数理解析研究所講究録 第2092巻 2018年 97-103



# On the digits in the base-$b$ expansion of smooth numbers

Yann Bugeaud
Institut de Recherche Mathématique Avancée, U.M.R. 7501
Université de Strasbourg et C.N.R.S.
Hajime Kaneko
Institute of Mathematics, University of Tsukuba
Research Core for Mathematical Sciences, University of Tsukuba

**Abstract**

Erdős [4] conjectured that, for any integer $m \geq 9$, the digit 2 appears at least once in the ternary expansion of $2^m$. More precisely, Dupuy and Weirich [3] conjectured that, for any sufficiently large $m$, the digits 0,1, and 2 appear "uniformly" in the ternary expansion of $2^m$. This is still open. Stewart [10] obtained a lower bound for the number of nonzero digits in the ternary expansion of $2^m$, thus giving (very) partial results of "uniformity". In this report, we investigate the number of nonzero digits in the base-$b$ expansion of more general smooth numbers and introduce the main results established in [2].

## 1 Problems on the base-$b$ expansion of $a^n$

Throughout this survey, $b$ denotes an integer greater than 1. The main purpose of this report is the study of the uniformity of the digits in the base-$b$ expansion of smooth numbers. We now recall the definition of smooth numbers. We denote by $P[n]$ the greatest prime factor of an integer $n \geq 2$. For convenience, let $P[1] := 1$. Let $x$ be a positive real number. Recall that a positive integer $n$ is $x$-smooth if $P[n] \leq x$. For instance, let $a$ be an integer greater than 1. Then, $a^m$ is $P[a]$-smooth for any nonnegative integer $m$. In this section we review open problems related to the base-$b$ expansions of powers of integers.

In 1979, Erdős [4] conjectured that, if $m$ is an integer greater than 8, then the digit 2 appears at least once in the ternary expansion of $2^m$ (note that $2^8 = 3^5 + 3^2 + 3 + 1$). Let $T$ be a positive integer. Denote by $N(T)$ the number of integers $m$ with $0 \leq m \leq T$ such that the ternary expansion of $2^m$ omits 2. It is still unproven whether $N(T)$ is bounded as $T$ tends to infinity. In 1980, Narkiewicz [7] showed that

$$N(T) \leq 1.62T^{\log_3 2},$$

where $\log_3 2 = (\log 2)/(\log 3) \approx 0.63092$. Moreover, for any positive real number $\lambda$, let $N_\lambda(T)$ be the number of integers $m$ with $0 \leq m \leq T$ such that the ternary expansion of $\lfloor \lambda 2^m \rfloor$ omits 2. Lagarias [6] proved that

$$N_\lambda(T) \leq 25T^{0.9725},$$

for any $T \geq T_0(\lambda)$, where $T_0(\lambda)$ is a positive number depending only on $\lambda$.

Recall that two positive integers $a$ and $b$ are multiplicatively independent if $a^i b^j \neq 1$ for any integers $(i, j) \neq (0, 0)$. Let $a$ and $b$ be multiplicatively independent positive integers. Let $v = v_1 v_2 \ldots v_l$ be any finite word over the alphabet $\{0, 1, \ldots, b-1\}$ of length $l \geq 1$. Lagarias [6] conjectured that $v$ occurs at least once in the base-$b$ expansion of $a^m$ for any sufficiently large integer $m$. Erdős conjecture treats the case $l = 1$. We introduce a related result from [5]. Let $p$ be a prime number and $a \geq 2$ an integer coprime to $p$. Let $v = v_1 v_2 \ldots v_l$ be any finite word over the alphabet $\{0, 1, \ldots, p-1\}$ with length $l \geq 1$. Let $\gamma (\geq 2)$ be the number of circular shift occurrences of $v$ in $vv = v_1 v_2 \ldots v_l v_1 v_2 \ldots v_l$. Moreover, let $e_p(v; a^m)$ be the number of (possibly overlapping) occurrences of $v$ in the base-$p$ expansion of $a^m$. It is shown in [5] that we have

$$\limsup_{m \to \infty} \frac{e_p(v; a^m)}{\log m} \geq \frac{\gamma - 1}{l \log p}.$$

Refining Erdős conjecture, Dupuy and Weirich [3] proposed the following problem on the uniformity of digits. Let $p$ and $q$ be distinct prime numbers. Let $h$ be an integer with $0 \leq h \leq p-1$. Then Dupuy and Weirich [3] conjectured that

$$\lim_{m \to \infty} \frac{e_p(h; q^m)}{\log_p(q^m)} = \frac{1}{p}.$$

Moreover, they obtained the following partial results in the direction of the conjecture stated above. Let $k$ be a fixed positive integer. For any integer $m, h$ with $0 \leq m, 0 \leq h \leq p-1$, let

$$q^m = s_0^{(p)} + s_1^{(p)} p + \cdots + s_M^{(p)} p^M$$

be the base-$p$ expansion of $q^m$, where $M = \lfloor \log_p(q^m) \rfloor$ and $s_i^{(p)} \in \{0, 1, \ldots, p-1\}$ for $i = 0, \ldots, M$. Let $e_p(h, k; q^m)$ be the number of occurrences of $h$ in the word $s_{k-1}^{(p)} \ldots s_1^{(p)} s_0^{(p)}$. Then

$$\lim_{k \to \infty} \lim_{N \to \infty} \frac{1}{N} \sum_{m=1}^{N} \frac{e_p(h, k; q^m)}{k} = \frac{1}{p}.$$

It is in general very difficult to get some non-trivial information on the number of occurrences $e_p(h; q^m)$ of a fixed digit $h$ in $q^m$. In the next section we consider the number of nonzero digits, which also gives partial results for the uniformity of digits. In Section 2, we introduce various results on the number of nonzero digits of smooth numbers. In Section 3, we present the main new results obtained in [2], which improve and extend the results of Section 2.

## 2 Number of nonzero digits of smooth numbers

For any positive integer $n$, let $\lambda_b(n)$ be the number of nonzero digits in the base-$b$ expansion of $n$. If the conjecture by Dupuy and Weirich in Section 1 is true, then

$$\lim_{m\to\infty} \frac{\lambda_p(q^m)}{\log_p(q^m)} = \frac{p-1}{p}.$$

However, it is still unknown whether

$$\limsup_{m\to\infty} \frac{\lambda_p(q^m)}{\log_p(q^m)} > 0.$$

We introduce the lower bounds for the number of nonzero digits established by Stewart [10]. Let $a$ and $b$ be multiplicatively independent integers greater than 1. Let $\varepsilon$ be an arbitrary positive real number. Then Stewart [10] showed that there exists an effectively computable positive number $C_1(a,b,\varepsilon)$ such that

$$\lambda_a(n) + \lambda_b(n) \geq (1-\varepsilon)\frac{\log\log n}{\log\log\log n},$$

for any $n \geq C_1(a,b,\varepsilon)$. In particular, consider the case of $n = a^m$, where $m$ is a nonnegative integer. Since $\lambda_a(a^m) = 1$, we see that

$$\lambda_b(a^m) \geq (1-\varepsilon)\frac{\log\log a^m}{\log\log\log a^m}, \tag{2.1}$$

that is,

$$\lambda_b(a^m) \geq (1-\varepsilon)\frac{\log m}{\log\log m},$$

for any sufficiently large integer $m$. The proof rests on a subtle use of estimates for complex linear forms in the logarithms of rational numbers.

Lower bounds for the number of nonzero digits in the base-$b$ expansion of more general smooth numbers were discussed in [1]. Note that $\lambda_b(bn) = \lambda_b(n)$ for any positive integer $n$. Thus, we assume that $n$ is not divisible by $b$ in the rest of this section. It is easy to see that if $n \geq b+1$, then $\lambda_b(n) \geq 2$, which is a trivial lower bound. Thus, we first consider sufficient conditions for $\lambda_b(n) \geq 3$ and $\lambda_b(n) \geq 4$. Note that $\lambda_b(n) = 2$ if and only if $n$ has the form $n = t_1 b^m + t_0$, where $m$ is a positive integer and $t_1, t_0 \in \{1, 2, \ldots, b-1\}$. Applying the result on the greatest prime factor of linear recurrences established by Stewart [9], we obtain that there exists an effectively computable positive number $C_2(b)$, depending only on $b$, such that if an integer $n \geq C_2(b)$ satisfies

$$P[n] \leq (\log n)^{1/2} \exp\left(\frac{\log\log n}{105\log\log\log n}\right),$$

then $\lambda_b(n) \geq 3$. In the opposite direction, Schinzel [8] constructed arbitrary large integers $n$ such that

$$P[n] \leq n^{c/\log\log\log n} \text{ and } \lambda_b(n) = 2,$$

where $c$ is an absolute real number.

Furthermore, the following result was obtained in [1]. Let $\varepsilon$ be an arbitrary positive real number. Then there exists an effectively computable positive number $C_3(b,\varepsilon)$, depending only on $b,\varepsilon$, such that if $n \geq C_3(b,\varepsilon)$ satisfies

$$P[n] \leq (1-\varepsilon)(\log\log n)\frac{\log\log\log n}{\log\log\log\log n}, \tag{2.2}$$

then $\lambda_b(n) \geq 4$. In addition, it was also proved in [1], that, for any fixed integer $N \geq 4$, the greatest prime factor of $n$ tends to infinity, as $n$ tends to infinity and runs through the set of integers not divisible by $b$ and having at most $N$ nonzero digits in their base-$b$ expansion. Since the proof rests on the subspace theorem, no estimate for the speed of divergence can be derived, thus no sufficient condition on $P[n]$ ensuring that $\lambda_b(n) \geq 5$ was known until very recently. In the next section we give sufficient condition for $\lambda_b(n) \geq k$, where $k$ is an arbitrary positive integer.

## 3 Main results

In this section, we review without proof lower bounds for $\lambda_b(n)$ obtained in [2]. Throughout this section, $C_i(x,y\ldots)$ $(i=4,5,\ldots)$ denote effectively computable positive numbers depending only on $x,y,\ldots$.

**THEOREM 3.1.** *Let $k$ be an integer greater than 2. Let $\varepsilon$ be an arbitrary positive real number. Let $n \geq C_4(b,k,\varepsilon)$ be an integer not divisible by $b$. Suppose that*

$$P[n] \leq \left(\frac{1}{k-2}-\varepsilon\right)(\log\log n)\frac{\log\log\log n}{\log\log\log\log n}. \tag{3.1}$$

*Then we have $\lambda_b(n) \geq k+1$.*

Note that (3.1) generalizes (2.2). Now let $\mathcal{A}$ be the set of positive integers $n$ not divisible by $b$ such that $n$ is $\log\log n$-smooth. Applying Theorem 3.1, we see

$$\lim_{n\in\mathcal{A}, n\to\infty} \lambda_b(n) = \infty.$$

In what follows, we investigate quantitative version of this result, that is, we give lower bounds

$$\lambda_b(n) \geq \varphi(n) \tag{3.2}$$

for smooth numbers $n$, using suitable increasing functions $\varphi$ with $\lim_{n\to\infty}\varphi(n) = \infty$.

**THEOREM 3.2.** *For any $n \in \mathcal{A}$ with $n \geq C_5(b)$, we have*

$$\lambda_b(n) \geq \frac{1}{2}\cdot\frac{\log\log\log n}{\log\log\log\log n}.$$

Changing the condition of smoothness, we give another lower bound of the form (3.2).

**THEOREM 3.3.** *Let $n \geq C_6(b)$ be an integer not divisible by $b$. Suppose that*

$$P[n] \leq (\log\log n)^{1/2}\left(\frac{\log\log\log n}{\log\log\log\log n}\right)^{1/2}.$$

*Then, we have*

$$\lambda_b(n) \geq \frac{1}{3}(\log\log n)^{1/2}\left(\frac{\log\log\log n}{\log\log\log\log n}\right)^{1/2}.$$

We now generalize (2.1). Let $S$ be a non-empty finite set of prime numbers. Recall that a positive integer $n$ is an integral $S$-unit if all the prime factors of $n$ are in $S$. In particular, if $p$ denotes the maximal element of $S$, then any integral $S$-unit is $p$-smooth.

**THEOREM 3.4.** *Let $S$ be a non-empty finite set of prime numbers. Let $\varepsilon$ be an arbitrary positive real number. Let $n \geq C_7(b, S, \varepsilon)$ be an integral $S$-unit not divisible by $b$. Then we have*

$$\lambda_b(n) \geq (1-\varepsilon)\frac{\log\log n}{\log\log\log n}. \tag{3.3}$$

Let $a \geq 2$ be an integer coprime to $b$. Let $S$ be the set of prime divisors of $a$. Then, (2.1) follows from Theorem 3.4.

Changing the coefficient of the right-hand side of (3.3), we can extend (2.1) to more general smooth numbers.

**THEOREM 3.5.** *Let $n \geq C_8(b)$ be an integer not divisible by $b$. Suppose that*

$$P[n] \leq \frac{1}{2}(\log\log\log n)\frac{\log\log\log\log n}{\log\log\log\log\log n}.$$

*Then, we have*

$$\lambda_b(n) \geq \frac{1}{2}\frac{\log\log n}{\log\log\log n}. \tag{3.4}$$

Note that (3.4) is the best lower bound (upto the value $\frac{1}{2}$) for $\lambda_b(n)$ obtainable by our method. We conclude with a general statement giving lower bounds for $\lambda_b(n)$.

**THEOREM 3.6.** *Let $f$ be a positive real valued function defined over the set of positive integers. Assume that*

$$\lim_{n\to\infty} f(n) = \infty$$

*and that there exists a real number $0 < \delta < 1$ satisfying*

$$f(n) \leq (1-\delta)\frac{\log\log n}{\log\log\log n}$$

*for any sufficiently large integer $n$. Set*

$$\Psi_f(n) := \frac{\log\log n}{f(n)}$$

*and*

$$\delta_0 := \sup\left\{\delta > 0 \ : \ f(n) \leq (1-\delta)\frac{\log\log n}{\log\log\log n} \text{ for any sufficiently large } n\right\}.$$

*Let $\varepsilon$ be an arbitrary positive real number. Suppose that a sufficiently large integer $n$ not divisible by $b$ satisfies*

$$P[n] \leq (\delta_0 - \varepsilon)\Psi_f(n)\frac{\log \Psi_f(n)}{\log\log \Psi_f(n)}.$$

*Then we have*

$$\lambda_b(n) \geq f(n).$$

The proofs of all the results stated in this section depend on lower estimates for linear forms in the complex logarithms of rational numbers, combined with lower estimates for linear forms in the $p$-adic logarithms of rational numbers, where $p$ is a prime divisor of $b$.

# Acknowledgements

This work was supported by JSPS KAKENHI Grant Number 15K17505.

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