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On some parabolic systems arising from a nuclear reactor model *

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1 Introduction

We study the following initial-boundary value problem for a reaction diffusion system:

$$\text{(NR)} \qquad \begin{cases} \partial_t u_1 - \Delta u_1 = u_1 u_2 - b u_1, & x \in \Omega,\ t > 0, \\ \partial_t u_2 - \Delta u_2 = a u_1, & x \in \Omega,\ t > 0, \\ \partial_\nu u_1 + \alpha u_1 = \partial_\nu u_2 + \beta |u_2|^{\gamma-2} u_2 = 0, & x \in \partial\Omega,\ t > 0, \\ u_1(x,0) = u_{10}(x) \geq 0,\ u_2(x,0) = u_{20}(x) \geq 0, & x \in \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, ν denotes the unit outward normal vector on $\partial\Omega$ and $\partial_\nu u_i = \nabla u_i \cdot \nu$ $(i = 1, 2)$. u_1, u_2 are real-valued unknown functions, a and b are given positive constants. As for the parameters appearing in the boundary condition, we assume $\alpha \geq 0$, $\beta > 0$ and $\gamma \geq 2$. We note that the boundary condition for u_1 becomes the homogeneous Neumann boundary condition when $\alpha = 0$, and the boundary condition for u_2 gives the Robin boundary condition when $\gamma = 2$. Moreover, u_{10}, $u_{20} \in L^\infty(\Omega)$ are given initial data.

This system describes diffusion phenomena of neutrons and heat in nuclear reactors by taking the heat conduction into consideration, introduced by Kastenberg and Chambré [10]. In this model u_1 and u_2 represent the neutron density and the temperature in nuclear reactors respectively. There are many studies on this model under various boundary conditions, for example, [2], [3], [6], [7], [9], [17] and [18]. Many of them are concerned with the existence of positive steady-state solutions and the long-time behavior of solutions.

In [6], they study this system with the homogeneous Neumann boundary condition and Robin boundary condition:

$$(1.1) \qquad \begin{cases} \partial_t u_1 - \Delta u_1 = u_1 u_2 - b u_1, & x \in \Omega,\ t > 0, \\ \partial_t u_2 - \Delta u_2 = a u_1, & x \in \Omega,\ t > 0, \\ \partial_\nu u_1 = \partial_\nu u_2 + \beta u_2 = 0, & x \in \partial\Omega,\ t > 0, \\ u_1(x,0) = u_{10}(x),\ u_2(x,0) = u_{20}(x), & x \in \overline{\Omega}. \end{cases}$$

They showed the existence and the ordered uniqueness of positive stationary solution for $N \in [2, 5]$. They also investigated some threshold property to determine blow-up or globally existence. Moreover, in [18] the case where $\beta = 0$, that is, the homogeneous Neumann boundary condition for u_2 is studied. The author of [18] discussed the stability region and the instability region of (1.1) and give an upper bound and a lower bound on the blowing-up time for a solution which blows up in finite time.

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The following system with the homogeneous Dirichlet boundary conditions:

$$
\begin{cases}
\partial_t u_1 - \Delta u_1 = u_1 u_2^p - b u_1, & x \in \Omega,\ t > 0, \\
\partial_t u_2 - \Delta u_2 = a u_1, & x \in \Omega,\ t > 0, \\
u_1 = u_2 = 0, & x \in \partial\Omega,\ t > 0, \\
u_1(x,0) = u_{10}(x),\ u_2(x,0) = u_{20}(x), & x \in \Omega,
\end{cases}
\tag{1.2}
$$

is studied by [7] and [9]. In [7], they showed the existence of positive stationary solutions for the case where $p = 1$ and $N = 2$, 3 or Ω is bounded convex domain with $N \in [2,5]$. Furthermore, they obtained similar resluts of the threshold property in [6] when Ω is ball. In [9], the existence and ordered uniqueness of positive stationary solutions are considered for general $p > 0$ and some threshold reslut is obtained. Moreover the blow-up rate estimate is given for positive blowing-up solutions when Ω is ball and $p \geq 1$.

In this paper, we are concerned with the nonlinear boundary condition. From physical point of view it could be more natural to consider the nonlinear boundary condition than the homogeneous Dirichlet boundary condition or Neumann boundary condition. Indeed, if there is no control of the heat flux on the boundary, it is well known that the power type nonlinearity for u_2 is justified by Stefan-Boltzmann's law, which says that the heat energy radiation from the surface of the body is proportional to the fourth power of temperature when $N = 3$. In Section 3, we consider the stationary problem associated with (NR) and show the existence of positive solutions by applying abstract fixed point theorem based on Krasnosel'skii [11]. In Section 4, we disscus the large time behavior of solutions to (NR) and prove that every positive stationary solution plays a role of threshold to separate global solutions and finite time blowing-up solutions.

2 Preliminaries

First of all, we state several lemma to prove our results for (NR).

Lemma 2.1 (Krasnosel'skii-type fixed point theorem [11], [12]). *Suppose that E is a real Banach space with norm $\|\cdot\|$, $K \subset E$ is a positive cone, and $\Phi : K \to K$ is a compact mapping satisfying $\Phi(0) = 0$. Assume that there exists two constants $R > r > 0$ and an element $\varphi \in K \setminus \{0\}$, such that*

$$\text{(i) } u \neq \lambda\Phi(u),\quad \forall\lambda \in (0,1),\quad \text{if } u \in K \text{ and } \|u\| = r,$$

$$\text{(ii) } u \neq \Phi(u) + \lambda\varphi,\quad \forall\lambda \geq 0,\quad \text{if } u \in K \text{ and } \|u\| = R.$$

Then the mapping Φ possesses at least one fixed point in $K_1 := \{u \in K;\ 0 < r < \|u\| < R\}$.

Lemma 2.2 ([5]). *Let λ_1 and φ_1 be the first eigenvalue and the corresponding eigenfunction for the problem:*

$$
\begin{cases}
-\Delta\varphi = \lambda\varphi, & x \in \Omega, \\
\partial_\nu\varphi + \alpha\varphi = 0, & x \in \partial\Omega,
\end{cases}
$$

where Ω is smooth bounded domain in $\mathbb{R}^N$ and $\alpha > 0$. Then there exists a constant $C_\alpha > 0$ such that

$$\varphi_1(x) \geq C_\alpha \qquad x \in \overline{\Omega}.$$

Lemma 2.3 ([16]). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. For $p \in [1,\infty)$ there exists a constant $C = C(\Omega, p) > 0$ such that*

$$\left\| u - \frac{1}{|\partial\Omega|}\int_{\partial\Omega} u dS \right\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)} \qquad \forall u \in W^{1,p}(\Omega).$$

Lemma 2.4 ([4]). *Let $\gamma \geq 2$ and $N \in \mathbb{N}$. Then there exists $C_\gamma > 0$ such that*

$$(x-y)\cdot\left(|x|^{\gamma-2}x - |y|^{\gamma-2}y\right) \geq C_\gamma|x-y|^\gamma \tag{2.1}$$

for all $x,\ y \in \mathbb{R}^N$.

Lemma 2.5 ([15]). *Let Ω be any domain in $\mathbb{R}^N$ and assume that exists a number $r_0 \geq 1$ and a constant C independent of $r \in [r_0, \infty)$ such that*

$$\|u\|_{L^r(\Omega)} \leq C \qquad \forall r \in [r_0, \infty),$$

then u belongs to $L^\infty(\Omega)$ and the following property holds.

$$\lim_{r\to\infty} \|u\|_{L^r(\Omega)} = \|u\|_{L^\infty(\Omega)}. \tag{2.2}$$

Conversely, assume that $u \in L^{r_0}(\Omega) \cap L^\infty(\Omega)$ for some $r_0 \in [1, \infty)$, then u satisfies (2.2).

Lemma 2.6 ([15]). *Let $y(t)$ be a bounded measurable non-negative function on $[0, T]$ and suppose that there exists $y_0 \geq 0$ and a monotone non-decreasing function $m(\cdot) : [0, +\infty) \to [0, +\infty)$ such that*

$$y(t) \leq y_0 + \int_0^t m(y(s))ds \qquad a.e.\ t \in (0, T).$$

Then there exists a number $T_0 = T_0(y_0, m(\cdot)) \in (0, T]$ such that

$$y(t) \leq y_0 + 1 \qquad a.e.\ t \in [0, T_0].$$

3 Stationary Problem

First, we consider the following stationary problem:

$$\text{(S-NR)} \qquad \begin{cases} -\Delta u_1 = u_1 u_2 - b u_1, & x \in \Omega, \\ -\Delta u_2 = a u_1, & x \in \Omega, \\ \partial_\nu u_1 + \alpha u_1 = \partial_\nu u_2 + \beta |u_2|^{\gamma-2} u_2 = 0, & x \in \partial\Omega. \end{cases}$$

Since (S-NR) has no variational structure, it is hard to apply the variational method to (S-NR). Hence in order to show the existence of positive stationary solutions to (NR), we rely on the abstract fixed point theorem developed by Krasnosell'skii. The difficluty of proving the existence of positive stationary solutions is how to obtain L^∞-estimates for solutions.

3.1 Existence of positive solutions

Theorem 3.1. *Let $1 \leq N \leq 5$, suppose that either* (A) *or* (B) *is satisfyied* :

$$\begin{cases} \text{(A)} \qquad \gamma = 2, \quad \alpha \leq 2\beta, \\ \text{(B)} \qquad \gamma > 2. \end{cases}$$

Then (S-NR) *has at least one positive solution.*

We shall prove this theorem by Lemma 2.1. In order to apply Lemma 2.1 , we here fix our setting:

$$\begin{aligned} &E = C(\overline{\Omega}) \times C(\overline{\Omega}), && u = (u_1, u_2)^{\mathrm{T}} \in E, \\ &\|u\| = \|u_1\|_{C(\overline{\Omega})} + \|u_2\|_{C(\overline{\Omega})}, && K = \{u \in E; u_1 \geq 0, u_2 \geq 0\}. \end{aligned}$$

Set $\varphi = (\varphi_1, 0)^{\mathrm{T}} \in K \setminus \{0\}$, where λ_1 and φ_1 are the first eigenvalue and the corresponding eigenfunction of the eigenvalue problem:

$$\begin{cases} -\Delta\varphi = \lambda\varphi, & x \in \Omega, \\ \partial_\nu\varphi + \alpha\varphi = 0, & x \in \partial\Omega. \end{cases} \tag{3.1}$$

It is well known that $\lambda_1 > 0$. In this section, we normalize $\varphi_1(x)$ such that $\|\varphi_1\|_{L^2} = 1$. For given $u = (u_1, u_2)^{\mathrm{T}} \in K$, let $v = (v_1, v_2)^{\mathrm{T}} = \Psi(u)$ be the unique nonnegative solution (see Brézis [1]) of

$$\begin{cases} -\Delta v_1 + bv_1 = u_1u_2, & x \in \Omega, \\ -\Delta v_2 = au_1, & x \in \Omega, \\ \partial_\nu v_1 + \alpha v_1 = \partial_\nu v_2 + \beta|v_2|^{\gamma-2}v_2 = 0, & x \in \partial\Omega. \end{cases}$$

It is easy to see that $\Psi(0) = 0$ and $\Psi : K \to K$ is compact.

Thus in order to prove that (S-NR) has a positive solution, it suffices to show that Ψ has a fixed point in K. Therefore, for proving Theorem 3.1 we are going to check the conditions (i) and (ii) of Lemma 2.1.

We first check condition (i).

Lemma 3.2. *Let $r = \frac{b}{2}$. We see that $u \neq \lambda\Psi(u)$ for any $\lambda \in (0,1)$ and $u \in K$ satisfying $\|u\| = r$. That is, condition* (i) *of* Lemma 2.1 *with $\Phi = \Psi$ holds.*

Proof. We prove the statement by contradiction. Suppose that there exist $\lambda \in (0,1)$ and $u \in K$ with $\|u\| = r$ such that $u = \lambda\Psi(u)$, that is, u_1 and u_2 satisfy

$$\begin{cases} -\Delta u_1 + bu_1 = \lambda u_1u_2, & x \in \Omega, \\ -\Delta u_2 = \lambda au_1, & x \in \Omega, \\ \partial_\nu u_1 + \alpha u_1 = \partial_\nu u_2 + \beta\left|\dfrac{u_2}{\lambda}\right|^{\gamma-2}u_2 = 0, & x \in \partial\Omega. \end{cases} \tag{3.2}$$

Multiplying the first equation of (3.2) by u_1 and using integration by parts, we obtain

$$\begin{aligned} \|\nabla u_1\|^2_{L^2(\Omega)} + \alpha\int_{\partial\Omega} u_1^2 dS + b\|u_1\|^2_{L^2(\Omega)} &= \lambda\int_\Omega u_1^2u_2dx \\ &\leq \|u_2\|_{L^\infty(\Omega)}\|u_1\|^2_{L^2(\Omega)} \\ &\leq \frac{b}{2}\|u_1\|^2_{L^2(\Omega)}. \end{aligned}$$

Hence we have $u_1 = 0$. By the second equation of (3.2), we see that u_2 satisfies

$$\begin{cases} -\Delta u_2 = 0, & x \in \Omega, \\ \partial_\nu u_2 + \beta\left|\dfrac{u_2}{\lambda}\right|^{\gamma-2}u_2 = 0, & x \in \partial\Omega. \end{cases}$$

Multiplying this equation by u_2 and integration by parts, we obtain

$$\|\nabla u_2\|_{L^2(\Omega)} = 0, \quad u_2\,|_{\partial\Omega} = 0.$$

By using Poincaré's inequality, we get $u_2 = 0$. Thus $u_1 = u_2 = 0$. This contradicts the assumption $\|u\| = \frac{b}{2} > 0$. □

In order to verify condition (ii) of Lemma 2.1, we here claim the following lemma.

Lemma 3.3. *Let* $1 \le N \le 5$ *and suppose that either* (A) *or* (B) *is satisfyied :*

$$\begin{cases} \text{(A)} & \gamma = 2, \quad \alpha \le 2\beta, \\ \text{(B)} & \gamma > 2, \end{cases}$$

then there exists a constant $R(> r = \frac{b}{2})$ *such that the estimate*

$$\|u\| < R$$

holds for all $\lambda \ge 0$ *and* $u \in K$ *satisfying* $u = \Psi(u) + \lambda\varphi$.

Proof. We rewite $u = \Psi(u) + \lambda\varphi$ in terms of each component, that is:

$$\text{(3.3)} \qquad \begin{cases} -\Delta u_1 + bu_1 = u_1u_2 + \lambda(b+\lambda_1)\varphi_1, & x \in \Omega, \\ -\Delta u_2 = au_1, & x \in \Omega, \\ \partial_\nu u_1 + \alpha u_1 = \partial_\nu u_2 + \beta|u_2|^{\gamma-2}u_2 = 0, & x \in \partial\Omega. \end{cases}$$

Hereafter we denote by $C > 0$ a general constant. First, we derive H^1-estimate for u_2. Replacing u_1 in the first equation of (3.3) by $-\frac{1}{a}\Delta u_2$, we get

$$\text{(3.4)} \qquad \Delta^2 u_2 - b\Delta u_2 = -u_2\Delta u_2 + \lambda a(b+\lambda_1)\varphi_1.$$

By multipying (3.4) by φ_1 and using integration by parts, we have

$$\begin{aligned} (l.h.s) &= \int_\Omega \Delta^2 u_2\varphi_1 dx - b\int_\Omega \Delta u_2\varphi_1 dx \\ &= -\int_\Omega \nabla(\Delta u_2)\cdot\nabla\varphi_1 dx + \int_{\partial\Omega}(\partial_\nu\Delta u_2)\varphi_1 dS + b\int_\Omega \nabla u_2\cdot\nabla\varphi_1 dx - b\int_{\partial\Omega}(\partial_\nu u_2)\varphi_1 dS \\ &= -\lambda_1\int_\Omega \Delta u_2\varphi_1 dx - b\int_\Omega u_2\Delta\varphi_1 dx + b\int_{\partial\Omega} u_2(\partial_\nu\varphi_1)dS - b\int_{\partial\Omega}(\partial_\nu u_2)\varphi_1 dS \\ &= \lambda_1(b+\lambda_1)\int_\Omega u_2\varphi_1 dx + \beta(b+\lambda_1)\int_{\partial\Omega} u_2^{\gamma-1}\varphi_1 dS - \alpha(b+\lambda_1)\int_{\partial\Omega} u_2\varphi_1 dS, \end{aligned}$$

$$\begin{aligned} (r.h.s) &= -\int_\Omega u_2\Delta u_2\varphi_1 dx + \lambda a(b+\lambda_1)\|\varphi_1\|^2_{L^2(\Omega)} \\ &= \int_\Omega \nabla u_2\cdot\nabla(u_2\varphi_1)dx - \int_{\partial\Omega}(\partial_\nu u_2)u_2\varphi_1 dS + \lambda a(b+\lambda_1) \\ &= \int_\Omega |\nabla u_2|^2\varphi_1 dx + \frac{1}{2}\int_\Omega \nabla u_2^2\cdot\nabla\varphi_1 dx + \beta\int_{\partial\Omega} u_2^\gamma\varphi_1 dS + \lambda a(b+\lambda_1) \\ &= \int_\Omega |\nabla u_2|^2\varphi_1 dx - \frac{1}{2}\int_\Omega u_2^2\Delta\varphi_1 dx + \frac{1}{2}\int_{\partial\Omega} u_2^2(\partial_\nu\varphi_1)dS + \beta\int_{\partial\Omega} u_2^\gamma\varphi_1 dS + \lambda a(b+\lambda_1) \\ &= \int_\Omega |\nabla u_2|^2\varphi_1 dx + \frac{\lambda_1}{2}\int_\Omega u_2^2\varphi_1 dx + \beta\int_{\partial\Omega} u_2^\gamma\varphi_1 dS - \frac{\alpha}{2}\int_{\partial\Omega} u_2^2\varphi_1 dS + \lambda a(b+\lambda_1). \end{aligned}$$

Therefore we see that the following equality holds.

$$\text{(3.5)} \qquad \begin{aligned} \lambda_1(b+\lambda_1)\int_\Omega u_2\varphi_1 dx &= \int_\Omega |\nabla u_2|^2\varphi_1 dx + \frac{\lambda_1}{2}\int_\Omega u_2^2\varphi_1 dx + a(b+\lambda_1)\lambda \\ &\quad + \int_{\partial\Omega}\left\{\beta u_2^\gamma - \beta(b+\lambda_1)u_2^{\gamma-1} - \frac{\alpha}{2}u_2^2 + \alpha(b+\lambda_1)u_2\right\}\varphi_1 dS. \end{aligned}$$

Since (A) : $\gamma = 2$, $\alpha \leq 2\beta$ or (B) : $\gamma > 2$ holds,

$$\inf_{u_2 \geq 0} \left\{ \beta u_2^\gamma - \beta (b + \lambda_1) u_2^{\gamma-1} - \frac{\alpha}{2} u_2^2 + \alpha (b + \lambda_1) u_2 \right\} \geq -C > -\infty.$$

Moreover, since φ_1 is bounded (Lemma 2.2), we see that

$$\lambda_1(b + \lambda_1) \int_\Omega u_2 \varphi_1 dx \geq \int_\Omega |\nabla u_2|^2 \varphi_1 dx + \frac{\lambda_1}{2} \int_\Omega u_2^2 \varphi_1 dx + a(b + \lambda_1)\lambda - C.$$

By Schwarz's inequality and Young's inequality, it is easy to see that

$$\begin{aligned} \int_\Omega |\nabla u_2|^2 \varphi_1 dx + \frac{\lambda_1}{2} \int_\Omega u_2^2 \varphi_1 dx + a(b + \lambda_1)\lambda &\leq \lambda_1(b + \lambda_1) \int_\Omega u_2 \varphi_1 dx + C \\ &\leq \lambda_1(b + \lambda_1) \left(\int_\Omega u_2^2 \varphi_1 dx \right)^{\frac{1}{2}} \left(\int_\Omega \varphi_1 dx \right)^{\frac{1}{2}} + C \\ &\leq \frac{\lambda_1}{4} \int_\Omega u_2^2 \varphi_1 dx + C. \end{aligned}$$

Hence we obtain

$$\int_\Omega |\nabla u_2|^2 \varphi_1 dx \leq C, \qquad \int_\Omega u_2^2 \varphi_1 dx \leq C, \qquad \lambda \leq C, \tag{3.6}$$

and

$$\int_\Omega u_2 \varphi_1 dx \leq \left(\int_\Omega u_2^2 \varphi_1 dx \right)^{\frac{1}{2}} \left(\int_\Omega \varphi_1 dx \right)^{\frac{1}{2}} \leq C. \tag{3.7}$$

Furthermore it follows from Lemma 2.2 and (3.6)

$$\|u_2\|_{H^1(\Omega)} \leq C. \tag{3.8}$$

By (3.7) and (3.5), we also have

$$\int_{\partial\Omega} \left\{ \beta u_2^\gamma - \beta (b + \lambda_1) u_2^{\gamma-1} - \frac{\alpha}{2} u_2^2 + \alpha (b + \lambda_1) u_2 \right\} \varphi_1 dS \leq C.$$

Similarly, using Hölder's inequality and Young's inequality, we obtain

$$\begin{cases} \displaystyle\int_{\partial\Omega} u_2^\gamma dS \leq C, & (\gamma > 2 \ \text{ or } \ \gamma = 2, \ \alpha < 2\beta) \\ \displaystyle\int_{\partial\Omega} u_2 dS \leq C. & (\gamma = 2, \ \alpha = 2\beta) \end{cases} \tag{3.9}$$

Now, we derive H^1-estimate for u_1. Multipying the first equation of (3.3) by φ_1 and using integration by parts, we get

$$(\lambda_1 + b) \int_\Omega u_1 \varphi_1 dx = \int_\Omega u_1 u_2 \varphi_1 dx + \lambda(\lambda_1 + b) \tag{3.10}$$

Similarly, multipying the second equation of (3.3) by φ_1, we get

$$\lambda_1 \int_\Omega u_2 \varphi_1 dx + \beta \int_{\partial\Omega} u_2^{\gamma-1} \varphi_1 dS - \alpha \int_{\partial\Omega} u_2 \varphi_1 dS = a \int_\Omega u_1 \varphi_1 dx. \tag{3.11}$$

Then by (3.10), (3.11), (3.8) and (3.9), we obtain

$$\int_\Omega u_1\varphi_1 dx \le C, \qquad \int_\Omega u_1u_2\varphi_1 dx \le C. \tag{3.12}$$

We first suppose that $N = 3, 4, 5$ and let $\theta = \frac{6-N}{4} \in (0,1)$. Multiplying the frist equation of (3.3) by u_1 and using integration by parts, (3.9), Hölder's inequality and Sobolev's inequality, we obtain

$$\begin{aligned}
\|\nabla u_1\|^2_{L^2(\Omega)} + \alpha\int_{\partial\Omega} u_1^2 ds + b\|u_1\|^2_{L^2(\Omega)} &= \int_\Omega u_1^2u_2 dx + \lambda(b+\lambda_1)\int_\Omega u_1\varphi_1 dx \\
&\le \int_\Omega (u_1u_2)^\theta \left(u_1^{\frac{2-\theta}{1-\theta}}u_2\right)^{1-\theta} dx + C \\
&\le \left(\int_\Omega u_1u_2 dx\right)^\theta \left(\int_\Omega u_1^{\frac{2-\theta}{1-\theta}}u_2 dx\right)^{1-\theta} + C \\
&\le C\left(\int_\Omega u_1^{\frac{N+2}{N-2}}u_2 dx\right)^{\frac{N-2}{4}} + C \\
&\le C\|u_1\|^{\frac{N+2}{4}}_{L^{2^*}(\Omega)}\|u_2\|^{\frac{N-2}{4}}_{L^{2^*}(\Omega)} + C \\
&\le C\|u_1\|^{\frac{N+2}{4}}_{H^1(\Omega)} + C,
\end{aligned}$$

where $2^* = \frac{2N}{N-2}$ is critical Sobolev exponent. Since $N \in [3,5]$, we have $\frac{N+2}{4} < 2$. Hence we obtain

$$\|u_1\|_{H^1(\Omega)} \le C. \tag{3.13}$$

Finally, we derive L^∞-estimates for u_1 and u_2. Since (3.13), we know

$$\|u_1\|_{L^{2^*}(\Omega)} \le C.$$

From the second equation of (3.3) and the elliptic estimate , we have

$$\|u_2\|_{W^{2,2^*}(\Omega)} \le C.$$

Since $N \in [3,5]$, we have

$$2^* - 2 = \frac{4N}{N-2} > N.$$

Hence, Sobolev imbedding theorem gives

$$\|u_2\|_{L^\infty(\Omega)} \le C_2.$$

Similarly, we can get $\|u_1\|_{L^\infty} \le C_1$ from the first equation of (3.3). As for the cases $N = 1$, 2, we can show this result by slight modification and omit the details here. Choosing $R > C_1 + C_2$, we can see that the conclusion of this lemma holds. □

Proof of Theorem 3.1. By applying Lemma 3.2, Lemma 3.3 and Lemma 2.1, we can verify that Theorem 3.1 holds. □

Remark 3.4. If $\alpha = 0$, for $\gamma \in (1,2)$ we can derive H^1-estimate for u_2 by taking H^1 norm of u_2 as $\|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^1(\partial\Omega)}$ in the proof of Lemma 3.3. In fact, it is easy to see that this norm is equivalent to the usual $H^1(\Omega)$ norm by Lemma 2.3. Therefore it is easy to see that Theorem 3.1 holds in the case of $\alpha = 0$, $\beta > 0$ and $\gamma > 1$.

3.2 Ordered Uniqueness

Theorem 3.5. *Let* (u_1, u_2) *and* (v_1, v_2) *be two positive solutions of* (S-NR) *satisfying* $u_1 \le v_1$ *or* $u_2 \le v_2$. *Then* $u_1 \equiv v_1$ *and* $u_2 \equiv v_2$.

Proof. Suppose that $u_1 \not\equiv v_1$ or $u_2 \not\equiv v_2$. Without loss of generality, we only have to consider the case where $u_2 \not\equiv v_2$ and $u_2 \le v_2$. In fact, if $u_1 \le v_1$, by the second equation of (S-NR) we have

$$-\Delta(u_2 - v_2) = a(u_1 - v_1) \le 0. \tag{3.14}$$

Multipying (3.14) by $[u_2 - v_2]^+ := \max\{u_2 - v_2, 0\}$ and using integration by parts, we obtain

$$\|\nabla[u_2 - v_2]^+\|^2_{L^2(\Omega)} + \beta \int_{\partial\Omega} [u_2 - v_2]^+ \left(|u_2|^{\gamma-2}u_2 - |v_2|^{\gamma-2}v_2\right) dS \le 0. \tag{3.15}$$

By Lemma 2.4,

$$\begin{aligned}\int_{\partial\Omega} [u_2 - v_2]^+ \left(|u_2|^{\gamma-2}u_2 - |v_2|^{\gamma-2}v_2\right) dS &= \int_{\{u_2 \ge v_2\}} (u_2 - v_2)\left(|u_2|^{\gamma-2}u_2 - |v_2|^{\gamma-2}v_2\right) dS \\ &\ge \int_{\{u_2 \ge v_2\}} C_\gamma (u_2 - v_2)^\gamma dS \\ &= C_\gamma \int_{\partial\Omega} \left([u_2 - v_2]^+\right)^\gamma dS.\end{aligned}$$

By this inequality and (3.15),

$$\|\nabla[u_2 - v_2]^+\|^2_{L^2(\Omega)} + C_\gamma \int_{\partial\Omega} \left([u_2 - v_2]^+\right)^\gamma dS \le 0.$$

Therefore we get

$$\nabla[u_2 - v_2]^+ = 0,$$
$$[u_2 - v_2]^+\big|_{\partial\Omega} = 0.$$

Hence we deduce $[u_2 - v_2]^+ \equiv 0$, i.e., $u_2 \le v_2$. Next we consider the following eigenvalue problems:

$$\begin{cases} -\Delta w + (b - u_2(x))\, w = \mu' w & \text{in } \Omega, \\ \partial_\nu w + \alpha w = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.16}$$

and

$$\begin{cases} -\Delta w + (b - v_2(x))\, w = \eta' w & \text{in } \Omega, \\ \partial_\nu w + \alpha w = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.17}$$

If necessary, we take some nonnegative constant $L \ge 0$ and add both sides of equations of (3.16) and (3.17) by L, and we can assume $U(x) := b - u_2(x) + L \ge 1$ and $V(x) := b - v_2(x) + L \ge 1$. Thus we consider the following problems in stead of (3.16) and (3.17):

$$\begin{cases} -\Delta w + U(x) w = \mu w & \text{in } \Omega, \\ \partial_\nu w + \alpha w = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.18}$$

and

$$\text{(3.19)} \qquad \begin{cases} -\Delta w + V(x)w = \eta w & \text{in } \Omega, \\ \partial_\nu w + \alpha w = 0 & \text{on } \partial\Omega. \end{cases}$$

By applying the compactness argument for the associate Rayleigh's quotients of (3.18) and (3.19) , we know that the smallest positive eigenvalues of (3.18) and (3.19) are atteined and we denote them by μ_0 and η_0. Moreover, thanks to $u_2 \not\equiv v_2$ and $u_2 \le v_2$, we see that $\eta_0 < \mu_0$. On the other hand, since (u_1, u_2) and (v_1, v_2) are positive stationary solutions for (S-NR), $u_1 > 0$ and $v_1 > 0$ satisfy

$$\begin{cases} -\Delta u_1 + (b - u_2(x) + L)\, u_1 = Lu_1 & \text{in } \Omega, \\ \partial_\nu u_1 + \alpha u_1 = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\Delta v_1 + (b - v_2(x) + L)\, v_1 = Lv_1 & \text{in } \Omega, \\ \partial_\nu v_1 + \alpha v_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

By the fact that the eigenvalue corresponding positive eigenfunction is the smallest one, we deduce $\mu_0 = L = \eta_0$. This is in contradiction with $\eta_0 < \mu_0$. Thus the proof is completed. □

4 Nonstationary Problem

In this section, we investigate the large time behavior of solutions to (NR) and prove that the positive stationary solution plays a role of threshold to classify initial data into two groups; namely corresponding solutions of (NR) blow up in finite time or exist globally.

4.1 Local Well-posedness

First we state the local well-posedness of problem (NR).

Theorem 4.1. *Assume* $(u_{10}, u_{20}) \in L^\infty(\Omega) \times L^\infty(\Omega)$. *Then there exists* $T > 0$ *such that* (NR) *possesses a unique soltion* $(u_1, u_2) \in (L^\infty(0, T; L^\infty(\Omega)) \cap C([0, T); L^2(\Omega)))^2$ *satisfying*

$$\text{(4.1)} \qquad \sqrt{t}\partial_t u_1, \sqrt{t}\partial_t u_2, \sqrt{t}\Delta u_1, \sqrt{t}\Delta u_2 \in L^2(0, T; L^2(\Omega)).$$

Furthermore, if the initial data is nonnegative, then the local solution (u_1, u_2) *for* (NR) *is nonnegative.*

Proof. It is easy to see that (NR) has a unique local solution by the standard abstract theory [1] and L^∞-energy method [15]. In fact, we consider the following approximate problem:

$$\text{(4.2)} \qquad \begin{cases} \partial_t u_1 - \Delta u_1 = [u_1]_M [u_2]_M - b u_1, & x \in \Omega,\ t > 0, \\ \partial_t u_2 - \Delta u_2 = a u_1, & x \in \Omega,\ t > 0, \\ \partial_\nu u_1 + \alpha u_1 = \partial_\nu u_2 + \beta |u_2|^{\gamma-2} u_2 = 0, & x \in \partial\Omega,\ t > 0, \\ u_1(x,0) = u_{10}(x),\ u_2(x,0) = u_{20}(x), & x \in \Omega, \end{cases}$$

where, $M > 0$ is a given constant and cut-off function $[u]_M$ is defined by

$$[u]_M = \begin{cases} M, & u \ge M, \\ u, & |u| \le M, \\ -M, & u \le -M. \end{cases}$$

Since $u \mapsto [u]_M$ is Lipschitz continuous from $L^2(\Omega)$ into itself, it is well known that (4.2) has a unique global solution (u_1, u_2) satisfying (4.1) by applying the abstract theory on maximal monotone operators developed by Brézis [1].

By multipying the first equation of (4.2) by $|u_1|^{r-2}u_1$ and using integration by parts,

$$\frac{1}{r}\frac{d}{dt}\|u_1(t)\|_{L^r}^r + (r-1)\int_\Omega |\nabla u_1|^2 u_1^{r-2}dx + \alpha\int_{\partial\Omega} u_1^r dS = \int_\Omega u_1^{r-1}[u_1]_M[u_2]_M dx - b\int_\Omega u_1^r dx.$$

Hence

$$\frac{1}{r}\frac{d}{dt}\|u_1(t)\|_{L^r}^r \le \|u_2(t)\|_{L^\infty}\|u_1(t)\|_{L^r}^r.$$

Divide both sides by $\|u_1\|_{L^r}^{r-1}$ and integrate with respect to t on $[0,t]$, then we get

$$\|u_1(t)\|_{L^r} \le \|u_{10}\|_{L^r} + \int_0^t \|u_1(\tau)\|_{L^r}\|u_2(\tau)\|_{L^\infty}d\tau.$$

Letting r tend to ∞ (Lemma 2.4), we derive

$$\|u_1(t)\|_{L^\infty} \le \|u_{10}\|_{L^\infty} + \int_0^t \|u_1(\tau)\|_{L^\infty}\|u_2(\tau)\|_{L^\infty}d\tau.$$

Similarly, we can get the following L^∞ estimate for u_2 ;

$$\|u_2(t)\|_{L^\infty} \le \|u_{20}\|_{L^\infty} + \int_0^t a\|u_1(\tau)\|_{L^\infty}d\tau.$$

Therefore setting $y(t) = \|u_1(t)\|_{L^\infty(\Omega)} + \|u_2(t)\|_{L^\infty(\Omega)}$, we get

$$y(t) \le y(0) + \int_0^t \left(y^2(\tau) + ay(\tau)\right) d\tau.$$

Thus applying Lemma 2.5, we find that there exists a number $T > 0$ depending only on $\|u_{10}\|_{L^\infty(\Omega)}$ and $\|u_{20}\|_{L^\infty(\Omega)}$ such that

$$y(t) \le y(0) + 1 \qquad \text{a.e. } t \in [0,T].$$

In other words, we get

$$\|u_1(t)\|_{L^\infty(\Omega)} + \|u_2(t)\|_{L^\infty(\Omega)} \le \|u_{10}\|_{L^\infty(\Omega)} + \|u_{20}\|_{L^\infty(\Omega)} + 1 \qquad \text{a.e. } t \in [0,T].$$

Hence choosing $M > \|u_{10}\|_{L^\infty(\Omega)} + \|u_{20}\|_{L^\infty(\Omega)} + 1$, we can see that (u_1, u_2) gives a solution for (NR) on $[0,T]$ by the definition of cut-off function $[u]_M$.

To get the regularity estimate and the uniqueness of the solution for (NR) is easy and usual, so we omit the details. In order to prove that the solution for (NR) is nonnegative, we consider the following equations:

$$\text{(abs-NR)} \quad \begin{cases} \partial_t u_1 - \Delta u_1 = |u_1||u_2| - bu_1, & x \in \Omega,\ t > 0, \\ \partial_t u_2 - \Delta u_2 = au_1, & x \in \Omega,\ t > 0, \\ \partial_\nu u_1 + \alpha u_1 = \partial_\nu u_2 + \beta|u_2|^{\gamma-2}u_2 = 0, & x \in \partial\Omega,\ t > 0, \\ u_1(x,0) = u_{10}(x) \ge 0,\ u_2(x,0) = u_{20}(x) \ge 0, & x \in \Omega. \end{cases}$$

Just as before, we see that (abs-NR) has a unique local solution. Furthermore, multiplying the equations of (abs-NR) by $u_1^- := \max\{-u_1, 0\}$ and $u_2^- := \max\{-u_2, 0\}$ respectively, we get $u_1 \ge 0$ and $u_2 \ge 0$. Thus, we deduce from the uniqueness of the solution for (NR) that the solution u_1, u_2 for (NR) is nonnegative. □

4.2 Threshold Property

Finally, we study the threshold property and prove that every positive stationary solution for (NR) gives a threshold in the following sense.

Theorem 4.2. *Let $(\overline{u}_1, \overline{u}_2)$ be a positive stationary solution of* (NR)*, then the followings hold.*

(1) *Let $0 \le u_{10}(x) \le \overline{u}_1(x)$, $0 \le u_{20}(x) \le \overline{u}_2(x)$, then the solution (u_1, u_2) of* (NR) *exists globaly. In addition, if $0 \le u_{10}(x) \le l_1\overline{u}_1(x)$, $0 \le u_{20}(x) \le l_2\overline{u}_2(x)$ for some $0 < l_1 < l_2 \le 1$, then*

$$\lim_{t\to+\infty} (u_1(x,t), u_2(x,t)) = (0,0), \quad \text{pointwisely on } \overline{\Omega}.$$

(2) *Assume further $\alpha \le 2\beta$ and let $u_{10}(x) \ge l_1\overline{u}_1(x)$, $u_{20}(x) \ge l_2\overline{u}_2(x)$ for some $l_1 > l_2 > 1$, then the solution (u_1, u_2) of* (NR) *blows up in finite time.*

We first prove the following comparison theorem for the proof of Theorem 4.2.

Lemma 4.3 (Comparison theorem)**.** *If (u_{10}, u_{20}), (v_{10}, v_{20}) are two initial data for* (NR) *satisfying*

$$0 \le u_{10} \le v_{10}, \quad 0 \le u_{20} \le v_{20} \qquad \text{on } \overline{\Omega},$$

then the corresponding soltions (u_1, u_2), (v_1, v_2) remain in the initial data order in time interval where the solutions exist, i.e., $u_1(x,t) \le v_1(x,t)$ and $u_2(x,t) \le v_2(x,t)$ a.e. $x \in \Omega$ as long as (u_1, u_2) and (v_1, v_2) exist.

Proof. Let $w_1 = u_1 - v_1$, $w_2 = u_2 - v_2$. By (NR) we have

$$\text{(4.3)} \qquad \begin{cases} \partial_t w_1 - \Delta w_1 = w_1 u_2 + v_1 w_2 - b w_1, & x \in \Omega,\ t \in (0, T_m), \\ \partial_t w_2 - \Delta w_2 = a w_1, & x \in \Omega,\ t \in (0, T_m), \\ \partial_\nu w_1 + \alpha w_1 = \partial_\nu w_2 + \beta\left(|u_2|^{\gamma-2}u_2 - |v_2|^{\gamma-2}v_2\right) = 0, & x \in \partial\Omega,\ t \in (0, T_m), \\ w_1(x,0) \le 0, \quad w_2(x,0) \le 0, & x \in \overline{\Omega}, \end{cases}$$

where $T_m > 0$ is the maximum existence time for (u_1, u_2) and (v_1, v_2). We set

$$w^+ = w \vee 0, \quad w^- = (-w) \vee 0,$$

where $a \vee b = \max\{a, b\}$. It is easy to see that w^+, $w^- \ge 0$ and

$$w = w^+ - w^-, \quad |w| = w^+ + w^-.$$

Multiplying the first equation of (4.3) by w_1^+, we get

$$\int_\Omega \partial_t w_1 w_1^+ dx - \int_\Omega \Delta w_1 w_1^+ dx = \int_\Omega w_1 u_2 w_1^+ dx + \int_\Omega v_1 w_2 w_1^+ dx - b \int_\Omega w_1 w_1^+ dx.$$

Here, we see that

$$\int_\Omega \partial_t w_1 w_1^+ dx = \int_{\{w_1 \ge 0\}} \partial_t w_1 w_1 dx = \frac{1}{2}\frac{d}{dt}\int_{\{w_1 \ge 0\}} w_1^2 dx = \frac{1}{2}\frac{d}{dt}\int_\Omega \left(w_1^+\right)^2 dx.$$

Similarly,

$$\begin{aligned} -\int_\Omega \Delta w_1 w_1^+ dx &= \int_\Omega \nabla w_1 \cdot \nabla w_1^+ dx + \alpha \int_{\partial\Omega} w_1 w_1^+ dS \\ &= \int_{\{w_1 \ge 0\}} |\nabla w_1|^2 dx + \alpha \int_{\{w_1 \ge 0\}} w_1^2 dS = \int_\Omega |\nabla w_1^+|^2 dx + \alpha \int_{\partial\Omega} \left(w_1^+\right)^2 dS. \end{aligned}$$

Hence noting that $v_1 \geq 0$, we obtain for any $T \in (0, T_m)$

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega \left(w_1^+\right)^2 dx + \int_\Omega |\nabla w_1^+|^2 dx + \alpha \int_{\partial\Omega} \left(w_1^+\right)^2 dS &= \int_\Omega w_1 u_2 w_1^+ dx + \int_\Omega v_1 w_2 w_1^+ dx - b\int_\Omega w_1 w_1^+ dx \\
&= \int_\Omega \left(w_1^+ - w_1^-\right) u_2 w_1^+ dx \\
&\quad + \int_\Omega v_1 \left(w_2^+ - w_2^-\right) w_1^+ dx - b\int_\Omega \left(w_1^+\right)^2 dx \\
&\leq \|u_2\|_{L_T^\infty L^\infty} \int_\Omega \left(w_1^+\right)^2 dx \\
&\quad + \|v_1\|_{L_T^\infty L^\infty} \int_\Omega w_1^+ w_2^+ dx \\
&\leq C\left(\|w_1^+(t)\|_{L^2(\Omega)}^2 + \|w_2^+(t)\|_{L^2(\Omega)}^2\right),
\end{aligned}$$

where $L_T^\infty L^\infty := L^\infty(0, T; L^\infty(\Omega))$. Hence we get

$$\frac{1}{2}\frac{d}{dt}\|w_1^+(t)\|_{L^2(\Omega)}^2 \leq C\left(\|w_1^+(t)\|_{L^2(\Omega)}^2 + \|w_2^+(t)\|_{L^2(\Omega)}^2\right). \tag{4.4}$$

Next we do the same calculation for the second equation of (4.3). We also have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left(w_2^+\right)^2 dx + \int_\Omega |\nabla w_2^+|^2 dx - \int_{\partial\Omega} (\partial_\nu w_2) w_2^+ dS \leq \frac{a}{2}\left(\|w_1^+(t)\|_{L^2(\Omega)}^2 + \|w_2^+(t)\|_{L^2(\Omega)}^2\right),$$

and

$$\begin{aligned}
-\int_{\partial\Omega} (\partial_\nu w_2) w_2^+ dS &= \beta \int_{\partial\Omega} \left(|u_2|^{\gamma-2}u_2 - |v_2|^{\gamma-2}v_2\right) w_2^+ dS \\
&= \beta \int_{\{u_2 \geq v_2\}} \left(|u_2|^{\gamma-2}u_2 - |v_2|^{\gamma-2}v_2\right)(u_2 - v_2)\, dS \geq 0.
\end{aligned}$$

Therefore

$$\frac{1}{2}\frac{d}{dt}\|w_2^+(t)\|_{L^2(\Omega)}^2 \leq \frac{a}{2}\left(\|w_1^+(t)\|_{L^2(\Omega)}^2 + \|w_2^+(t)\|_{L^2(\Omega)}^2\right). \tag{4.5}$$

Thus by (4.4), (4.5) and Gronwall's inequality, we get

$$\|w_1^+(t)\|_{L^2(\Omega)}^2 + \|w_2^+(t)\|_{L^2(\Omega)}^2 \leq \left(\|w_1^+(0)\|_{L^2(\Omega)}^2 + \|w_2^+(0)\|_{L^2(\Omega)}^2\right) e^{Ct}, \qquad \forall\, t \in [0, T_m).$$

Since $w_1^+(0) = w_2^+(0) = 0$, the above inequality means $w_1^+ = w_2^+ = 0$. Hence, we have the desired result. □

Proof of Theorem 4.2. (1) If $0 \leq u_{10} \leq \overline{u}_1$ and $0 \leq u_{20} \leq \overline{u}_2$, then since $(\overline{u}_1, \overline{u}_2)$ is a global solution for (NR), $0 \leq u_1(x,t) \leq \overline{u}_1(x)$ and $0 \leq u_2(x,t) \leq \overline{u}_2(x)$ follow directly from Lemma 4.3. That is, we have

$$\sup_{t \in [0,T)} \|u_i(\cdot, t)\|_{L^\infty(\Omega)} \leq \|\overline{u}_i\|_{L^\infty(\Omega)}. \quad (i = 1, 2)$$

Hence the solution (u_1, u_2) exists globally.

In addition, let $u_{10}(x) \leq l_1\overline{u}_1(x)$, $u_{20}(x) \leq l_2\overline{u}_2(x)$ for some $0 < l_1 < l_2 \leq 1$. Since the comparison theorem holds, without loss of generality, we can assume that $u_{10}(x) = l_1\overline{u}_1(x)$, $u_{20}(x) = l_2\overline{u}_2(x)$ and $l_1 < l_2 \leq 1$. We consider $\delta u_1 := u_1(t+h) - u_1(t)$ and $\delta u_2 := u_2(t+h) - u_2(t)$ for $h > 0$ and get the following equations from (NR).

$$
(4.6)\qquad \begin{cases} \partial_t(\delta u_1) - \Delta(\delta u_1) = (\delta u_1)\, u_2(t+h) + u_1(t)\,(\delta u_2) - b\,(\delta u_1), \\ \partial_t(\delta u_2) - \Delta(\delta u_2) = a\,(\delta u_1), \\ \partial_\nu(\delta u_1) + \alpha\,(\delta u_1) = \partial_\nu(\delta u_2) + \beta\left(|u_2(t+h)|^{\gamma-2}u_2(t+h) - |u_2(t)|^{\gamma-2}u_2(t)\right) = 0, \\ \delta u_1(0) = u_1(0+h) - u_1(0), \quad \delta u_2(0) = u_2(0+h) - u_2(0). \end{cases}
$$

Multipying the first and second equation of (4.6) by $[\delta u_1]^+$ and $[\delta u_2]^+$ respectively and using integration by parts and repeating the same argument as for (4.4), we obtain the following inequality:

$$
\|[\delta u_1]^+\|^2_{L^2(\Omega)} + \|[\delta u_2]^+\|^2_{L^2(\Omega)} \le \left(\|[\delta u_1(0)]^+\|^2_{L^2(\Omega)} + \|[\delta u_2(0)]^+\|^2_{L^2(\Omega)}\right) e^{Ct}
$$

We divide both sides of this inequality by h^2:

$$
\left\|\left[\frac{\delta u_1}{h}\right]^+\right\|^2_{L^2(\Omega)} + \left\|\left[\frac{\delta u_2}{h}\right]^+\right\|^2_{L^2(\Omega)} \le \left(\left\|\left[\frac{\delta u_1(0)}{h}\right]^+\right\|^2_{L^2(\Omega)} + \left\|\left[\frac{\delta u_2(0)}{h}\right]^+\right\|^2_{L^2(\Omega)}\right) e^{Ct}.
$$

Since we know that u_1, u_2 is differentiable on *a.e.* t by the regularity results of Theorem 4.1, by letting $h \searrow 0$, we obtain

$$
\|[\partial_t u_1]^+\|^2_{L^2(\Omega)} + \|[\partial_t u_2]^+\|^2_{L^2(\Omega)} \le \left(\|[\partial_t u_1(0)]^+\|^2_{L^2(\Omega)} + \|[\partial_t u_2(0)]^+\|^2_{L^2(\Omega)}\right) e^{Ct}.
$$

We here note that since $(l_1\overline{u}_1, l_2\overline{u}_2)$ is strict upper solution for (S-NR), it holds that

$$
\begin{aligned}
\partial_t u_1(0) &= \Delta u_{10} + u_{10}u_{20} - b u_{10} \\
&= l_1 \Delta \overline{u}_1 + l_1 l_2 \overline{u}_1 \overline{u}_2 - b l_1 \overline{u}_1 \\
&\le l_1\left(\Delta \overline{u}_1 + \overline{u}_1\overline{u}_2 - b\overline{u}_1\right) = 0, \\
\partial_t u_2(0) &= \Delta u_{20} + a u_{10} \\
&= l_2 \Delta \overline{u}_2 + a l_1 \overline{u}_1 \\
&< l_2\left(\Delta \overline{u}_2 + a\overline{u}_1\right) = 0,
\end{aligned}
$$

which imply that $[\partial_t u_1(0)]^+ = [\partial_t u_2(0)]^+ = 0$. Hence we find that $\partial_t u_1 \le 0$ and $\partial_t u_2 \le 0$, i.e., $u_1(x,t)$ and $u_2(x,t)$ are monotone decreasing in t for a.e. $x \in \Omega$. Thus

$$
\lim_{t\to\infty} (u_1(x,t), u_2(x,t)) =: (\tilde{u}_1(x), \tilde{u}_2(x))
$$

exists and $(\tilde{u}_1, \tilde{u}_2)$ is a nonnegative stationary solution of (NR) satisfying $(0,0) \le (\tilde{u}_1, \tilde{u}_2) \le (l_1\overline{u}_1, l_2\overline{u}_2) < (\overline{u}_1, \overline{u}_2)$. By the ordered uniqueness of positive stationary solutions (Lemma 3.5), $(\tilde{u}_1(x), \tilde{u}_2(x))$ is nothing but $(0,0)$.

(2) Let $\gamma = 2$ and $\alpha \le 2\beta$. By the comparison theorem, we can assume without loss of generality that $u_{10}(x) = l_1\overline{u}_1(x)$, $u_{20}(x) = l_2\overline{u}_2(x)$ for some $l_1 > l_2 > 1$. Suppose that the solution (u_1, u_2) for (NR) exists globally, i.e.,

$$
(4.7)\qquad \sup_{t\in[0,T]} \|u_i(\cdot,t)\|_{L^\infty(\Omega)} < \infty, \qquad (i=1,2) \quad \forall\, T > 0.
$$

Now we are going to constract a subsolution. For this purpose, we first note that there exists a sufficiently small number $\varepsilon > 0$ such that

$$\begin{cases} a(l_2 - l_1)\overline{u}_1 + \varepsilon l_2 \overline{u}_2 < 0 & \text{on } \overline{\Omega}, \\ \varepsilon + (1 - l_2)\overline{u}_2 < 0 & \text{on } \overline{\Omega}. \end{cases} \tag{4.8}$$

Here we used the fact that $\overline{u}_1(x) > 0$, $\overline{u}_2(x) > 0$ on $\overline{\Omega}$, which is assured by Hopf's type maximum principle. Let $u_1^*(x,t) = l_1 e^{\varepsilon t}\overline{u}_1(x)$ and $u_2^*(x,t) = l_2 e^{\varepsilon t}\overline{u}_2(x)$. Then using (4.8), we get

$$\begin{aligned} \partial_t u_1^* - \Delta u_1^* - u_1^* u_2^* + b u_1^* &= \varepsilon l_1 e^{\varepsilon t}\overline{u}_1 - l_1 e^{\varepsilon t}\Delta \overline{u}_1 - l_1 e^{\varepsilon t}\overline{u}_1 l_2 e^{\varepsilon t}\overline{u}_2 + b l_1 e^{\varepsilon t}\overline{u}_1 \\ &= \varepsilon l_1 e^{\varepsilon t}\overline{u}_1 + l_1 e^{\varepsilon t}\left(\overline{u}_1\overline{u}_2 - b\overline{u}_1\right) - l_1 e^{\varepsilon t}\overline{u}_1 l_2 e^{\varepsilon t}\overline{u}_2 + b l_1 e^{\varepsilon t}\overline{u}_1 \\ &\le \varepsilon l_1 e^{\varepsilon t}\overline{u}_1 + l_1 e^{\varepsilon t}\overline{u}_1\overline{u}_2 - l_1 l_2 e^{\varepsilon t}\overline{u}_1\overline{u}_2 = \{\varepsilon + (1 - l_2)\,\overline{u}_2\}\, l_1 e^{\varepsilon t}\overline{u}_1 < 0, \\ \partial_t u_2^* - \Delta u_2^* - a u_1^* &= \varepsilon l_2 e^{\varepsilon t}\overline{u}_2 - l_2 e^{\varepsilon t}\Delta\overline{u}_2 - a l_1 e^{\varepsilon t}\overline{u}_1 \\ &= \varepsilon l_2 e^{\varepsilon t}\overline{u}_2 + l_2 e^{\varepsilon t} a\overline{u}_1 - a l_1 e^{\varepsilon t}\overline{u}_1 \\ &= \{\varepsilon l_2 \overline{u}_2 + a\,(l_2 - l_1)\,\overline{u}_1\}\, e^{\varepsilon t} < 0. \end{aligned}$$

Moreover $\partial_\nu u_1^* + \alpha u_1^* = 0$, $\partial_\nu u_2^* + \beta u_2^* = 0$ on $\partial\Omega$ and $u_1^*(x,0) = l_1\overline{u}_1(x)$, $u_2^*(x,0) = l_2\overline{u}_2(x)$. Hence by the comparison principle, we have

$$u_1^*(x,t) \le u_1(x,t), \quad u_2^*(x,t) \le u_2(x,t). \tag{4.9}$$

Multiplication of equations in (NR) by φ_1 and integration by parts yield

$$\frac{d}{dt}\left(\int_\Omega u_1\varphi_1 dx\right) + (b + \lambda_1)\int_\Omega u_1\varphi_1 dx = \int_\Omega u_1 u_2 \varphi_1 dx, \tag{4.10}$$

$$\frac{d}{dt}\left(\int_\Omega u_2\varphi_1 dx\right) + \lambda_1 \int_\Omega u_2\varphi_1 dx + (\beta - \alpha)\int_{\partial\Omega} u_2\varphi_1 dS = a\int_\Omega u_1\varphi_1 dx, \tag{4.11}$$

where λ_1 and φ_1 are the first eigenvalue and the corresponding eigenfunction for (3.1). We here normalize φ_1 so that $\|\varphi_1\|_{L^1(\Omega)} = 1$. Substituting (4.11) and $u_1 = \frac{1}{a}(\partial_t u_2 - \Delta u_2)$ in (4.10) and using integration by parts, we get

$$\begin{aligned} &\frac{d}{dt}\left\{\frac{d}{dt}\left(\int_\Omega u_2\varphi_1 dx\right) + \lambda_1\int_\Omega u_2\varphi_1 dx + (\beta - \alpha)\int_{\partial\Omega} u_2\varphi_1 dS\right\} \\ &\quad + (b + \lambda_1)\left\{\frac{d}{dt}\left(\int_\Omega u_2\varphi_1 dx\right) + \lambda_1\int_\Omega u_2\varphi_1 dx + (\beta - \alpha)\int_{\partial\Omega} u_2\varphi_1 dS\right\} \\ &= \frac{1}{2}\frac{d}{dt}\int_\Omega u_2^2\varphi_1 dx + \int_\Omega |\nabla u_2|^2\varphi_1 dx + \frac{\lambda_1}{2}\int_\Omega u_2^2\varphi_1 dx + \left(\beta - \frac{\alpha}{2}\right)\int_{\partial\Omega} u_2^2\varphi_1 dS, \end{aligned} \tag{4.12}$$

where we note that

$$\begin{aligned} -\int_\Omega (\Delta u_2) u_2\varphi_1 dx &= \int_\Omega \nabla u_2 \cdot \nabla(u_2\varphi_1) dx - \int_{\partial\Omega} (\partial_\nu u_2) u_2 \varphi_1 dS \\ &= \int_\Omega |\nabla u_2|^2\varphi_1 dx + \int_\Omega u_2 \nabla u_2 \cdot \nabla\varphi_1 dx + \beta\int_{\partial\Omega} u_2^2\varphi_1 dS \\ &= \int_\Omega |\nabla u_2|^2\varphi_1 dx + \frac{1}{2}\int_\Omega \nabla u_2^2 \cdot \nabla\varphi_1 dx + \beta\int_{\partial\Omega} u_2^2\varphi_1 dS \\ &= \int_\Omega |\nabla u_2|^2\varphi_1 dx + \frac{\lambda_1}{2}\int_\Omega u_2^2\varphi_1 dx - \frac{\alpha}{2}\int_{\partial\Omega} u_2^2\varphi_1 dS + \beta\int_{\partial\Omega} u_2^2\varphi_1 dS. \end{aligned}$$

Here we assume $\beta - \alpha > 0$. (For the case $\beta - \alpha \leq 0$, we can prove the same result by the slight modification.) From (4.9), it holds that

$$\begin{aligned}\frac{\lambda_1}{2}\int_\Omega u_2^2\varphi_1 dx - (b+\lambda_1)\lambda_1\int_\Omega u_2\varphi_1 dx &= \frac{\lambda_1}{4}\int_\Omega u_2^2\varphi_1 dx + \lambda_1\int_\Omega\left\{\frac{1}{4}u_2 - (b+\lambda_1)\right\}u_2\varphi_1 dx\\ &\geq \frac{\lambda_1}{4}\int_\Omega u_2^2\varphi_1 dx + \lambda_1\int_\Omega\left\{\frac{1}{4}u_2^* - (b+\lambda_1)\right\}u_2\varphi_1 dx\\ &\geq \frac{\lambda_1}{4}\int_\Omega u_2^2\varphi_1 dx + \lambda_1\int_\Omega\left\{\frac{1}{4}me^{\varepsilon t} - (b+\lambda_1)\right\}u_2\varphi_1 dx,\end{aligned}$$

where $m := \min_{x\in\overline{\Omega}} l_2\overline{u}_2(x) > 0$. Hence there exists $t_1 > 0$ such that

$$\frac{\lambda_1}{2}\int_\Omega u_2^2\varphi_1 dx - (b+\lambda_1)\lambda_1\int_\Omega u_2\varphi_1 dx \geq \frac{\lambda_1}{4}\int_\Omega u_2^2\varphi_1 dx \qquad \forall\, t \geq t_1. \tag{4.13}$$

Similarly, since we see that

$$\begin{aligned}&\left(\beta - \frac{\alpha}{2}\right)\int_{\partial\Omega} u_2^2\varphi_1 dS - (b+\lambda_1)(\beta-\alpha)\int_{\partial\Omega} u_2\varphi_1 dS\\ =&\frac{1}{2}\left(\beta - \frac{\alpha}{2}\right)\int_{\partial\Omega} u_2^2\varphi_1 dS + \int_{\partial\Omega}\left\{\frac{1}{2}\left(\beta - \frac{\alpha}{2}\right)u_2 - (b+\lambda_1)(\beta-\alpha)\right\}u_2\varphi_1 dS\\ \geq&\frac{1}{2}\left(\beta - \frac{\alpha}{2}\right)\int_{\partial\Omega} u_2^2\varphi_1 dS + \int_{\partial\Omega}\left\{\frac{1}{2}\left(\beta - \frac{\alpha}{2}\right)u_2^* - (b+\lambda_1)(\beta-\alpha)\right\}u_2\varphi_1 dS\\ \geq&\frac{1}{2}\left(\beta - \frac{\alpha}{2}\right)\int_{\partial\Omega} u_2^2\varphi_1 dS + \int_{\partial\Omega}\left\{\frac{1}{2}\left(\beta - \frac{\alpha}{2}\right)me^{\varepsilon t} - (b+\lambda_1)(\beta-\alpha)\right\}u_2\varphi_1 dS,\end{aligned}$$

there exists $t_2 > 0$ such that

$$\left(\beta - \frac{\alpha}{2}\right)\int_{\partial\Omega} u_2^2\varphi_1 dS - (b+\lambda_1)(\beta-\alpha)\int_{\partial\Omega} u_2\varphi_1 dS \geq \frac{1}{2}\left(\beta - \frac{\alpha}{2}\right)\int_{\partial\Omega} u_2^2\varphi_1 dS \qquad \forall\, t\geq t_2. \tag{4.14}$$

Therefore by (4.13), (4.14) and (4.12), we have

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{d}{dt}\left(\int_\Omega u_2\varphi_1 dx\right)\right\} + (b+2\lambda_1)\frac{d}{dt}\left(\int_\Omega u_2\varphi_1 dx\right) + (\beta-\alpha)\frac{d}{dt}\left(\int_{\partial\Omega} u_2\varphi_1 dS\right)\\ \geq&\frac{1}{2}\frac{d}{dt}\left(\int_\Omega u_2^2\varphi_1 dx\right) + \frac{\lambda_1}{4}\int_\Omega u_2^2\varphi_1 dx + \frac{1}{2}\left(\beta - \frac{\alpha}{2}\right)\int_{\partial\Omega} u_2^2\varphi_1 dS \qquad \forall\, t\geq t_3 := t_1\vee t_2.\end{aligned} \tag{4.15}$$

Now we integrate (4.15) with respect to t over $[t_3, t]$, so we get

$$\begin{aligned}&\frac{d}{dt}\left\{\int_\Omega u_2\varphi_1 dx + (\beta-\alpha)\int_{t_3}^t\int_{\partial\Omega} u_2\varphi_1 dS d\tau\right\}\\ \geq&\frac{1}{2}\int_\Omega u_2^2\varphi_1 dx - (b+2\lambda_1)\int_\Omega u_2\varphi_1 dx - \frac{1}{2}\int_\Omega u_2^2(t_3)\varphi_1 dx + \frac{1}{2}\left(\beta - \frac{\alpha}{2}\right)\int_{t_3}^t\int_{\partial\Omega} u_2^2\varphi_1 dS d\tau\\ &+\int_\Omega \partial_t u_2(t_3)\varphi_1 dx,\end{aligned} \tag{4.16}$$

where we neglected positive terms. Moreover we can see that there exists $t_4 > t_3$ such that

$$\frac{1}{2}\int_\Omega u_2^2\varphi_1 dx - (b+2\lambda_1)\int_\Omega u_2\varphi_1 dx - \frac{1}{2}\int_\Omega u_2^2(t_3)\varphi_1 dx + \int_\Omega \partial_t u_2(t_3)\varphi_1 dx \geq \frac{1}{4}\int_\Omega u_2^2\varphi_1 dx \tag{4.17}$$

for $t \geq t_4$ by the same argument as before. Therefore from (4.16) and (4.17), we have

$$\frac{d}{dt}\left\{\int_\Omega u_2\varphi_1 dx + (\beta-\alpha)\int_{t_3}^t\int_{\partial\Omega} u_2\varphi_1 dSd\tau\right\} \geq \frac{1}{4}\int_\Omega u_2^2\varphi_1 dx + \frac{1}{2}\left(\beta-\frac{\alpha}{2}\right)\int_{t_3}^t\int_{\partial\Omega} u_2^2\varphi_1 dSd\tau. \tag{4.18}$$

By Schwarz's inequality and $\|\varphi_1\|_{L^1(\Omega)} = 1$, we get

$$\frac{1}{4}\int_\Omega u_2^2\varphi_1 dx \geq \frac{1}{4}\left(\int_\Omega u_2\varphi_1 dx\right)^2,$$

and

$$\begin{aligned}\frac{1}{2}\left(\beta-\frac{\alpha}{2}\right)\int_{t_3}^t\int_{\partial\Omega} u_2^2\varphi_1 dSd\tau &\geq \frac{1}{2}\left(\beta-\frac{\alpha}{2}\right)\frac{1}{\|\varphi_1\|_{L^\infty(\Omega)}|\partial\Omega|}\frac{1}{t-t_3}\left\{\int_{t_3}^t\int_{\partial\Omega} u_2\varphi_1 dSd\tau\right\}^2 \\ &= \frac{1}{2}\frac{\beta-\frac{\alpha}{2}}{\|\varphi_1\|_{L^\infty(\Omega)}|\partial\Omega|(\beta-\alpha)^2}\frac{1}{t-t_3}\left\{(\beta-\alpha)\int_{t_3}^t\int_{\partial\Omega} u_2\varphi_1 dSd\tau\right\}^2.\end{aligned}$$

By the above inequalities and (4.18), for $t \geq t_5 := t_4 \vee (t_3+1)$, we finally get

$$\begin{aligned}&\frac{d}{dt}\left\{\int_\Omega u_2\varphi_1 dx + (\beta-\alpha)\int_{t_3}^t\int_{\partial\Omega} u_2\varphi_1 dSd\tau\right\} \\ \geq&\frac{1}{4}\int_\Omega u_2^2\varphi_1 dx + \frac{1}{2}\left(\beta-\frac{\alpha}{2}\right)\int_{t_3}^t\int_{\partial\Omega} u_2^2\varphi_1 dSd\tau \\ \geq&\frac{1}{4}\left(\int_\Omega u_2\varphi_1 dx\right)^2 + \frac{1}{2}\frac{\beta-\frac{\alpha}{2}}{\|\varphi_1\|_{L^\infty(\Omega)}|\partial\Omega|(\beta-\alpha)^2}\frac{1}{t-t_3}\left\{(\beta-\alpha)\int_{t_3}^t\int_{\partial\Omega} u_2\varphi_1 dSd\tau\right\}^2 \\ \geq&C'\frac{1}{t-t_3}\left\{\left(\int_\Omega u_2\varphi_1 dx\right)^2 + \left((\beta-\alpha)\int_{t_3}^t\int_{\partial\Omega} u_2\varphi_1 dSd\tau\right)^2\right\} \\ \geq&C\frac{1}{t-t_3}\left\{\int_\Omega u_2\varphi_1 dx + (\beta-\alpha)\int_{t_3}^t\int_{\partial\Omega} u_2\varphi_1 dSd\tau\right\}^2.\end{aligned}$$

Set $y(t) := \int_\Omega u_2\varphi_1 dx + (\beta-\alpha)\int_{t_3}^t\int_{\partial\Omega} u_2\varphi_1 dSd\tau$, then this inequality is transformed into the form of the following differential inequality:

$$\begin{cases}\frac{d}{dt}\{y(t)\} \geq \frac{C}{t-t_3}y^2(t) & t \geq t_5, \\ y(t_5) > 0.\end{cases}$$

By the direct calculation, it is easy to see that there exists $T^* > t_5$ such that

$$\lim_{t\to T^*} y(t) = +\infty.$$

This contradicts the assumption that (u_1, u_2) exists globally.

Remark 4.4. Since we use a contradiction in order to prove that solutions blow up in finite time on the proof, we obtain there exists $T > 0$ such that

$$\lim_{t\to T}\|u_1(t)\|_{L^\infty(\Omega)} = \infty \quad \text{or} \quad \lim_{t\to T}\|u_2(t)\|_{L^\infty(\Omega)} = \infty.$$

However we can show easily that these L^∞-norms of u_1 and u_2 blow up in same time, i.e., it holds that there exists $T > 0$ such that

$$\lim_{t\to T} \|u_1(t)\|_{L^\infty(\Omega)} = \infty \quad \text{and} \quad \lim_{t\to T} \|u_2(t)\|_{L^\infty(\Omega)} = \infty.$$

□

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