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Translation-invariant quantum Markov states

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1 Introduction

The notion of quantum Markov states was first introduced by Accardi and Frigerio ([1], [3]), and was further discussed from a somewhat different viewpoint by Fannes, Nachtergaele and Werner ([5]). A Markov state by Accardi and Frigerio is defined on a UHF algebra and is determined by an initial state and a family of completely positive quasi-conditional expectations. However, thanks to [2], [3] and also [6], conditional expectations can be used in place of quasi-conditional expectations. Although Accardi and Frigerio defined their Markov states without translation-invariance, we restrict ourselves to translation-invariant ones and clarify their fine structure.

In [4] it was implicitly stated that any translation-invariant Markov state in the sense of [3] is determined by a single conditional expectation (so that it is a C*-finitely correlated state in [5]), and an explicit form of translation-invariant Markov states was given. In Section 2 we make the relation between two notions of quantum Markov states more precise and consider the question concerning the commutativity of local density matrices of a Markov state. In Section 3 we see explicit form of quantum Markov states due to [4].

2 Characterization of translation-invariant Markov states

Let $A_i = M_d = M_d(C)$, the $d \times d$ complex matrix algebra, for $i \in N$ and $A$ be the infinite $C^*$-tensor product $\bigotimes_{i=1}^{\infty} A_i$. We denote $A_{\Lambda} = \bigotimes_{n \in \Lambda} A_n$ for arbitrary subset $\Lambda \subset N$. The translation $\gamma$ is the right shift on $A$. We write $\phi_{[1,n]}$ for the localization $\phi|A_{[1,n]}$, and in particular $\phi_1$ for $n = 1$. The following definition is from [3] with a slight modification.

**Definition 2.1** A state $\phi$ on $A$ is called a (quantum) Markov state if for each $n \in N$ there exists a conditional expectation $E_n$ from $A_{[1,n+1]}$ into $A_{[1,n]}$ such that $E_n(A_{[1,n+1]}) \supset A_{[1,n-1]}$ and $\phi_{[1,n+1]} = \phi_{[1,n]} \circ E_n$. A Markov state is said to be translation-invariant if $\phi \circ \gamma = \phi$. 
Although the above definition is a bit different from the original one of Accardi and Frigerio in [3], it is known that both definition are equivalent ([2], [3] and also [6]).

We assume that \( \phi \) is locally faithful, i.e. \( \phi_{[1,n]} \) is faithful for all \( n \in \mathbb{N} \). The next theorem was implicitly stated in [4]; we here give a proof.

**Theorem 2.2** Let \( \phi \) be a state on \( \mathfrak{A} \). Then the following are equivalent.

(i) \( \phi \) is a translation-invariant Markov state,

(ii) There exists a conditional expectation \( E \) from \( M_{d} \otimes M_{d} \) into \( M_{d} \) such that \( \phi_{1} \circ E(I \otimes A) = \phi(A) \) for all \( A \in M_{d} \) and

\[
\phi(A_{1} \otimes \cdots \otimes A_{n}) = \phi_{1}(E(A_{1} \otimes E(A_{2} \otimes \cdots \otimes E(A_{n-1} \otimes A_{n}) \cdots)))
\]

for all \( A_{1}, \ldots, A_{n} \in M_{d} \).

**Proof.** (ii) \( \Rightarrow \) (i). Assume (ii), and define conditional expectation \( E_{n} : \mathfrak{A}_{[1,n+1]} \to \mathfrak{A}_{[1,n]} \), \( n \in \mathbb{N} \), by

\[
E_{n}(X \otimes A) = X \otimes E(A)
\]

for \( X \in \mathfrak{A}_{[1,n-1]} \) and \( A \in \mathfrak{A}_{[n,n+1]} \). Then for \( A_{1}, \ldots, A_{n} \in M_{d} \),

\[
\phi(A_{1} \otimes \cdots \otimes A_{n}) = \phi_{1} \circ E(A_{1} \otimes E(A_{2} \otimes \cdots \otimes E(A_{n-1} \otimes A_{n}) \cdots)) = \phi(A_{1} \otimes \cdots \otimes A_{n-2} \otimes E(A_{n-1} \otimes A_{n})) = \phi \circ E_{n-1}(A_{1} \otimes \cdots \otimes A_{n})
\]

and

\[
\phi(I \otimes A_{1} \otimes \cdots \otimes A_{n}) = \phi_{1} \circ E(I \otimes E(A_{1} \otimes E(A_{2} \otimes \cdots \otimes E(A_{n-1} \otimes A_{n}) \cdots)) = \phi_{1} \circ E(A_{1} \otimes E(A_{2} \otimes \cdots \otimes E(A_{n-1} \otimes A_{n}) \cdots)) = \phi(A_{1} \otimes \cdots \otimes A_{n}).
\]

So \( \phi \) is a translation-invariant Markov state.

(i) \( \Rightarrow \) (ii). We fix \( n \in \mathbb{N} \), and define \( F_{n-1} = \gamma^{-1} \circ E_{n} \circ \gamma \). This is well defined. Indeed, for any \( A \in \mathfrak{A}_{[1,n-1]} \) and \( B \in \mathfrak{A}_{1} \),

\[
E_{n}(I \otimes A) \cdot B \otimes I_{\otimes n-1}^{\otimes} = E_{n}(B \otimes A) = B \otimes I_{\otimes n-1}^{\otimes} \cdot E_{n}(I \otimes A).
\]

Hence, \( E_{n}(I \otimes A) \in \mathfrak{A}_{[2,n]} \). Similarly, we define \( F_{i} = \gamma_{(n-i)}^{-1} \circ E_{n} \circ \gamma^{n-i} \) (1 \( \leq i \leq n-1 \). Then for 1 \( \leq i \leq n \) and \( A_{1}, \ldots, A_{i+1} \in M_{d} \), we have

\[
F_{i}(A_{1} \otimes \cdots \otimes A_{i+1}) = (A_{1} \otimes \cdots \otimes A_{i-1} \otimes I_{\otimes 2}) \cdot F_{i}(I_{\otimes i-1} \otimes A_{i} \otimes A_{i+1}) = A_{1} \otimes \cdots \otimes A_{i-1} \otimes F_{i}(A_{i} \otimes A_{i+1}).
\]

Now, let \( \mathfrak{F}_{n} \) denote the set of all conditional expectations \( F : M_{d} \otimes M_{d} \to M_{d} \) such that if we define \( F_{i}(A_{1} \otimes \cdots \otimes A_{i+1}) = A_{1} \otimes \cdots \otimes A_{i-1} \otimes F(A_{i} \otimes A_{i+1}) \), for \( A_{1}, \ldots, A_{i+1} \in M_{d} \), then \( \phi_{[1,i]} \circ F_{i} = \phi_{[1,i+1]} \) for each 1 \( \leq i \leq n \). Then the
above argument guarantees the non-emptyness of $\mathfrak{F}_n$. Since $\mathfrak{F}_n$'s are compact and $\mathfrak{F}_1 \supseteq \mathfrak{F}_2 \cdots$, it follows that $\bigcap_{n \in \mathbb{N}} \mathfrak{F}_n$ is not empty. Choose $E \in \bigcap_{n \in \mathbb{N}} \mathfrak{F}_n$ and define

$$E_n(A_1 \otimes \cdots \otimes A_{n+1}) = A_1 \otimes \cdots \otimes A_{n-1} \otimes E(A_n \otimes A_{n+1})$$

for $A_1, \ldots, A_{n+1} \in M_d$. Then

$$\phi(A_1 \otimes \cdots \otimes A_n) = \phi_1 \circ E_1 \circ \cdots \circ E_n(A_1 \otimes \cdots \otimes A_n) = \phi_1 \circ E(A_1 \otimes E(A_2 \otimes \cdots \otimes E(A_{n-1} \otimes A_n) \cdots)),$$

and

$$\phi_1 \circ E(I \otimes A) = \phi(I \otimes A) = \phi_1(A)$$

for $A \in M_d$. \qed

The following definition is from [5].

**Definition 2.3** A state $\phi$ on $\mathfrak{A}$ is called a $C^*$-finitely correlated state if there exist a finite dimensional $C^*$-algebra $\mathfrak{B}$, a completely positive map $E : M_d \otimes \mathfrak{B} \to \mathfrak{B}$ and a state $\rho$ on $\mathfrak{B}$ such that

$$\rho(E(I_d \otimes B)) = \rho(B)$$

for all $B \in \mathfrak{B}$ and

$$\phi(A_1 \otimes \cdots \otimes A_n) = \rho(E(A_1 \otimes E(A_2 \otimes \cdots \otimes E(A_{n-1} \otimes I_{\mathfrak{B}}) \cdots)))$$

for all $A_1, \ldots, A_n \in M_d$.

Let $\phi$ be a translation-invariant Markov state, and $E$ be as in (ii) of Theorem 2.2. We set $\hat{\mathfrak{B}} = E(M_d \otimes M_d)$ and $\hat{E} = E|_{M_d \otimes \mathfrak{B}}$. Then $\phi$ is a $C^*$-finitely correlated state with a triple $(\mathfrak{B}, \hat{E}, \phi|\mathfrak{B})$. Hence any translation-invariant Markov state becomes a $C^*$-finitely correlated state.

Now, let $q_1, \ldots, q_k$ be minimal central projections of $\mathfrak{B}$, so that $\mathfrak{B}q_i \cong M_{d_i}$ for some $d_i \in \mathbb{N}$. Let $m_i$ be the multiplicity of $M_{d_i}$ in $M_d$. Then

$$\mathfrak{B} = \bigoplus_{i=1}^k \mathfrak{B}q_i = \bigoplus_{i=1}^k M_{d_i}.$$  

Moreover, we set

$$\mathfrak{C} = \bigoplus_{i=1}^k M_{d_i} \otimes M_{m_i}$$

and let $E_\mathfrak{C} : M_d \to \mathfrak{C}$ be the pinching $A \mapsto \sum_{i=1}^k q_i A q_i$.

The next proposition is included in [4].
Proposition 2.4 There exist positive linear functionals $\rho_{ij}$ on $M_{m_i} \otimes M_{d_j}$ ($1 \leq i, j \leq k$) such that

$$\hat{E} = (\bigoplus_{i,j=1}^{k} \text{id}_{M_{d_i}} \otimes \rho_{ij}) (E_{\mathcal{C}} \otimes \text{id}_{\mathfrak{B}}).$$

We remark that the unitality of $\hat{E}$ is equivalent to the condition that $\bigoplus_{j=1}^{k} \rho_{\dot{0}j}$ is a state on $M_{m_i} \otimes \mathfrak{B}$ for each $1 \leq i \leq k$. Furthermore, the condition that $\phi_{i} \circ \hat{E}(I \otimes B) = \phi_{i}(B)$ for all $B \in \mathfrak{B}$ is equivalent to the condition that for $B_j \in M_{d_j}$ ($1 \leq j \leq k$),

$$\sum_{i=1}^{k} \psi_{i}(q_i) \rho_{ij}(I_{m_i} \otimes B_j) = \psi_{j}(B_j), \quad (1)$$

where $\phi|\mathfrak{B} = \bigoplus_{i=1}^{k} \psi_i$. Set $\pi_{ij} = \rho_{ij}(I_{m_i} \otimes q_j)$ and $\alpha_i = \psi_i(q_i)$; then the equation (1) says

$$\begin{bmatrix} \alpha_1 & \cdots & \alpha_k \\ \pi_{11} & \cdots & \pi_{1k} \\ \vdots & \ddots & \vdots \\ \pi_{k1} & \cdots & \pi_{kk} \end{bmatrix} = \begin{bmatrix} \alpha_1 & \cdots & \alpha_k \end{bmatrix}.$$  

The unitality of $\hat{E}$ means that the matrix $[\pi_{ij}]$ is a stochastic matrix. The faithfullness of $\hat{E}$ guarantees $\pi_{ij} > 0$ for all $1 \leq i, j \leq k$. Hence, $\{\alpha_i\}$ is uniquely determined by $\{\pi_{ij}\}$ from the Perron-Frobenius theorem. So, by (1), $\{\psi_i\}$ is also uniquely determined by $\{\rho_{ij}\}$.

By Proposition 2.4, we get the next corollary.

Corollary 2.5 Let $S_i$ and $T_{ij}$ be the density matrices of $\psi_i$ and $\rho_{ij}$, respectively. Then the density matrix $\hat{D}_n$ of $\phi|\mathfrak{A}_{[1,n-1]} \otimes \mathfrak{B}$ is

$$\bigoplus_{i_1, \ldots, i_n} S_{i_1} \otimes T_{i_1i_2} \otimes \cdots \otimes T_{i_{n-1}i_n}.$$ 

In the above the density matrix $\hat{D}_n$ is taken with respect to the usual trace on $\mathfrak{A}_{[1,n-1]} \otimes \mathfrak{B}$, i.e. the trace having the value 1 for each rank one projection. If all summands of $\mathfrak{B}$ are of multiplicity one, then the density in the above corollary is actually the density matrix $D_n$ of $\phi_{[1,n]}$. Hence, the densities $D_n$ are all commuting in this case (see [6]).

Consider the case $d = 2$. Subalgebras $\mathfrak{B}$ of $M_2$ are $M_2$ or $\mathbb{C} \oplus \mathbb{C}$ or $\mathbb{C}$. If $\mathfrak{B}$ is $M_2$ or $\mathbb{C}$, any translation-invariant Markov state relative to $\mathfrak{B}$ is a product state. If $\mathfrak{B}$ is $\mathbb{C} \oplus \mathbb{C}$, all summands of $\mathfrak{B}$ are of multiplicity one. Hence, the density matrices $D_n$'s are commuting in this case (see [3]).

The following is the simplest example where the densities $\{D_n\}$ are non-commuting.
Example 2.6 We set $d = 3$ and $\mathcal{B} = \mathbb{C} \oplus \mathbb{C} = \mathbb{C}(e_{11} + e_{22}) + \mathbb{C}e_{33}$, where $e_{ij}$ $(1 \leq i, j \leq 3)$ are the matrix unit of $M_3$. Assume that the density matrix of $\bigoplus_{i,j=1}^{3} \rho_{ij}$ is $T_{11} \oplus T_{12} \oplus T_{21} \oplus T_{22} = A_1 \otimes A_2 \oplus c_1 \oplus c_2 \in M_2 \oplus M_2 \oplus \mathbb{C} \oplus \mathbb{C}$, where $A_1, A_2 \in M_2$, $A_1, A_2 \geq 0$, $\text{Tr}(A_1 + A_2) = 1$, and $c_1, c_2 \in \mathbb{R}^+$, $c_1 + c_2 = 1$. We define

$$\psi_1(e_{11} + e_{22}) = \frac{c_1}{c_1 + \text{Tr}(A_2)}, \quad \psi_2(e_{33}) = \frac{\text{Tr}(A_2)}{c_1 + \text{Tr}(A_2)},$$

then it is easily seen that the condition (1) is satisfied. In this case, the density matrix $D_n$ of $\mathfrak{A}_{[1,n]}$ is

$$\bigoplus_{i_1, \ldots, i_{n-1}} S_{i_1} \otimes T_{i_1i_2} \otimes \cdots \otimes T_{i_{n-2}i_{n-1}} \otimes (T_{i_{n-1}1} \otimes (A_1 + A_2) \oplus T_{n-1} \cdot 2 \otimes (c_1 + c_2)).$$

So, $D_n$'s are non-commuting if so are $A_1$ and $A_2$.

3 Disintegration of quantum Markov states

In this section, we survey the explicit form of Markov states due to [4]. Let $\phi$ be a translation-invariant Markov state as in Section 2. We put $\Omega_n = \{1, \ldots k\}$.

$$\Omega = \prod_{n \in \mathbb{N}} \Omega_n$$

and

$$(x_1, x_2, \ldots, x_n) = \{(y_1, y_2, \ldots) \in \Omega \mid y_i = x_i, 1 \leq i \leq n\}.$$ 

We define the measure $\nu$ on $\Omega$ by

$$\nu((x_1, x_2, \ldots, x_n)) = \phi(q_{x_1} \otimes q_{x_2} \otimes \cdots q_{x_n})$$

$$= \alpha_{x_1} \cdot \prod_{i=1}^{n-1} \pi_{x_i x_{i+1}}.$$ 

Then $\nu$ is a probability measure on $\Omega$. For an arbitrary element $\omega = (\omega_1, \omega_2, \ldots) \in \Omega$, we set

$$\mathcal{B}_\omega = M_{d_{\omega_1}} \otimes M_{d_{\omega_2}} \cdots$$

and the state $\psi_\omega$ on $\mathcal{B}_\omega$ by

$$\psi_\omega = \tilde{\psi}_{\omega_1} \otimes \bigotimes_{i=1}^{\infty} \tilde{\rho}_{\omega_i \omega_{i+1}},$$

where $\tilde{\psi}_i = \psi_i/\alpha_i$ and $\tilde{\rho}_{ij} = \rho_{ij}/\pi_{ij}$. Let $E_\omega : \mathfrak{A} \rightarrow \mathcal{B}_\omega$ be a completely positive map defined by

$$E_\omega(A_1 \otimes A_2 \otimes \cdots \otimes A_n) = q_{\omega_1} A_1 q_{\omega_1} \otimes \cdots \otimes q_{\omega_n} A_n q_{\omega_n},$$

for any $A_1, \ldots, A_n \in M_d$, and $\phi_\omega = \psi_\omega \circ E_\omega.$
Theorem 3.1 Define $\Omega$, $\nu$ and $\phi_\omega$ as above. Then
\[
\phi = \int_\Omega \phi_\omega \nu(d\omega).
\]

Proof. If $\omega, \omega' \in (x_1, \ldots, x_n)$, then
\[
\begin{align*}
\phi_\omega(A_1 \otimes \cdots \otimes A_{n-1}) &= \psi_\omega(q_{x_1}A_{x_1} \otimes \cdots \otimes q_{x_n}A_{n-1}q_{x_n}) \\
&= \frac{1}{\nu((x_1, \ldots, x_n))} \cdot \left(\psi_{x_1} \otimes \bigotimes_{i=1}^{n-1} \rho_{x_i}A_{x_i+1}\right) \cdot (q_{x_1}A_{x_1} \otimes \cdots \otimes q_{x_{n-1}}A_{n-1}q_{x_{n-1}}) \\
&= \phi_{\omega'}(A_1 \otimes \cdots \otimes A_{n-1})
\end{align*}
\]
for any $A_1, \ldots, A_{n-1} \in M_d$. Therefore,
\[
\begin{align*}
\phi(A_1 \otimes \cdots \otimes A_{n-1}) &= \phi \circ E_\epsilon(A_1 \otimes \cdots \otimes A_{n-1}) \\
&= \sum_{x_1, \ldots, x_n} \left(\psi_{x_1} \otimes \bigotimes_{i=1}^{n-1} \rho_{x_i}A_{x_i+1}\right) \cdot (q_{x_1}A_{x_1} \otimes \cdots \otimes q_{x_{n-1}}A_{n-1}q_{x_{n-1}}) \\
&= \sum_{x_1, \ldots, x_n} \nu((x_1, \ldots, x_n)) \cdot \phi_{\omega(x_1, \ldots, x_n)}(A_1 \otimes \cdots \otimes A_{n-1}) \\
&= \int_\Omega \phi_\omega(A_1 \otimes \cdots \otimes A_{n-1}) \nu(d\omega),
\end{align*}
\]
where $\omega(x_1, \ldots, x_n)$ is an arbitraly element in $(x_1, \ldots, x_n)$.

References


