

# Algebraic independence properties of the values of Hecke-Mahler series and its derivatives

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## 1 Introduction

This article is based on [11], joint work with Professor Taka-aki Tanaka. Let  $\omega$  be a real number. We denote by  $[x]$  the integral part of the real number  $x$ , namely the largest integer not exceeding  $x$ . Hecke-Mahler series, the generating function of the sequence  $\{[k\omega]\}_{k=1}^{\infty}$ , is defined by

$$h_{\omega}(z) = \sum_{k=1}^{\infty} [k\omega] z^k,$$

where  $z$  is complex with  $|z| < 1$ . Hecke [2] proved that, if  $\omega$  is an irrational number, then  $h_{\omega}(z)$  has the unit circle  $|z| = 1$  as its natural boundary. Mahler [5] proved that, if  $\omega$  is a quadratic irrational number, then the value  $h_{\omega}(\alpha)$  is transcendental, where  $\alpha$  is a nonzero algebraic number inside the unit circle.

In what follows, let  $\omega$  be a real quadratic irrational number. We denote by  $h_{\omega}^{(l)}(z)$  the derivative of  $h_{\omega}(z)$  of order  $l$ . Nishioka proved the algebraic independence of the values of  $h_{\omega}(z)$  and its derivative of any order at any fixed nonzero algebraic number inside the unit circle.

**Theorem 1** (Nishioka [8]). *If  $\alpha$  is an algebraic number with  $0 < |\alpha| < 1$ , then the infinite set of the values  $\{h_{\omega}^{(l)}(\alpha) \mid l \geq 0\}$  is algebraically independent.*

On the other hand, Masser proved the algebraic independence of the values of  $h_{\omega}(z)$  at any nonzero distinct algebraic numbers inside the unit circle.

**Theorem 2** (Masser [6]). *The infinite set of the values  $\{h_{\omega}(\alpha) \mid \alpha \in \overline{\mathbb{Q}}, 0 < |\alpha| < 1\}$  is algebraically independent.*

We denote by  $\omega'$  the conjugate of the real quadratic irrational number  $\omega$ . The following is the main theorem of this article.

**Theorem 3** (with Tanaka [11]). *Suppose that  $\omega$  satisfies  $|\omega - \omega'| > 2$ . Then the infinite set of the values  $\{h_\omega^{(l)}(\alpha) \mid l \geq 0, \alpha \in \overline{\mathbb{Q}}, 0 < |\alpha| < 1\}$  is algebraically independent.*

**Corollary 1.** *Suppose that  $\omega$  is an algebraic integer. Then the infinite set of the values  $\{h_\omega^{(l)}(\alpha) \mid l \geq 0, \alpha \in \overline{\mathbb{Q}}, 0 < |\alpha| < 1\}$  is algebraically independent.*

**Corollary 2.** *Let  $m > 1$  be a square-free integer and  $r$  a rational number. Put  $\omega = r\sqrt{m}$ . If  $|\omega| > 1$ , then the infinite set of the values  $\{h_\omega^{(l)}(\alpha) \mid l \geq 0, \alpha \in \overline{\mathbb{Q}}, 0 < |\alpha| < 1\}$  is algebraically independent.*

For more general irrational number  $\omega$ , some results on the arithmetic properties of the values of  $h_\omega(z)$  can be found in for example [3], [7], [1].

In the next section we consider the case where  $\omega$  is rational. In Section 3 we give the sketch of the proof of Theorem 3.

## 2 On the case where $\omega$ is rational

For any positive number  $\omega$ , we define

$$H_\omega(z_1, z_2) = \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{[k_1\omega]} z_1^{k_1} z_2^{k_2}.$$

As mentioned in the previous section, if  $\omega$  is an irrational number, then  $h_\omega(z)$  is transcendental over  $\mathbb{C}(z)$ . In the rest of this section, let  $\omega$  be a rational number, not necessarily positive. We assume that  $\omega$  is expanded in the finite continued fraction

$$\omega = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_r}}}} =: [a_0; a_1, a_2, \dots, a_r],$$

where  $a_0 = [\omega]$  and  $a_1, \dots, a_r$  are positive integers. We denote by  $\{x\}$  the fractional part of the real number  $x$ . Put  $\chi = \{\omega\} = [0; a_1, a_2, \dots, a_r]$ . Define positive integers  $s_\mu, t_\mu$  ( $0 \leq \mu \leq r-1$ ) by  $\chi = s_0/t_0$ ,

$$\frac{s_\mu}{t_\mu} = \frac{1}{a_{\mu+1} + s_{\mu+1}/t_{\mu+1}} \quad (0 \leq \mu \leq r-2)$$

and  $s_{r-1}/t_{r-1} = 1/a_r$  with  $s_\mu$  and  $t_\mu$  relatively prime for any  $\mu$ . Define positive integers  $p_\mu, q_\mu$  ( $0 \leq \mu \leq r$ ) by

$$\begin{pmatrix} p_\mu & q_\mu \\ p_{\mu-1} & q_{\mu-1} \end{pmatrix} = \begin{pmatrix} a_\mu & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}.$$

For any positive integer  $a$ , we have

$$\begin{aligned}
H_{a+\omega}(z_1, z_2) &= \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{[k_1 a + k_1 \omega]} z_1^{k_1} z_2^{k_2} \\
&= \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{k_1 a + [k_1 \omega]} z_1^{k_1} z_2^{k_2} \\
&= \sum_{k_1=1}^{\infty} \left( z_1^{k_1} \sum_{k_2=1}^{a k_1} z_2^{k_2} + (z_1 z_2^a)^{k_1} \sum_{k_2=1}^{[k_1 \omega]} z_2^{k_2} \right) \\
&= \sum_{k_1=1}^{\infty} z_1^{k_1} \frac{z_2 - z_2^{a k_1 + 1}}{1 - z_2} + H_{\omega}(z_1 z_2^a, z_2) \\
&= \frac{z_1 z_2}{(1 - z_1)(1 - z_2)} - \frac{z_1 z_2^{a+1}}{(1 - z_2)(1 - z_1 z_2^a)} + H_{\omega}(z_1 z_2^a, z_2). \tag{1}
\end{aligned}$$

For any rational number  $p/q$ , where  $p, q$  are relatively prime positive integers,

$$\begin{aligned}
H_{p/q}(z_1, z_2) + H_{q/p}(z_2, z_1) &= \sum_{\substack{k_1 \geq 1, k_2 \geq 1 \\ k_2 \leq k_1 p/q}} z_1^{k_1} z_2^{k_2} + \sum_{\substack{k_1 \geq 1, k_2 \geq 1 \\ k_2 \leq k_1 q/p}} z_2^{k_1} z_1^{k_2} \\
&= \sum_{\substack{k_1 \geq 1, k_2 \geq 1 \\ k_2 \leq k_1 p/q}} z_1^{k_1} z_2^{k_2} + \sum_{\substack{k_1 \geq 1, k_2 \geq 1 \\ k_1 \leq k_2 p/q}} z_1^{k_2} z_2^{k_1} \\
&= \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} z_1^{k_1} z_2^{k_2} + \sum_{k=1}^{\infty} (z_1^q z_2^p)^k \\
&= \frac{z_1 z_2}{(1 - z_1)(1 - z_2)} + \frac{z_1^q z_2^p}{1 - z_1^q z_2^p}. \tag{2}
\end{aligned}$$

Hence

$$\begin{aligned}
H_{s_{\mu}/t_{\mu}}(z_1, z_2) &= \frac{z_1 z_2}{(1 - z_1)(1 - z_2)} + \frac{z_1^{t_{\mu}} z_2^{s_{\mu}}}{1 - z_1^{t_{\mu}} z_2^{s_{\mu}}} - H_{a_{\mu+1} + s_{\mu+1}/t_{\mu+1}}(z_2, z_1) \\
&= \frac{z_1^{a_{\mu+1}+1} z_2}{(1 - z_1)(1 - z_1^{a_{\mu+1}} z_2)} + \frac{z_1^{t_{\mu}} z_2^{s_{\mu}}}{1 - z_1^{t_{\mu}} z_2^{s_{\mu}}} - H_{s_{\mu+1}/t_{\mu+1}}(z_1^{a_{\mu+1}} z_2, z_1).
\end{aligned}$$

Therefore, noting the definition of  $p_\mu$  and  $q_\mu$  ( $0 \leq \mu \leq 2$ ), we see that

$$\begin{aligned}
H_\chi(z_1, z_2) &= H_{s_0/t_0}(z_1, z_2) \\
&= \frac{z_1^{a_1+1} z_2}{(1-z_1)(1-z_1^{a_1} z_2)} + \frac{z_1^{t_0} z_2^{s_0}}{1-z_1^{t_0} z_2^{s_0}} - H_{s_1/t_1}(z_1^{a_1} z_2, z_1) \\
&= \frac{z_1^{p_1+p_0} z_2^{q_1}}{(1-z_1^{p_0})(1-z_1^{p_1} z_2^{q_1})} + \frac{z_1^{t_0} z_2^{s_0}}{1-z_1^{t_0} z_2^{s_0}} - H_{s_1/t_1}(z_1^{p_1} z_2^{q_1}, z_1^{p_0}) \\
&= \frac{z_1^{p_1+p_0} z_2^{q_1}}{(1-z_1^{p_0})(1-z_1^{p_1} z_2^{q_1})} + \frac{z_1^{t_0} z_2^{s_0}}{1-z_1^{t_0} z_2^{s_0}} - \frac{(z_1^{p_1} z_2^{q_1})^{a_2+1} z_1^{p_0}}{(1-z_1^{p_1} z_2^{q_1})(1-(z_1^{p_1} z_2^{q_1})^{a_2} z_1^{p_0})} \\
&\quad - \frac{(z_1^{p_1} z_2^{q_1})^{t_1} z_1^{p_0 s_1}}{1-(z_1^{p_1} z_2^{q_1})^{t_1} z_1^{p_0 s_1}} + H_{s_2/t_2}((z_1^{p_1} z_2^{q_1})^{a_2} z_1^{p_0}, z_1^{p_1} z_2^{q_1}) \\
&= \frac{z_1^{p_1+p_0} z_2^{q_1}}{(1-z_1^{p_0})(1-z_1^{p_1} z_2^{q_1})} + \frac{z_1^{t_0} z_2^{s_0}}{1-z_1^{t_0} z_2^{s_0}} - \frac{z_1^{p_2+p_1} z_2^{q_2+q_1}}{(1-z_1^{p_1} z_2^{q_1})(1-z_1^{p_2} z_2^{q_2})} \\
&\quad - \frac{z_1^{p_1 t_1 + p_0 s_1} z_2^{q_1 t_1}}{1-z_1^{p_1 t_1 + p_0 s_1} z_2^{q_1 t_1}} + H_{s_2/t_2}(z_1^{p_2} z_2^{q_2}, z_1^{p_1} z_2^{q_1}).
\end{aligned}$$

Continuing this process, we see that

$$\begin{aligned}
&H_\chi(z_1, z_2) \\
&= \sum_{\mu=0}^{r-2} (-1)^\mu \left( \frac{z_1^{p_{\mu+1}+p_\mu} z_2^{q_{\mu+1}+q_\mu}}{(1-z_1^{p_{\mu+1}} z_2^{q_{\mu+1}})(1-z_1^{p_\mu} z_2^{q_\mu})} + \frac{z_1^{p_\mu t_\mu + p_{\mu-1} s_\mu} z_2^{q_\mu t_\mu + q_{\mu-1} s_\mu}}{1-z_1^{p_\mu t_\mu + p_{\mu-1} s_\mu} z_2^{q_\mu t_\mu + q_{\mu-1} s_\mu}} \right) \\
&\quad + (-1)^{r-1} H_{s_{r-1}/t_{r-1}}(z_1^{p_{r-1}} z_2^{q_{r-1}}, z_1^{p_{r-2}} z_2^{q_{r-2}}).
\end{aligned}$$

Since

$$\begin{aligned}
H_{s_{r-1}/t_{r-1}}(z_1, z_2) &= \frac{z_1 z_2}{(1-z_1)(1-z_2)} + \frac{z_1^{t_{r-1}} z_2^{s_{r-1}}}{1-z_1^{t_{r-1}} z_2^{s_{r-1}}} - H_{a_r}(z_2, z_1) \\
&= \frac{z_1^{a_r+1} z_2}{(1-z_1)(1-z_1^{a_r} z_2)} + \frac{z_1^{t_{r-1}} z_2^{s_{r-1}}}{1-z_1^{t_{r-1}} z_2^{s_{r-1}}}
\end{aligned}$$

by (2),  $s_{r-1}/t_{r-1} = 1/a_r$  and (1) with  $\omega = 0$ , we have

$$\begin{aligned}
&H_\chi(z_1, z_2) \\
&= \sum_{\mu=0}^{r-1} (-1)^\mu \left( \frac{z_1^{p_{\mu+1}+p_\mu} z_2^{q_{\mu+1}+q_\mu}}{(1-z_1^{p_{\mu+1}} z_2^{q_{\mu+1}})(1-z_1^{p_\mu} z_2^{q_\mu})} + \frac{z_1^{p_\mu t_\mu + p_{\mu-1} s_\mu} z_2^{q_\mu t_\mu + q_{\mu-1} s_\mu}}{1-z_1^{p_\mu t_\mu + p_{\mu-1} s_\mu} z_2^{q_\mu t_\mu + q_{\mu-1} s_\mu}} \right).
\end{aligned}$$

Noting that

$$H_\chi(z, 1) = h_\chi(z) = h_\omega(z) - \frac{a_0 z}{(1-z)^2}$$

by  $\chi = \omega - a_0$ , we see that

$$h_\omega(z) = \frac{a_0 z}{(1-z)^2} + \sum_{\mu=0}^{r-1} (-1)^\mu \left( \frac{z^{p_{\mu+1}+p_\mu}}{(1-z^{p_{\mu+1}})(1-z^{p_\mu})} + \frac{z^{p_\mu t_\mu + p_{\mu-1} s_\mu}}{1-z^{p_\mu t_\mu + p_{\mu-1} s_\mu}} \right) \in \mathbb{Q}(z).$$

Hence we see that  $h_\omega(z)$  is a rational function if  $\omega$  is rational.

### 3 Proof of Theorem 3

Let  $\Omega = (\omega_{ij})$  be an  $n \times n$  matrix with nonnegative integer entries. For  $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$ , we define a multiplicative transformation  $\Omega : \mathbb{C}^n \rightarrow \mathbb{C}^n$  by

$$\Omega \mathbf{z} = \left( \prod_{j=1}^n z_j^{\omega_{1j}}, \prod_{j=1}^n z_j^{\omega_{2j}}, \dots, \prod_{j=1}^n z_j^{\omega_{nj}} \right). \quad (3)$$

Then the iterates  $\Omega^k \mathbf{z}$  ( $k = 0, 1, 2, \dots$ ) are well-defined.

For any positive irrational number  $\chi$ , we see that

$$\begin{aligned} H_\chi(z_1, z_2) + H_{1/\chi}(z_2, z_1) &= \sum_{\substack{h_1 \geq 1, h_2 \geq 1 \\ h_2 < h_1 \chi}} z_1^{h_1} z_2^{h_2} + \sum_{\substack{h_1 \geq 1, h_2 \geq 1 \\ h_1 > h_2 \chi}} z_1^{h_2} z_2^{h_1} \\ &= \frac{z_1}{1 - z_1} \frac{z_2}{1 - z_2}. \end{aligned} \quad (4)$$

Let  $D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  and  $E(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$  for any positive integer  $a$ . Define  $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \chi = (p\chi + q)/(r\chi + s)$ , where  $p, q, r, s$  are nonnegative integers. Then we see that

$$H_{D\chi}(z_1, z_2) \equiv -H_\chi(D(z_1, z_2)) \pmod{\mathbb{Q}(z_1, z_2)} \quad (5)$$

and that

$$H_{E(a)\chi}(z_1, z_2) \equiv H_\chi(E(a)(z_1, z_2)) \pmod{\mathbb{Q}(z_1, z_2)} \quad (6)$$

by (4) and (1), respectively, where  $D(z_1, z_2)$  and  $E(a)(z_1, z_2)$  are defined by (3).

**Sketch of the proof of Theorem 3.** Since

$$\begin{aligned} h_\omega(z) + h_{-\omega}(z) &= \sum_{k=1}^{\infty} [k\omega] z^k + \sum_{k=1}^{\infty} [-k\omega] z^k \\ &= \sum_{k=1}^{\infty} [k\omega] z^k + \sum_{k=1}^{\infty} (-[k\omega] - 1) z^k = -\frac{z}{1-z}, \end{aligned}$$

we see that the algebraic independency of  $\{h_\omega^{(l)}(\alpha) \mid l \geq 0, \alpha \in \overline{\mathbb{Q}}, 0 < |\alpha| < 1\}$  is equivalent to that of  $\{h_{-\omega}^{(l)}(\alpha) \mid l \geq 0, \alpha \in \overline{\mathbb{Q}}, 0 < |\alpha| < 1\}$ . Hence, considering  $-\omega$  instead of  $\omega$  if necessary, we may assume that  $\omega > \omega'$ . Since  $|\omega - \omega'| > 2$ , there exists an integer  $a_0$  such that  $0 < \omega - a_0 < 1$  and  $\omega' - a_0 < -1$ . Let  $\chi = \omega - a_0$ . By

$$h_\chi(z) = \sum_{k=1}^{\infty} [k(\omega - a_0)] z^k = h_\omega(z) - \frac{a_0 z}{(1-z)^2},$$

we may consider  $\chi$  instead of  $\omega$  by the same reason as above. Then  $\chi$  is reduced and so expanded in a purely periodic continued fraction as follows:

$$\chi = [0; a_1, a_2, \dots] = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}}$$

where  $a_1, a_2, \dots$  are positive integers. Let  $\nu$  be its even period. Then

$$\chi = [0; a_1, a_2, \dots, a_\nu, \chi] = DE(a_1)DE(a_2) \cdots DE(a_\nu)\chi.$$

Let  $T^{(1)} = E(a_\nu)DE(a_{\nu-1})D \cdots E(a_1)D$ . Then by (5) and (6) we have

$$H_\chi(z_1, z_2) \equiv H_\chi(T^{(1)}(z_1, z_2)) \pmod{\mathbb{Q}(z_1, z_2)}.$$

Let  $\alpha_1, \dots, \alpha_n$  be any nonzero distinct algebraic numbers with  $|\alpha_1|, \dots, |\alpha_n| < 1$ . It is enough to show that  $\{h_\chi^{(l)}(\alpha_i) \mid 0 \leq l \leq L, 1 \leq i \leq n\}$  is algebraically independent for any sufficiently large  $L$ . Let  $g_l(z) = \sum_{k=1}^{\infty} k^l [k\chi] z^k$ . Then the algebraic independency of  $\{h_\chi^{(l)}(\alpha_i) \mid 0 \leq l \leq L, 1 \leq i \leq n\}$  is equivalent to that of  $\{g_l(\alpha_i) \mid 0 \leq l \leq L, 1 \leq i \leq n\}$ . For the  $\alpha_1, \dots, \alpha_n$ , there exist multiplicatively independent algebraic numbers  $\beta_1, \dots, \beta_m$  with  $0 < |\beta_j| < 1$  ( $1 \leq j \leq m$ ) such that

$$\alpha_i = \zeta_i \prod_{j=1}^m \beta_j^{\ell_{ij}} \quad (1 \leq i \leq n),$$

where  $\zeta_i$  ( $1 \leq i \leq n$ ) are roots of unity and  $\ell_{ij}$  ( $1 \leq i \leq n, 1 \leq j \leq m$ ) are nonnegative integers (cf. [4, Lemma 3]). We define

$$T^{(m)} = \text{diag} \left( \underbrace{T^{(1)}, \dots, T^{(1)}}_m \right).$$

Let  $\mathbf{x} = (x_1, \dots, x_m)$ ,  $\mathbf{y} = (y_1, \dots, y_m)$  be variables. Let

$$\mathbf{z}_0 = (\beta_1, 1, \beta_2, 1, \dots, \beta_m, 1). \quad (7)$$

and  $M_i(\mathbf{x}) = x_1^{\ell_{i1}} \cdots x_m^{\ell_{im}}$ . Define

$$\begin{aligned} G_i(\mathbf{z}) &= G(\zeta_i, M_i, \mathbf{z}) := H_\chi(\zeta_i M_i(\mathbf{x}), M_i(\mathbf{y})) \\ &= \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{[k_1\chi]} (\zeta_i M_i(\mathbf{x}))^{k_1} M_i(\mathbf{y})^{k_2} \quad (1 \leq i \leq n), \end{aligned} \quad (8)$$

where  $\mathbf{z} = (x_1, y_1, x_2, y_2, \dots, x_m, y_m)$ . By (8) we see that

$$D_{j_i}^l G_i(\mathbf{z}_0) = \sum_{h=1}^{\infty} \ell_{ij_i}^l h^l [h\chi] \alpha_i^h,$$

where  $\ell_{ij_i} > 0$ . Hence the algebraic independency of  $\{g_l(\alpha_i) \mid 0 \leq l \leq L, 1 \leq i \leq n\}$  is equivalent to that of  $\{D_{j_i}^l G_i(\mathbf{z}_0) \mid 0 \leq l \leq L, 1 \leq i \leq n\}$ .

Similarly to  $H_\chi$ , each  $G_i$  satisfies a functional equation:

**Lemma 1** (Masser [6, Lemma 3.3]). *There exists a positive power  $T$  of  $T^{(m)}$  such that*

$$G_i(\mathbf{z}) \equiv G_i(T\mathbf{z}) \pmod{\overline{\mathbb{Q}}(\mathbf{z})}$$

for any  $i$  ( $1 \leq i \leq n$ ).

The matrix  $T$  in Lemma 1 can be written as

$$T = \text{diag} \left( \underbrace{\left( \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}, \dots, \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \right)}_m \right).$$

Let  $D_j = x_j \partial / \partial x_j$  and  $D'_j = y_j \partial / \partial y_j$  ( $1 \leq j \leq m$ ). Since

$$\begin{aligned} D_j G_i(\mathbf{z}) &\equiv x_j \frac{\partial G_i}{\partial x_j}(T\mathbf{z}) t_{11} x_j^{t_{11}-1} y_j^{t_{12}} + x_j \frac{\partial G_i}{\partial y_j}(T\mathbf{z}) t_{21} x_j^{t_{21}-1} y_j^{t_{22}} \pmod{\overline{\mathbb{Q}}(\mathbf{z})} \\ &\equiv t_{11} D_j G_i(T\mathbf{z}) + t_{21} D'_j G_i(T\mathbf{z}) \pmod{\overline{\mathbb{Q}}(\mathbf{z})} \end{aligned}$$

and

$$\begin{aligned} D'_j G_i(\mathbf{z}) &\equiv y_j \frac{\partial G_i}{\partial x_j}(T\mathbf{z}) t_{12} x_j^{t_{11}} y_j^{t_{12}-1} + y_j \frac{\partial G_i}{\partial y_j}(T\mathbf{z}) t_{22} x_j^{t_{21}} y_j^{t_{22}-1} \pmod{\overline{\mathbb{Q}}(\mathbf{z})} \\ &\equiv t_{12} D_j G_i(T\mathbf{z}) + t_{22} D'_j G_i(T\mathbf{z}) \pmod{\overline{\mathbb{Q}}(\mathbf{z})} \end{aligned}$$

for  $1 \leq i \leq n$ ,  $1 \leq j \leq m$ , we see that  $D_1^{k_1} D'_1{}^{k'_1} \dots D_m^{k_m} D'_m{}^{k'_m} G_i(\mathbf{z})$  ( $0 \leq k_1, k'_1, \dots, k_m, k'_m \leq L$ ,  $1 \leq i \leq n$ ) satisfy a system of functional equations of the form

$$\begin{aligned} &\begin{pmatrix} G_i(\mathbf{z}) \\ D_1 G_i(\mathbf{z}) \\ \vdots \\ (D_1 D'_1 \dots D_m D'_m)^L G_i(\mathbf{z}) \end{pmatrix} \\ &\equiv A \begin{pmatrix} G_i(T\mathbf{z}) \\ D_1 G_i(T\mathbf{z}) \\ \vdots \\ (D_1 D'_1 \dots D_m D'_m)^L G_i(T\mathbf{z}) \end{pmatrix} \pmod{(\overline{\mathbb{Q}}(\mathbf{z}))^{(L+1)2m}} \end{aligned}$$

for  $1 \leq i \leq n$  and for any  $L \geq 0$ , where  $A$  is an  $(L+1)^{2m} \times (L+1)^{2m}$  matrix with rational entries. In order to prove the algebraic independency of  $\{D_j^l G_i(\mathbf{z}_0) \mid 0 \leq l \leq L, 1 \leq i \leq n\}$ , we use the following criterion:

**Lemma 2** (Nishioka [9]). *Let  $K$  be an algebraic number field. Suppose that  $f_1(\mathbf{z}), \dots, f_M(\mathbf{z}) \in K[[z_1, \dots, z_N]]$  converge in an  $N$ -polydisc  $U$  around the origin of  $\mathbb{C}^N$  and satisfy the system of functional equations of the form*

$$\begin{pmatrix} f_1(\mathbf{z}) \\ \vdots \\ f_M(\mathbf{z}) \end{pmatrix} = A \begin{pmatrix} f_1(\Omega\mathbf{z}) \\ \vdots \\ f_M(\Omega\mathbf{z}) \end{pmatrix} + \begin{pmatrix} b_1(\mathbf{z}) \\ \vdots \\ b_M(\mathbf{z}) \end{pmatrix},$$

where  $A$  is an  $M \times M$  matrix with entries in  $K$  and  $b_i(\mathbf{z}) \in K(z_1, \dots, z_N)$  ( $1 \leq i \leq M$ ). Let  $\alpha$  be a point in  $U$  whose components are nonzero algebraic numbers. Assume that  $\Omega$  and  $\alpha$  satisfy suitable conditions. Then, if  $f_1(\mathbf{z}), \dots, f_r(\mathbf{z})$  ( $r \leq M$ ) are linearly independent over  $K$  modulo  $K(z_1, \dots, z_N)$ , then  $f_1(\alpha), \dots, f_r(\alpha)$  are algebraically independent.

We can find that the matrix  $T$  and the point  $\mathbf{z}_0$  satisfy the condition in above lemma. Therefore it suffices to show that  $\{D_{j_i}^l G_i(\mathbf{z}) \mid 0 \leq l \leq L, 1 \leq i \leq n\}$  is linearly independent over  $\overline{\mathbb{Q}}$  modulo  $\overline{\mathbb{Q}}(\mathbf{z})$ .

On the contrary, we assume that  $\{D_{j_i}^l G_i(\mathbf{z}) \mid 0 \leq l \leq L, 1 \leq i \leq n\}$  is linearly dependent over  $\overline{\mathbb{Q}}$  modulo  $\overline{\mathbb{Q}}(\mathbf{z})$ . Then there exist algebraic integers  $\lambda_{il}$  ( $1 \leq i \leq n, 0 \leq l \leq L$ ), not all zero, and a rational function  $R(\mathbf{z}) \in \overline{\mathbb{Q}}(\mathbf{z})$  such that

$$\sum_{i=1}^n \sum_{l=0}^L \lambda_{il} D_{j_i}^l G_i(\mathbf{z}) = R(\mathbf{z}). \quad (9)$$

Substituting 1 into  $y_1, y_2, \dots, y_m$ , we obtain

$$\sum_{i=1}^n \sum_{l=0}^L \lambda_{il} \ell_{ij_i}^l \sum_{h=1}^{\infty} h^l [h\chi](\zeta_i M_i(\mathbf{x}))^h = R'(x_1, x_2, \dots, x_m) \in \overline{\mathbb{Q}}(\mathbf{x}).$$

We take a sufficiently large positive integer  $t$  and attempt a specialization of the form

$$\mathbf{x} = (w^t, w^{t^2}, \dots, w^{t^m})$$

for a single variable  $w$ . Let  $t_i = \sum_{j=1}^m \ell_{ij} t^j$  ( $1 \leq i \leq n$ ). Then

$$w^{t_i} = M_i(w^t, w^{t^2}, \dots, w^{t^m}).$$

We take  $t$  so large that, if  $M_i \neq M_j$ , then  $t_i \neq t_j$  ( $1 \leq i < j \leq n$ ) and that the denominator of  $R^*(w) := R'(w^t, w^{t^2}, \dots, w^{t^m}) \in \overline{\mathbb{Q}}(w)$  does not vanish. Let  $\lambda'_{il} = \lambda_{il} \ell_{ij_i}^l$  ( $1 \leq i \leq n, 0 \leq l \leq L$ ). Then we have

$$\sum_{i=1}^n \sum_{l=0}^L \lambda'_{il} \sum_{h=1}^{\infty} h^l [h\chi](\zeta_i w^{t_i})^h = \sum_{k=0}^{\infty} a_k w^k = R^*(w),$$

where

$$\begin{aligned} a_k &= \sum_{\substack{1 \leq i \leq n \\ t_i | k}} \sum_{l=0}^L \lambda'_{il} \frac{k^l}{t_i^l} \left[ \frac{k\chi}{t_i} \right] \zeta_i^{k/t_i} \\ &= \sum_{\substack{1 \leq i \leq n \\ t_i | k}} \sum_{l=0}^L \lambda'_{il} \frac{k^l}{t_i^l} \left( \frac{k\chi}{t_i} - \left\{ \frac{k\chi}{t_i} \right\} \right) \zeta_i^{k/t_i} \end{aligned}$$

and  $\{\cdot\}$  denotes the fractional part. Since  $R^*(w) \in \overline{\mathbb{Q}}(w)$  and since  $a_k$  ( $k \geq 0$ ) are algebraic integers, we can find

$$a_k = P_1(k)\xi_1^k + \cdots + P_M(k)\xi_M^k \quad (k \geq k_0), \quad (10)$$

where  $k_0$  is a sufficiently large integer,  $P_1(x), \dots, P_M(x) \in \overline{\mathbb{Q}}[x]$  and  $\xi_1, \dots, \xi_M$  are algebraic integers. Then by  $a_k = O(k^{L+1})$ , we see that  $\xi_1, \dots, \xi_M$  are roots of unity (cf. [10, proof of Theorem 3.4.8]). Let  $N$  be a positive integer such that  $\xi_1^N = \cdots = \xi_M^N = \zeta_1^N = \cdots = \zeta_n^N = 1$ . Let  $\{t'_1, \dots, t'_r\}$  be the maximum subset of  $\{t_1, \dots, t_n\}$  with  $t'_i \neq t'_j$  ( $1 \leq i < j \leq r$ ). Let  $T_i = \{j \mid t_j = t'_i\}$  ( $1 \leq i \leq r$ ). Then  $\zeta_j$  ( $j \in T_i$ ) are distinct for each  $i$ , since  $\alpha_1, \dots, \alpha_n$  are distinct. Put  $s = t'_1 \cdots t'_r N$  and  $s_i = s/t'_i$  ( $1 \leq i \leq r$ ). Noting that  $\{1, \dots, n\}$  is a disjoint union of  $T_1, \dots, T_r$ , for any  $k \geq k_0$  and for any fixed positive integer  $h$ , we see that

$$\begin{aligned} a_{ks+h} &= \sum_{\substack{1 \leq i \leq r \\ t'_i | h}} \sum_{l=0}^L \left( \sum_{j \in T_i} \lambda'_{jl} \zeta_j^{h/t'_i} \right) \frac{(ks+h)^l}{t_i^l} \left( \frac{(ks+h)\chi}{t_i} - \left\{ \frac{(ks+h)\chi}{t_i} \right\} \right) \\ &= k^{L+1} \sum_{i=1}^r \lambda_{iL}^{(h)} s_i^{L+1} \chi \\ &\quad + k^L \sum_{i=1}^r \left( (L+1)h \lambda_{iL}^{(h)} s_i^L \chi / t'_i + \lambda_{iL-1}^{(h)} s_i^L \chi - \lambda_{iL}^{(h)} s_i^L \left\{ \frac{(ks+h)\chi}{t_i} \right\} \right) \\ &\quad + \cdots - \sum_{i=1}^r \lambda_{i0}^{(h)} \left\{ \frac{(ks+h)\chi}{t_i} \right\}, \end{aligned} \quad (11)$$

where

$$\lambda_{iL}^{(h)} = \begin{cases} \sum_{j \in T_i} \lambda'_{jl} \zeta_j^{h/t'_i}, & \text{if } t'_i | h, \\ 0, & \text{otherwise,} \end{cases}$$

for  $1 \leq i \leq r$ . On the other hand, by (10) we have

$$a_{ks+h} = c_{L+1}^{(h)} k^{L+1} + c_L^{(h)} k^L + \cdots + c_0^{(h)}, \quad (12)$$

where  $c_0^{(h)}, \dots, c_{L+1}^{(h)}$  are algebraic numbers. We can take  $L$  such that  $\lambda'_{iL}$  ( $1 \leq i \leq n$ ) are not all zero.

Let  $\mathbf{a} = (s_1\chi, s_2\chi, \dots, s_r\chi)$ . Renumbering  $\alpha_1, \dots, \alpha_n$ , we may assume that  $t'_1 < t'_2 < \cdots < t'_r$ . Then  $s_1 > \cdots > s_r$ . Let  $\mathbf{p} = (\lambda_{1L}^{(h)} s_1^L, \dots, \lambda_{rL}^{(h)} s_r^L)$  and put  $c_L^{(h)'} = \sum_{i=1}^r \left( (L+1)h \lambda_{iL}^{(h)} s_i^L \chi / t'_i + \lambda_{iL-1}^{(h)} s_i^L \chi \right) - c_L^{(h)}$ .

**Lemma 3.** *If  $\mathbf{p} \neq \mathbf{0}$ , then there exists a real number  $\tau_0$  such that*

$$\mathbf{p} \cdot (\mathbf{a}\tau_0 - ([\tau_0 s_1 \chi], \dots, [\tau_0 s_r \chi])) \neq c_L^{(h)'}$$

*Proof.* We see that

$$\begin{aligned} [0, 1]^r &\ni \mathbf{a}\tau_0 - ([\tau_0 s_1 \chi], \dots, [\tau_0 s_r \chi]) \\ &= \begin{cases} \mathbf{a}\tau_0, & \tau_0 \in [0, 1/(s_1 \chi)), \\ \mathbf{a}\tau_0 - (*, \dots, *, \overset{i}{\dot{1}}, 0, \dots, 0), & \tau_0 \in [1/(s_i \chi), 1/(s_{i+1} \chi)), \end{cases} \end{aligned}$$

where  $s_{r+1} = s_r/2$ . Since  $(*, \dots, *, \overset{i}{\dot{1}}, 0, \dots, 0)$  ( $1 \leq i \leq r$ ) are linearly independent and  $\mathbf{p} \neq \mathbf{0}$ , there exists an  $i$  such that  $\mathbf{p} \cdot (*, \dots, *, \overset{i}{\dot{1}}, 0, \dots, 0) \neq 0$ . If  $\mathbf{p} \cdot \mathbf{a} \neq 0$ , then  $\mathbf{p} \cdot \mathbf{a}\tau_0$  takes at least two values when  $\tau_0$  varies in the interval  $[0, 1/(s_1 \chi))$ . If  $\mathbf{p} \cdot \mathbf{a} = 0$ , then  $\mathbf{p} \cdot (\mathbf{a}\tau_0 - ([\tau_0 s_1 \chi], \dots, [\tau_0 s_r \chi]))$  takes at least two values when  $\tau_0$  varies in the interval  $[0, 1/(s_{r+1} \chi))$ . Hence we can choose  $\tau_0 \in \mathbb{R}$  such that

$$\mathbf{p} \cdot (\mathbf{a}\tau_0 - ([\tau_0 s_1 \chi], \dots, [\tau_0 s_r \chi])) \neq c_L^{(h)'}. \quad \square$$

**Lemma 4.** For any real number  $\tau$  there exists an increasing sequence  $\{k_\nu\}_{\nu \geq 0}$  of positive integers such that

$$\lim_{\nu \rightarrow \infty} (\{k_\nu s_1 \chi\}, \dots, \{k_\nu s_r \chi\}) = \mathbf{a}\tau - ([\tau s_1 \chi], \dots, [\tau s_r \chi]),$$

where each component of the left-hand side approaches the corresponding component of the right-hand side from the right.

*Proof.* First we consider the case of  $\tau \geq 0$ . For any  $\varepsilon > 0$ , there exist positive integers  $p_\varepsilon$  and  $q_\varepsilon$  such that

$$0 < q_\varepsilon \chi - p_\varepsilon < \frac{\varepsilon}{s_1 \sqrt{r}},$$

since there are strictly increasing sequences  $\{p_\nu\}_{\nu \geq 0}$  and  $\{q_\nu\}_{\nu \geq 0}$  of positive integers such that  $0 < q_\nu \chi - p_\nu < 1/q_\nu$ . Let  $\mathbf{e}_i = (0, \dots, 0, \overset{i}{-1}, 0, \dots, 0)$  ( $1 \leq i \leq r$ ). Then, every component of  $q_\varepsilon \mathbf{a} + p_\varepsilon s_1 \mathbf{e}_1 + \dots + p_\varepsilon s_r \mathbf{e}_r$  is positive and less than  $\varepsilon/\sqrt{r}$ , and so  $\|q_\varepsilon \mathbf{a} + p_\varepsilon s_1 \mathbf{e}_1 + \dots + p_\varepsilon s_r \mathbf{e}_r\| < \varepsilon$ . Hence  $\{\mu(q_\varepsilon \mathbf{a} + p_\varepsilon s_1 \mathbf{e}_1 + \dots + p_\varepsilon s_r \mathbf{e}_r) \mid \mu \in \mathbb{N}\}$  is distributed on the half-line  $\mathbf{a}\mathbb{R}_{>0}$  with equal intervals of length less than  $\varepsilon$ . Therefore there exists a positive integer  $\mu_\varepsilon$  such that

$$\|\mu_\varepsilon(q_\varepsilon \mathbf{a} + p_\varepsilon s_1 \mathbf{e}_1 + \dots + p_\varepsilon s_r \mathbf{e}_r) - \mathbf{a}\tau\| < \varepsilon \quad (13)$$

and every component of  $\mu_\varepsilon(q_\varepsilon \mathbf{a} + p_\varepsilon s_1 \mathbf{e}_1 + \dots + p_\varepsilon s_r \mathbf{e}_r) - \mathbf{a}\tau$  is nonnegative. Let  $\mu_\varepsilon q_\varepsilon = k_\varepsilon$  and  $\mu_\varepsilon p_\varepsilon = k'_\varepsilon$ . It is clear that  $\mathbf{a}\tau - ([\tau s_1 \chi], \dots, [\tau s_r \chi]) = \mathbf{a}\tau + [\tau s_1 \chi] \mathbf{e}_1 + \dots + [\tau s_r \chi] \mathbf{e}_r \in [0, 1]^r$ . By (13) we have

$$\begin{aligned} &\|k_\varepsilon \mathbf{a} + (k'_\varepsilon s_1 + [\tau s_1 \chi]) \mathbf{e}_1 + \dots + (k'_\varepsilon s_r + [\tau s_r \chi]) \mathbf{e}_r \\ &\quad - (\mathbf{a}\tau + [\tau s_1 \chi] \mathbf{e}_1 + \dots + [\tau s_r \chi] \mathbf{e}_r)\| < \varepsilon \end{aligned} \quad (14)$$

and hence we can choose  $\varepsilon$  so small that  $k_\varepsilon \mathbf{a} + (k'_\varepsilon s_1 + [\tau s_1 \chi]) \mathbf{e}_1 + \cdots + (k'_\varepsilon s_r + [\tau s_r \chi]) \mathbf{e}_r \in (0, 1)^r$ . Since  $k'_\varepsilon s_1 + [\tau s_1 \chi], \dots, k'_\varepsilon s_r + [\tau s_r \chi] \in \mathbb{Z}$ , by the uniqueness of the fractional part, we see that

$$k_\varepsilon \mathbf{a} + (k'_\varepsilon s_1 + [\tau s_1 \chi]) \mathbf{e}_1 + \cdots + (k'_\varepsilon s_r + [\tau s_r \chi]) \mathbf{e}_r = (\{k_\varepsilon s_1 \chi\}, \dots, \{k_\varepsilon s_r \chi\}).$$

Hence by (14) there exists an increasing sequence  $\{k_\nu\}_{\nu \geq 0}$  of positive integers such that

$$\lim_{\nu \rightarrow \infty} (\{k_\nu s_1 \chi\}, \dots, \{k_\nu s_r \chi\}) = \mathbf{a}\tau - ([\tau s_1 \chi], \dots, [\tau s_r \chi]),$$

where each component of the left-hand side approaches the corresponding component of the right-hand side from the right.

Next we consider the case of  $\tau < 0$ . For any  $\varepsilon > 0$ , there exist positive integers  $p_\varepsilon$  and  $q_\varepsilon$  such that

$$-\frac{\varepsilon}{s_1 \sqrt{r}} < q_\varepsilon \chi - p_\varepsilon < 0,$$

since there are strictly increasing sequences  $\{p_\nu\}_{\nu \geq 0}$  and  $\{q_\nu\}_{\nu \geq 0}$  of positive integers such that  $-1/q_\nu < q_\nu \chi - p_\nu < 0$ . Then  $\{\mu(q_\varepsilon \mathbf{a} + p_\varepsilon s_1 \mathbf{e}_1 + \cdots + p_\varepsilon s_r \mathbf{e}_r) \mid \mu \in \mathbb{N}\}$  is distributed on the half-line  $\mathbf{a}\mathbb{R}_{<0}$  with equal intervals of length less than  $\varepsilon$ . By the same way as above we can take an increasing sequence  $\{k_\nu\}_{\nu \geq 0}$  of positive integers such that

$$\lim_{\nu \rightarrow \infty} (\{k_\nu s_1 \chi\}, \dots, \{k_\nu s_r \chi\}) = \mathbf{a}\tau - ([\tau s_1 \chi], \dots, [\tau s_r \chi]),$$

where each component of the left-hand side approaches the corresponding component of the right-hand side from the right. This completes the proof.  $\square$

We assume that  $\lambda_{iL}^{(h)}$  ( $1 \leq i \leq r$ ) are not all zero. Then  $\mathbf{p} \neq \mathbf{0}$ . By Lemmas 3 and 4, we see that there exist a real number  $\tau_0$  and an increasing sequence  $\{k_\nu\}_{\nu \geq 0}$  of positive integers such that

$$\mathbf{p} \cdot (\mathbf{a}\tau_0 - ([\tau_0 s_1 \chi], \dots, [\tau_0 s_r \chi])) \neq c_L^{(h)'} \quad (15)$$

and

$$\begin{aligned} \lim_{\nu \rightarrow \infty} (\{k_\nu s_1 \chi\}, \dots, \{k_\nu s_r \chi\}) &= \mathbf{a}\tau'_0 - ([\tau'_0 s_1 \chi], \dots, [\tau'_0 s_r \chi]) \\ &= (\{\tau'_0 s_1 \chi\}, \dots, \{\tau'_0 s_r \chi\}), \end{aligned} \quad (16)$$

where each component of the left-hand side approaches the corresponding component of the right-hand side from the right and  $\tau'_0 = \tau_0 - h/s$ . By (16) we see that

$$\begin{aligned} &\lim_{\nu \rightarrow \infty} (\{(k_\nu s + h)\chi/t'_1\}, \dots, \{(k_\nu s + h)\chi/t'_r\}) \\ &= \lim_{\nu \rightarrow \infty} (\{k_\nu s_1 \chi + h\chi/t'_1\}, \dots, \{k_\nu s_r \chi + h\chi/t'_r\}) \\ &= (\{(\tau_0 - h/s)s_1 \chi + h\chi/t'_1\}, \dots, \{(\tau_0 - h/s)s_r \chi + h\chi/t'_r\}) \\ &= (\{\tau_0 s_1 \chi\}, \dots, \{\tau_0 s_r \chi\}). \end{aligned} \quad (17)$$

Since  $\lim_{k \rightarrow \infty} a_{ks+h}/k^{L+1} = c_{L+1}^{(h)}$  by (12) and  $\lim_{k \rightarrow \infty} a_{ks+h}/k^{L+1} = \sum_{i=1}^r \lambda_{iL}^{(h)} s_i^{L+1} \chi$  by (11), we have

$$c_{L+1}^{(h)} = \sum_{i=1}^r \lambda_{iL}^{(h)} s_i^{L+1} \chi. \tag{18}$$

By (12)

$$\lim_{k \rightarrow \infty} \frac{a_{ks+h} - c_{L+1}^{(h)} k^{L+1}}{k^L} = c_L^{(h)}.$$

On the other hand, by (11), (15), (17) and (18), we have

$$\begin{aligned} & \lim_{\nu \rightarrow \infty} \frac{a_{k_\nu s+h} - c_{L+1}^{(h)} k_\nu^{L+1}}{k_\nu^L} \\ &= \lim_{\nu \rightarrow \infty} \frac{k_\nu^L \left( c_L^{(h)} + c_L^{(h)'} - \sum_{i=1}^r \lambda_{iL}^{(h)} s_i^L \{ (k_\nu s + h) \chi / t_i' \} \right) + \dots}{k_\nu^L} \\ &= \lim_{\nu \rightarrow \infty} \left( c_L^{(h)} + c_L^{(h)'} - \mathbf{p} \cdot (\{ (k_\nu s + h) \chi / t_1' \}, \dots, \{ (k_\nu s + h) \chi / t_r' \}) \right) \\ &\neq c_L^{(h)}, \end{aligned}$$

which is a contradiction. Hence we see that  $\lambda_{iL}^{(h)} = 0$  ( $1 \leq i \leq r$ ) for any positive integer  $h$ . Hence for  $h = t_i' k$  with  $k \geq 0$  and for  $i$  with  $1 \leq i \leq r$  we have

$$\sum_{j \in T_i} \lambda_{jL}' \zeta_j^k = 0.$$

Since  $\zeta_j$  ( $j \in T_i$ ) are distinct, by non-vanishing of the Vandermonde determinant, we see that  $\lambda_{jL}' = 0$  ( $1 \leq i \leq n$ ), which is a contradiction, and the proof of the theorem is completed. □

## References

- [1] Y. Z. Flicker. *Algebraic independence by a method of Mahler*, J. Austral. Math. Soc. Ser. A **27** (1979), 173–188.
- [2] E. Hecke. *Über analytische Funktionen und die Verteilung von Zahlen mod Eines*, Abh. Math. Sem. Hamburg **1** (1921), 54–76.
- [3] J. H. Loxton and A. J. van der Poorten. *Arithmetic properties of certain functions in several variables III*, Bull. Austral. Math. Soc. **16** (1977), 15–47.
- [4] J. H. Loxton and A. J. van der Poorten. *Algebraic independence properties of the Fredholm series*, J. Austral. Math. Soc. Ser. A **26** (1978), 31–45.
- [5] K. Mahler. *Arithmetische Eigenschaften der Lösungen einer Klasse von Funktionalgleichungen*, Math. Ann. **101** (1929), 342–366.

- [6] D. W. Masser. *Algebraic independence properties of the Hecke-Mahler series*, Q. J. Math. **50** (1999), 207–230.
- [7] K. Nishioka. *Evertse theorem in algebraic independence*, Arch. Math. **53** (1989), 159–170.
- [8] K. Nishioka. *Note on a paper by Mahler*, Tsukuba J. Math. **17** (1993), 455–459.
- [9] K. Nishioka. *Algebraic independence of Mahler functions and their values*, Tohoku Math. J. **48** (1996), 51–70.
- [10] K. Nishioka. *Mahler functions and transcendence*, LNM **1631**, Springer (1996).
- [11] T. Tanaka and Y. Tanuma. *Algebraic independence of the values of the Hecke-Mahler series and its derivatives at algebraic numbers*, Int. J. Number Theory (accepted).