,T_c} \int_{\varepsilon/(2T_c)}^{\hbar\omega_D/(2T_c)} v(2\,T_c\,\eta)^2 g(\eta)\, d\eta \quad (> 0).$$

*Remark* 2.6. Putting $\varepsilon = 0$ gives

$$\Delta C_V = -\frac{N_0}{8\,T_c} \int_0^{\hbar\omega_D/(2T_c)} v(2\,T_c\,\eta)^2 g(\eta)\, d\eta \quad (> 0). \tag{2.2}$$

*Remark* 2.7. If the potential $U(\cdot,\, \cdot)$ of the BCS-Bogoliubov gap equation (1.2) is a positive constant, then putting $\varepsilon = 0$ gives

$$\Delta C_V = N_0\, v \tanh \frac{\hbar\omega_D}{2k_B T_c} \quad (> 0). \tag{2.3}$$

*Remark* 2.8. If the potential $U(\cdot,\, \cdot)$ of the BCS-Bogoliubov gap equation is not a constant but a function, then we have (2.2). On the other hand, if the potential $U(\cdot,\, \cdot)$ is reduced to a constant, then we have (2.3). As far as the present author knows, no one obtained (2.2) and no one obtained (2.3). In previous physics literature, one obtained only an approximate expression that approximates (2.3). But, this time, we obtain the exact and explicit expression (2.2) from the viewpoint of operator theory.

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