数理解析研究所講究録 第2090巻 2018年 60-69

On the plasticity models with some threshold functions

Risei Kano
Research and Education Faculty, Humanities and Social Science Cluster
Kochi University
2-5-1 Akebono-cho, Kochi 780-8520 Japan
E-mail: kano@kochi-u.ac.jp

1 Introduction

In this paper, we introduce for the construction of hardening model and the results related to the hardening problem. At the first, we introduce the plastic deformation of the materials in working process. We consider a metal rod that is subjected to a traction force σ and that in consequence undergoes a relative elongation ε. In the figure 1, we consider first that, starting with a point M of the open segment OS, we let ε undergo small changes of arbitrary sign. We observe that the point moves in neighborhood of M while staying on the segment OS. It represents a region where the behavior of material is linear and reversible, that is to say elastic. Now, we place at a point P of the line SZ and let ε decrease. Then, we observe that the point (ε, σ) describes a line segment starting from P, parallel to OS, say PQ. At the point P, the behavior of the material is then no longer reversible; the line SZ represents a plastic region. If we continue the segment PQ to Q_1, we again find an open segment PQ_1 on which the behavior of the material is reversible. Furthermore, PQ_1 >OS as long as the line SZ is not a half-line parallel to $O\varepsilon$. This is the phenomenon of work hardening. As we know, in order to describe the hardening process the slope of line SZ is very important.

2 Hardening model in the working process

2.1 The strain for plasiticity

As we have mentioned above section, we recall the mathematical model of the plasticity. If the stress σ is a very small, the stain ε is linear by σ. Namely, if σ is smaller than the threshold value and σ goes to 0, then ε goes to 0, too. It is to say elastic. However, if σ is beyond the threshold value and σ goes to 0, the elastic strain ε_e goes to 0 but the plastic strain ε_p is remained. This effect is said "Hysteresis" that is the dependence of the state of a system on its history. In the mathematical formulations, since the relations of parameters is not one-to-one, the hysteresis is very difficult. To get the mathematical formulation of the constitutive law by the plasticity, we have to get the date $\{\sigma(s)\}_{s\in[0,t]}$ that it is only the initial date $\sigma(0)$ but also the all time date.

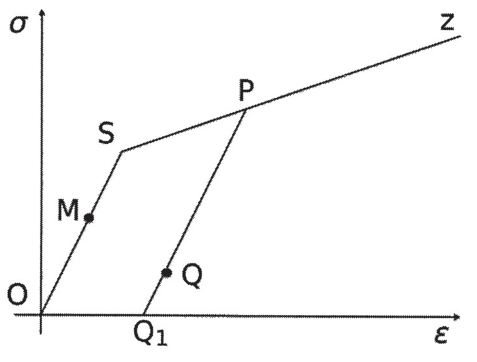

Figure 1: The relation of stress and strain

We define the yield function F that determines whether the stress is yielding situation, by

$$F(\sigma) < g \quad \Rightarrow \text{ only elastic,} \tag{2.1}$$
$$F(\sigma) = g \quad \Rightarrow \text{ yielding situation,} \tag{2.2}$$

where, g is a non-negative function that is called the threshold function. For example,

$$F(\sigma) = \frac{1}{2}|\tau^D|^2$$

where, $\tau_{ij}^D := \tau_{ij} - (1/3)\sum_{k=1}^{3} \tau_{kk}\delta_{ij}$ for $i, j = 1, 2, 3$ and $|\tau|^2 := \sum_{i,j=1}^{3} \tau_{ij}\tau_{ij}$ for all $\tau \in \mathbb{R}_{\text{sym}}^{3\times 3}$, the symbol δ_{ij} is the Kronecker delta.

We call the perfect plasticity model, if The threshold function is independent in ε (Figure 2). In this case, the relation $\{(2.1),(2.2)\}$ is presented by the following formulation;

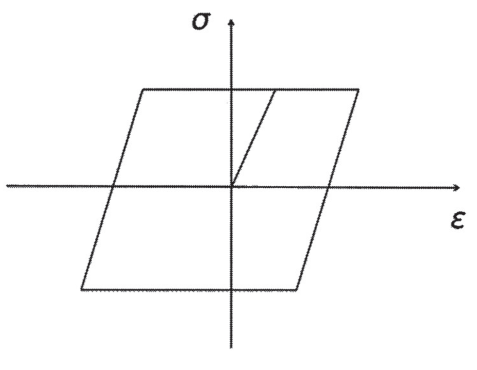

Figure 2: The relation of stress and strain for the perfect plasticity

$$\begin{cases} F(\sigma) < g & \Rightarrow \dfrac{\partial \varepsilon_p}{\partial t} = 0, \\ F(\sigma) = g \text{ and } \dfrac{\partial \sigma}{\partial t} = 0 & \Rightarrow \dfrac{\partial \varepsilon_p}{\partial t} \geq 0. \end{cases}$$

Namely, we get the following the Moreau sweeping process form

$$\frac{\partial \varepsilon_p}{\partial t} \in \partial I_Z(\sigma), \tag{2.3}$$

where, Z is a time-dependent closed convex set defined by $Z := \{\tau \in \mathbb{R}^{3\times 3}_{\text{sym}} : (1/2)|\tau^D|^2 \leq g\}$. Moreover, I_Z is the indicator function of Z, ∂I_Z is the subdifferential of I_Z.

Moreover, we consider the two conditions for the strain. The first condition is the additive decomposition of strains, namely, the strain can be expressed by the decomposition of elastic and plastic parts;

$$\varepsilon(\boldsymbol{u}) = \varepsilon_e + \varepsilon_p. \tag{2.4}$$

The second condition is that the elastic strain is linear with respect to stress;

$$\varepsilon_e = L\sigma, \tag{2.5}$$

where, L is the 3×3 matrix. Hence, we combine three conditions (2.3), (2.4) and (2.5), after that we can get the equation of σ. Indeed, taking the time derivative of first equation using the second equation, and then the third equation becomes the following inclusion;

$$L\frac{\partial \sigma}{\partial t} + \partial I_Z(\sigma) \ni \varepsilon\left(\frac{\partial \boldsymbol{u}}{\partial t}\right). \tag{2.6}$$

2.2 Hardening

Next, we recall the hardening problem in one dimensional space, which is derived by Visintin [5]. His idea is the characteristic of yielding situation. We put a and b are slopes of strain for stress, respectively with $0 \leq a < b$. When the strain consists of only the elastic strain that is on the elastic region, the slope is equal to b. When the strain consists of the elastic strain and the plastic strain that is on the plastic region, the slope is equal to a (Figure 3). In this case, the threshold value is equal to $g + a\varepsilon$. Therefore, we can get the following new expression of the plastic strain

$$\frac{\partial \varepsilon_p}{\partial t} \in \partial I_Z(\sigma - a\varepsilon), \tag{2.7}$$

indeed, these conditions are equivalent,

$$\sigma - a\varepsilon \in Z \quad \Leftrightarrow \quad \sigma \in Z + a\varepsilon.$$

Hence, we combine three conditions (2.7), (2.4) and (2.5), we get the new equation of σ;

$$\frac{\partial \sigma}{\partial t} + \partial I_Z(\sigma - a\varepsilon(u)) \ni b\varepsilon\left(\frac{\partial u}{\partial t}\right). \tag{2.8}$$

As the remark, in 1D case, on the previous perfect plasticity model (3.2), we take $a = 0$ and $b = 1/L$.

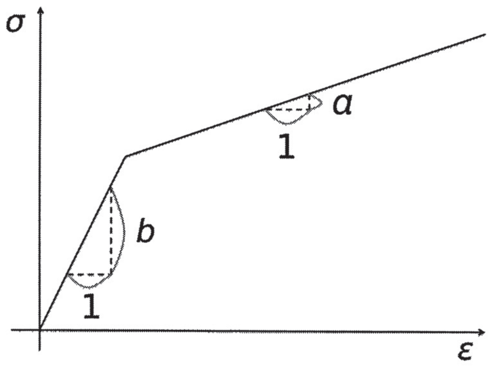

Figure 3: The relation of stress and strain in hardening process

3 The Known result of the perfect plasticity case

In this section, we introduce the existence theorem for the perfect plasticity problems of some type. For each theorem, we used by the method of variational inequality. The constraint is described by the Moreau sweeping process form which is well-known in the abstract evolution equations (c.f.[4]).

3.1 The classical problem by Duvaut–Lions

In this section, we recall the problem of perfect plasticity is derived by Duvaut and Lions [1]. The domain Ω is the bounded set in $\mathbb{R}^3$ with the smooth boundary $\Gamma = \partial\Omega$ which consists of $\Gamma = \Gamma_D \cup \Gamma_N$, and $\Gamma_D \cap \Gamma_N = \emptyset$ with $|\Gamma_D| > 0$ and $|\Gamma_N| > 0$. $\boldsymbol{\nu}$ denotes the unit normal vector outward from Γ. The unknown functions $\boldsymbol{u} = (u_1, u_2, u_3)$, $\sigma = \{\sigma_{ij}\}_{i,j=1,2,3}$, describe the displacement and the stress tensor, respectively. The strain $\varepsilon(\boldsymbol{u}) = \{\varepsilon_{i,j}\}_{i,j=1,2,3}$ depending by displacement $\boldsymbol{u}$, is defined by

$$\varepsilon_{i,j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

for $i, j = 1, 2, 3$.

We consider the model of perfect plasticity. To find $\boldsymbol{v} := \partial \boldsymbol{u}/\partial t$ and σ satisfying

$$\frac{\partial \boldsymbol{v}}{\partial t} = \mathbf{div}\sigma + \boldsymbol{f} \quad \text{in } Q := (0,T) \times \Omega, \tag{3.9}$$

$$\frac{\partial \sigma}{\partial t} + \partial I_Z(\sigma + \sigma_*) \ni \varepsilon(\boldsymbol{v}) + h \quad \text{in } Q, \tag{3.10}$$

where $\boldsymbol{f} : Q \to \mathbb{R}^3$, $h : Q \to \mathbb{R}^{3\times3}_{\text{sym}}$, and $\sigma_* : Q \to \mathbb{R}^{3\times3}_{\text{sym}}$ are given functions in Q, $\mathbb{R}^{3\times3}_{\text{sym}}$ stands for the 3×3 symmetric matrix. With the help of h and σ_*, we can translate the problem to the homogeneous boundary value problem. The operator **div** is defined by $\mathbf{div}\tau := (\text{div}\tau_{1\cdot}, \text{div}\tau_{2\cdot}, \text{div}\tau_{3\cdot})$ for all $\tau \in \mathbb{R}^{3\times3}_{\text{sym}}$, where $\text{div}\tau_{i\cdot} := \sum_{j=1}^{3} \partial\tau_{ij}/\partial x_j$ for $i = 1, 2, 3$. The first equation (3.9) is derived by the conservation law of momentum. The second equation (3.10) ensures the property of perfect plasticity, where we assume the additive decomposition of strain as Section 2.1.

We use the following notation: $\boldsymbol{H} := L^2(\Omega)^3$, $\boldsymbol{V} := \{\boldsymbol{z} \in H^1(\Omega)^3 : \boldsymbol{z} = \boldsymbol{0} \text{ a.e. on } \Gamma_D\}$, with their inner products $(\cdot,\cdot)_{\boldsymbol{H}}$, $(\cdot,\cdot)_{\boldsymbol{V}}$, and the norm $|\cdot|_{\boldsymbol{H}}$, where $|\cdot|_{\boldsymbol{V}}$ is defined by

$$|\boldsymbol{z}|_{\boldsymbol{V}} := \left\{ \sum_{i,j=1}^{3} \int_\Omega \left| \frac{\partial z_i}{\partial x_j} \right|^2 dx \right\}^{\frac{1}{2}} \quad \text{for all } \boldsymbol{z} \in \boldsymbol{V}.$$

Denote the dual space of $\boldsymbol{V}$ by $\boldsymbol{V}^*$ with the duality pair $\langle \cdot,\cdot \rangle_{\boldsymbol{V}^*,\boldsymbol{V}}$. Moreover, we define the following bilinear form: $(\!(\cdot,\cdot)\!) : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$

$$(\!(\boldsymbol{z}, \tilde{\boldsymbol{z}})\!) := \sum_{i,j=1}^{3} \int_\Omega \frac{\partial z_i}{\partial x_j} \frac{\partial \tilde{z}_i}{\partial x_j} dx \quad \text{for all } \boldsymbol{z}, \tilde{\boldsymbol{z}} \in \boldsymbol{V}.$$

We also define $\mathbb{H} := \{\tau := \{\tau_{ij}\} : \tau_{ij} \in L^2(\Omega),\ \tau_{ij} = \tau_{ji}\}$, $\mathbb{V} := \{\tau \in \mathbb{H} : \mathbf{div}\tau \in \boldsymbol{H}, \tau_{i\cdot} \cdot \boldsymbol{\nu} = 0 \text{ a.e. on } \Gamma_N\}$ with their inner products

$$(\tau, \tilde{\tau})_{\mathbb{H}} := \sum_{i,j=1}^{3} \int_\Omega \tau_{ij} \tilde{\tau}_{ij} dx \quad \text{for all } \tau, \tilde{\tau} \in \mathbb{H},$$

$$(\tau, \tilde{\tau})_{\mathbb{V}} := (\tau, \tilde{\tau})_{\mathbb{H}} + (\mathbf{div}\tau, \mathbf{div}\tilde{\tau})_{\boldsymbol{H}} = (\tau, \tilde{\tau})_{\mathbb{H}} + \sum_{i,j=1}^{3} \int_\Omega \frac{\partial \tau_{ij}}{\partial x_j} \frac{\partial \tilde{\tau}_{ij}}{\partial x_j} dx \quad \text{for all } \tau, \tilde{\tau} \in \mathbb{V}.$$

The following convex constraint plays an important role in this paper. For each $t \in [0, T]$,

$$\tilde{K}(t) := \left\{ \tau \in \mathbb{H} : \frac{1}{2} \left| \tau^D(x) \right|^2 \le g(t, x) \quad \text{for a.a. } x \in \Omega \right\}, \quad K(t) := \tilde{K}(t) - \sigma_*(t).$$

Finally, we recall an important relation. For each $\boldsymbol{z} \in \boldsymbol{V}$, $\tau \in \mathbb{V}$, the following relation holds:

$$\big(\varepsilon(\boldsymbol{z}), \tau\big)_{\mathbb{H}} + (\mathbf{div}\tau, \boldsymbol{z})_{\boldsymbol{H}} = 0. \tag{3.11}$$

This is called the Gauss–Green relation.

In the paper [1], the threshold function g is a constant function. Duvaut and Lions showed the existence of solutions for the perfect plasticity model.

Proposition 3.1. We assume that the following conditions hold;

$$\boldsymbol{f} \in W^{1,2}(0, T; \boldsymbol{H}),$$

$$h \in W^{1,2}(0, T; \mathbb{H}),$$

and σ_* is independent of time t. Then there exists an unique pair of functions $(\boldsymbol{v}, \sigma)$ such that

$$\boldsymbol{v}, \boldsymbol{v}' \in L^\infty(0, T; \boldsymbol{H}),$$

$$\sigma, \sigma' \in L^\infty(0, T; \mathbb{H}),$$

$$v_i \in L^\infty(0,T;L^2(\Omega)),$$
$$\sigma_{i,j} \in L^\infty(0,T;L^2(\Omega)),$$

and that satisfy

$$\big(\boldsymbol{v}'(t),\boldsymbol{z}\big)_{\boldsymbol{H}} - \big(\mathbf{div}(\sigma(t)),\boldsymbol{z}\big)_{\boldsymbol{H}} = \big(\boldsymbol{f}(t),\boldsymbol{z}\big)_{\boldsymbol{H}} \quad \textit{for all } \boldsymbol{z} \in \boldsymbol{V},$$

and

$$\big(\sigma'(t),\sigma(t)-\tau\big)_{\mathbb{H}} - \big(\varepsilon(\boldsymbol{v}(t)),\sigma(t)-\tau\big)_{\mathbb{H}} \le \big(h(t),\sigma(t)-\tau\big)_{\mathbb{H}} \quad \textit{for all } \tau \in \mathbb{V}$$

for a.a. $t \in (0,T)$ with $\boldsymbol{v}(0) = \boldsymbol{v}_0$ in $\boldsymbol{H}$ and $\sigma(0) = \sigma_0$ in $\mathbb{H}$.

3.2 The perfect plasticity problem with the threshold depending on time

We consider the case of the threshold function $g = g(t)$ depending on time, because, the threshold function depend on the unknown function in the target hardening problem. We use the same notation in the above section.

Definition 3.1. *For each* $\kappa \in (0,1]$ *and* $\nu \in (0,1]$, *the pair* $(\boldsymbol{v},\sigma)$ *is called a solution of modified problem for* (3.9) *and* (3.10) *in the sense of variational inequality if*

$$\boldsymbol{v} \in H^1(0,T;\boldsymbol{H}) \cap L^\infty(0,T;\boldsymbol{V}) \cap L^2\big(0,T;H^2(\Omega)^3\big),$$
$$\sigma \in H^1(0,T;\mathbb{H}) \cap L^2(0,T;\mathbb{V}), \quad \sigma(t) \in K(t) \quad \textit{for all } t \in [0,T],$$

and they satisfy

$$\big(\boldsymbol{v}'(t),\boldsymbol{z}\big)_{\boldsymbol{H}} + \nu\big((\boldsymbol{v}(t),\boldsymbol{z})\big) - \big(\mathbf{div}(\sigma(t)),\boldsymbol{z}\big)_{\boldsymbol{H}} = \big(\boldsymbol{f}(t),\boldsymbol{z}\big)_{\boldsymbol{H}} \quad \textit{for all } \boldsymbol{z} \in \boldsymbol{V},$$
$$\big(\sigma'(t),\sigma(t)-\tau\big)_{\mathbb{H}} + \kappa\big(\sigma(t),\sigma(t)-\tau\big)_{\mathbb{V}} - \big(\varepsilon(\boldsymbol{v}(t)),\sigma(t)-\tau\big)_{\mathbb{H}}$$
$$\le \big(h(t),\sigma(t)-\tau\big)_{\mathbb{H}} \quad \textit{for all } \tau \in K(t) \cap \mathbb{V}$$

for a.a. $t \in (0,T)$ *with* $\boldsymbol{v}(0) = \boldsymbol{v}_0$ *in* $\boldsymbol{H}$ *and* $\sigma(0) = \sigma_0$ *in* $\mathbb{H}$.

Proposition 3.2. *We assume that* (A1)–(A5) *hold;*

(A1) $\boldsymbol{f} \in L^2(0,T;\boldsymbol{H})$ and $h \in L^2(0,T;\mathbb{H})$;

(A2) $\boldsymbol{v}_0 \in \boldsymbol{V}$ and $\sigma_0 \in K(0) \cap \mathbb{V}$;

(A3) $\sigma_* \in H^1(0,T;\mathbb{V})$;

(A4) $g \in H^1(0,T;C(\overline{\Omega})) \cap C(\overline{Q})$;

(A5) There exist two constants $C_1, C_2 > 0$ such that

$$0 < C_1 \le g(t,x) \le C_2 \quad \text{for all } (t,x) \in \overline{Q}.$$

Then, there exists a unique solution $(\boldsymbol{v}, \sigma)$ *of modified problem for* (3.9) *and* (3.10) *in the sense of variational inequality.*

Let us remove the parameter $\kappa \in (0,1]$. In this case, the problem is the same as the *Moreau sweeping process.*

Definition 3.2. *For each* $\nu \in (0,1]$, *the pair* $(\boldsymbol{v}, \sigma)$ *is called a solution of the viscous perfect plasticity model for* (3.9) *and* (3.10) *if*

$$\boldsymbol{v} \in H^1(0,T;\boldsymbol{V}^*) \cap L^\infty(0,T;\boldsymbol{H}) \cap L^2(0,T;\boldsymbol{V}),$$
$$\sigma \in H^1(0,T;\mathbb{H}), \quad \sigma(t) \in K(t) \quad \text{for all } t \in [0,T],$$

and they satisfy

$$\langle \boldsymbol{v}'(t), \boldsymbol{z} \rangle_{\boldsymbol{V}^*,\boldsymbol{V}} + \nu\big(\!\big(\boldsymbol{v}(t), \boldsymbol{z}\big)\!\big) + \big(\sigma(t), \varepsilon(\boldsymbol{z})\big)_{\mathbb{H}} = \big(\boldsymbol{f}(t), \boldsymbol{z}\big)_{\boldsymbol{H}} \quad \text{for all } \boldsymbol{z} \in \boldsymbol{V},$$
$$\big(\sigma'(t), \sigma(t) - \tau\big)_{\mathbb{H}} - \big(\varepsilon\big(\boldsymbol{v}(t)\big), \sigma(t) - \tau\big)_{\mathbb{H}} \leq \big(h(t), \sigma(t) - \tau\big)_{\mathbb{H}} \quad \text{for all } \tau \in K(t)$$

for a.a. $t \in (0,T)$ *with* $\boldsymbol{v}(0) = \boldsymbol{v}_0$ *in* $\boldsymbol{H}$ *and* $\sigma(0) = \sigma_0$ *in* $\mathbb{H}$.

We replace (A3) by (A3′):

(A3′) $\sigma_* \in H^1(0,T;\mathbb{H})$.

Proposition 3.3. *Under assumptions* (A1), (A2), (A3′), (A4), *and* (A5), *there exists a unique solution* $(\boldsymbol{v}, \sigma)$ *of the viscous perfect plasticity model for* (3.9) *and* (3.10).

The proposition 3.2 and 3.3 is showed by Fukao and Kano in [2].

3.3 The perfect plasticity weakly problem with the threshold depending on time

To relax assumption (A4) on g with respect to time regularity, we recall the concept of the weak variational formulation:

Definition 3.3. *For each* $\kappa \in (0,1]$ *and* $\nu \in (0,1]$, *the pair* $(\boldsymbol{v}, \sigma)$ *is called a solution of modified problem for* (3.9) *and* (3.10) *in the sense of weak variational inequality if*

$$\boldsymbol{v} \in H^1(0,T;\boldsymbol{H}) \cap L^\infty(0,T;\boldsymbol{V}) \cap L^2\big(0,T;H^2(\Omega)^3\big),$$
$$\sigma \in C\big([0,T];\mathbb{H}\big) \cap L^2(0,T;\mathbb{V}), \quad \sigma(t) \in K(t) \quad \text{for a.a. } t \in [0,T],$$

and they satisfy

$$\big(\boldsymbol{v}'(t),\boldsymbol{z}\big)_{\mathbb{H}}+\nu\big((\boldsymbol{v}(t),\boldsymbol{z})\big)-\big(\mathbf{div}\sigma(t),\boldsymbol{z}\big)_{\boldsymbol{H}}=\big(\boldsymbol{f}(t),\boldsymbol{z}\big)_{\boldsymbol{H}}\quad \text{for all } \boldsymbol{z}\in\boldsymbol{V}, \tag{3.12}$$

$$\begin{aligned}&\int_0^t\big(\eta'(s),\sigma(s)-\eta(s)\big)_{\mathbb{H}}ds+\kappa\int_0^t\big(\sigma(s),\sigma(s)-\eta(s)\big)_{\mathbb{V}}ds\\&\quad-\int_0^t\big(\varepsilon(\boldsymbol{v}(s)),\sigma(s)-\eta(s)\big)_{\mathbb{H}}ds+\frac{1}{2}\big|\sigma(t)-\eta(t)\big|^2_{\mathbb{H}}\\&\le\int_0^t\big(h(s),\sigma(s)-\eta(s)\big)_{\mathbb{H}}ds+\frac{1}{2}\big|\sigma_0-\eta(0)\big|^2_{\mathbb{H}}\quad\text{for all } \eta\in\mathcal{K}_0,\end{aligned} \tag{3.13}$$

for a.a. $t\in(0,T)$ *with* $\boldsymbol{v}(0)=\boldsymbol{v}_0$ *in* $\boldsymbol{H}$ *and* $\sigma(0)=\sigma_0$ *in* $\mathbb{H}$, *where*

$$\mathcal{K}_0:=\big\{\eta\in H^1(0,T;\mathbb{H})\cap L^2(0,T;\mathbb{V}):\eta(t)\in K(t)\quad\text{for a.a. } t\in(0,T)\big\}.$$

We assume the weaker condition (A4′) in place of (A4):

(A4′) $g\in C(\overline{Q})$.

Proposition 3.4. *Under assumptions* (A1)–(A3), (A4′), *and* (A5), *there exists a unique solution* $(\boldsymbol{v},\sigma)$ *of modified problem for* (3.9) *and* (3.10) *in the sense of weak variational inequality.*

The proposition 3.4 is showed by Fukao and Kano in [2], too.

4 Hardening case

By the Section 2.2, our hardening problem in one-dimensional case $\{(4.14)$–$(4.18)\}$ is expressed by the following formulation:

$$\frac{\partial v}{\partial t}=\frac{\partial \sigma}{\partial x}+f,\quad \text{in } Q:=(0,T)\times(0,L), \tag{4.14}$$

$$\frac{\partial \sigma}{\partial t}+\partial I_Z\left(\sigma+\sigma_*-a\frac{\partial u}{\partial x}\right)\ni b\frac{\partial v}{\partial x}+h,\quad \text{in } Q, \tag{4.15}$$

$$\frac{\partial u}{\partial t}=v,\quad \text{in } Q, \tag{4.16}$$

$$\sigma(0)=\sigma_0,\ v(0)=v_0,\ u(0)=0\quad \text{in } (0,L), \tag{4.17}$$

$$v(0)=0,\ \sigma(L)=0\quad \text{in } (0,T), \tag{4.18}$$

where, $0<T<\infty$, $0<L<\infty$ and $b>a\ge 0$ are given constants. σ_* and h are given functions in Q, indeed, in order to consider the homogeneous Dirichlet boundary conditions, we change the variable σ. f is also given. g is the threshold function in Q, σ_0 and v_0 are given functions in $(0,L)$, the set $Z:=\{r\in\mathbb{R}\mid(1/2)|r|^2\le g(t)\}$ is constraint set, the I_Z is indicator function of Z and the ∂I_Z is subdifferential operator of I_Z. To discuss the existence of solutions, we can use the method of quasi-variational inequality. Quasi-variational inequality can be discussed by the following evolution equation with subdifferential operator of convex functions $\varphi^t(u;\cdot)$ in the Hilbert space H;

$$u'(t) + \partial\varphi^t(u;u(t)) \ni f(t) \quad \text{in } H.$$

Some results are known to quasi-variational inequality, this problem is one of the application for Kano–Kenmochi–Murase [3].

5 Future problem

We are going to consider the next two problems. The first problem is the hardening problem in the 3-demensional case. Namely, that problem is expressed by the following formulation:

$$\frac{\partial \boldsymbol{v}}{\partial t} = \mathbf{div}\sigma + \boldsymbol{f} \quad \text{in } Q := (0,T)\times\Omega, \tag{5.19}$$

$$\frac{\partial \sigma}{\partial t} + \partial I_Z\left(\sigma + \sigma_* - A\varepsilon\left(\boldsymbol{u}\right)\right) \ni B\varepsilon(\boldsymbol{v}) + h \quad \text{in } Q, \tag{5.20}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{v}, \quad \text{in } Q, \tag{5.21}$$

$$\sigma(0) = \sigma_0,\ v(0) = \boldsymbol{v}_0,\ u(0) = 0 \quad \text{in } \Omega, \tag{5.22}$$

$$v(0) = 0, \quad \text{on } (0,T)\times\Gamma_D, \tag{5.23}$$

$$\sigma = 0 \quad \text{on } (0,T)\times\Gamma_N, \tag{5.24}$$

where, T, Ω, Γ, Γ_N and Γ_D are the same one as Section 3.1. σ_* and h are given functions in Q. $\boldsymbol{f}$ is also given. g is the threshold function in Q, σ_0 and $\boldsymbol{v}_0$ are given functions in Ω. Z, I_Z and ∂I_Z are the same one as Section 2.1. A and B are the 3×3 matrix. As the remark, A and B make no physical meanings.

The second problem is the hardening problem with non-linear hardening.

$$\frac{\partial \boldsymbol{v}}{\partial t} = \mathbf{div}\sigma + \boldsymbol{f} \quad \text{in } Q := (0,T)\times\Omega, \tag{5.25}$$

$$\frac{\partial \sigma}{\partial t} + \partial I_Z\left(\sigma + \sigma_* - \alpha\left(\varepsilon\left(\boldsymbol{u}\right)\right)\right) \ni B\varepsilon(\boldsymbol{v}) + h \quad \text{in } Q, \tag{5.26}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{v}, \quad \text{in } Q, \tag{5.27}$$

$$\sigma(0) = \sigma_0,\ v(0) = \boldsymbol{v}_0,\ u(0) = 0 \quad \text{in } \Omega, \tag{5.28}$$

$$v(0) = 0, \quad \text{on } (0,T)\times\Gamma_D, \tag{5.29}$$

$$\sigma = 0 \quad \text{on } (0,T)\times\Gamma_N, \tag{5.30}$$

where some notations are the same one as above problem. $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$, is the non-linear smooth function.

We think that we can discuss the solvability of these problems using the quasi-variational inequality, too.

References

[1] G. Duvaut and J. L. Lions, *Inequalities in mechanics and physics*, Springer-Verlag, 1976.

[2] T. Fuako and R. Kano, *Time-dependence of the threshold function in the perfect plasticity model,* preprint arXiv:1610.08577v1 [math.AP] (2016), pp. 1–25.

[3] R. Kano, N. Kenmochi and Y. Murase, *Nonlinear evolution equations generated by subdifferentials with nonlocal constraints,* Banach Center Publications, 86 (2009),175-194.

[4] J. J. Moreau, Rafle par un convexe variable (Premiére partie), pp. 1–43 in *Travaux du Séminaire d'Analyse Convexe*, Montpellier, 1971.

[5] A. Visintin, *Mathematical models of hysteresis,* The science of hysteresis, pp.1-114, Vol. 1, 2006.