数理解析研究所講究録 第2092巻 2018年 122-129

A NEW APPROACH TO THE STUDY OF $D(-1)$–QUADRUPLES

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ABSTRACT. Problems found in Diophantus's books on Arithmetics prompted intense research activity in recent years. The modern treatment evolved from the unifying notion of $D(n)$–m–sets. After a short overview of the main questions in this area, we concentrate on $D(-1)$–sets. It is conjectured that no $D(-1)$–quadruples exist, but the issue is still open. In the literature one finds several properties a hypothetical $D(-1)$–quadruple necessarily has. After sketching the strategy producing such results, a novel approach is pointed out. This survey is based on work in progress performed jointly with N. C. Bonciocat (Bucharest, Romania) and M. Mignote (Strasbourg, France).

1. INTRODUCTION

1.1. **Two old problems.** About 1750 years ago, Diophantus of Alexandria wrote 13 books on Arithmetics. Today only six of them are known from later copies, see [3, 26] — or at least this is the prevailing view on this obscure part of the (hi)story (more details on the controversy surrounding Diophantus's life and work are found on the web page [33] and in references given there). There one finds the first documented appearance of several fertile ideas. It suffices to mention the geometric construction that underlines the doubling operation on the set of rational points on an elliptic curve, *cf.* [6]. Below we focus on two of Diophantus's problems that prompted intense work in recent years.

A first problem is to find sets with the property that the product of any two distinct elements of each of them is one less than a perfect square. By that time, and for a long time after, a problem was considered 'solved' by giving a (few) numerical example(s). Diophantus himself provided the set $\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}$ satisfying the required condition. Fermat found in the XVIIth century the first set consisting of four integers, viz. $\{1, 3, 8, 120\}$. As Euler showed in 1700's, Fermat's set can be enlarged by joining $\frac{777480}{8288641}$ without loosing the defining property. By a nice twist of fate, during RIMS 2017 Workshop on Analytic Number Theory and Related Areas appeared a preprint of Stoll [30] in which one finds, among other notable results, a proof that the only possibility to extend Fermat's set by a positive rational is the one obtained by Euler.

A second problem asks for sets whose elements have the property that the product of any two of them increased by the sum of the factors is a perfect square. Two such sets found by Diophantus are $\{4, 9, 28\}$ and $\{\frac{3}{10}, \frac{21}{5}, \frac{7}{10}\}$. Euler gave an example of a set with four elements: $\{\frac{5}{2}, \frac{9}{56}, \frac{9}{224}, \frac{65}{224}\}$. Further answers to the same problems are provided in 1999 by Dujella [9], whose example $\{-\frac{27}{40}, \frac{17}{8}, \frac{27}{10}, 9, \frac{493}{40}\}$ shows that any combination rational number-integer number, positive-negative is possible, and Gibbs [23], who produced several sets of cardinality six, one of which being $\{-\frac{11}{192}, \frac{35}{192}, \frac{155}{27}, \frac{512}{27}, \frac{1235}{48}, \frac{180873}{16}\}$.

A contemporary mathematician would like to know the context in which such problems emerged, the motivation behind them. This is a last minute concern, speaking at historical scale. Up to a few decades ago, the mere formulation of a question was a good enough reason to try to answer it, irrespective of its connections or foreseable applications. Since the available parts of Diophantus's writings offer no help for a possible reconstruction of the context in which the two problems discussed here appeared, one can only speculate on their origins. It is plausible that geometric intuition was, as in many other cases, the driving force acting behind the scene. Some musings on these lines can be found on the web page [34].

1.2. **A unifying framework.** At first glance the two problems do not have much in common. However, the identity $xy + x + y = (x + 1)(y + 1) - 1$ suggests a unifying viewpoint. Most papers published in this area in recent years adopt the following terminology.

Definition 1. *For any integer n, a set of positive integers $\{a_1, a_2, \ldots, a_m\}$ is called a $D(n)$–m–set or $D(n)$–m–tuple if $a_i a_j + n$ is a perfect square for any $1 \le i < j \le m$.*

If $n = 1$, one usually speaks of Diophantine m–sets. When $m = 2$ (3, 4, 5, 6, respectively), one prefers the terminology $D(n)$–pair (triple, quadruple, quintuple, sextuple, respectively).

A word of caution: the term 'Diophantine set' was previously introduced in work prompted by Hilbert's tenth problem. However, the chances of misunderstandings and confusions are slim because the two research areas have loose connections and the peaks of activity are rather distant in time.

Nowadays there are known more general variants of Definition 1. A possibility is to consider a_j elements of a commutative ring R instead of positive integers. In the same vein, even n can be taken in R. Dealing with such notions requires different techniques than those useful in the study of sets identified as in Definition 1 and opens the prospects of uncovering phenomena absent in the case delimited by Definition 1.

Diophantus himself was aware of such an extension of the initial problems. For instance, he provides examples of $D(12)$–triple as well as of $D(-10)$–triple.

1.3. **Main questions.** The very first question which pops up when facing a notion is to see the object of interest exists. We mentioned above several sporadic examples from the plethora of published ones. Producing own $D(n)$–pairs is a boring play: it suffices to take a preferred square r^2 and to find a factor a of $r^2 - n$; this factor together with its cofactor $(r^2 - n)/a$ form a $D(n)$–pair.

For $n = 0$ the game is even more uninteresting: each $D(0)$–tuple is obtained from a unique set of squares by multiplication with an arbitrary positive integer. It is therefore clear that $D(0)$–sets of arbitrary finite cardinality as well as infinite ones exist. For this reason, below we shall always assume $n \neq 0$.

The most successful strategy of constructing $D(n)$–tuples is unchanged since Euler. It is based on the idea of extending a known $D(n)$–set by adjoining an extra element without losing the property of interest.

The initial step — finding a $D(n)$–pair — presents no difficulty. A completely different situation emerges when searching for $D(n)$–triples $\{a, b, c\}$. According to the definition, there exist positive integers r, s, t such that $ab + n = r^2$, $ac + n = s^2$, and $bc + n = t^2$. Supposing that a and b are already known, finding c is equivalent to determination of either of s or t. Elimination of c results in the Pellian equation $bs^2 - at^2 = (b - a)n$. Solving such an equation is not trivial in practice, although the underlying theory is completely satisfactory. It is known that the all-or-nothing principle applies: if the equation has a solution in positive integers, then it has infinitely many such solutions. Obviously, each solution gives rise to a sought-for triple.

The same approach works when looking for $D(n)$–quadruples. It is clear that in this case three new Pellian equations are generated. By the work of Thue [31], such a system has only finitely many integer solutions. Finding any of them remains a very difficult task even for the present day technology.

In principle, this modus operandi is valid no matter the cardinality of the $D(n)$–m–set, but the explosion of the number of Pellian equations thus generated renders the technique unsuitable for larger values of m. This is the price to be paid for a strategy that generates *all* positive integers a_{m+1} that extend a fixed $D(n)$–m–set without losing the property required in Definition 1. The alternative is to come up with a formula that produces only one such a_{m+1}.

Euler showed that an arbitrary Diophantine pair $\{a, b\}$ can be extended to a Diophantine triple by taking $c = a + b + 2r$, where $r = \sqrt{ab+1}$, and even to a Diophantine quadruple if one puts $d = 4r(a + r)(b + r)$. This construction has been refined by Arkin, Hoggatt, and Strauss [1] and independently by Gibbs [22]: they associated to each Diophantine triple $\{a, b, c\}$ the integers

$$d_\pm = a + b + c + 2abc \pm 2\sqrt{(ab+1)(bc+1)(ca+1)} \tag{1}$$

and showed that $\{a, b, c, d_\pm\}$ is a Diophantine quadruple. The formula above has been generalised by Dujella to produce $D(n^2)$–quadruples.

All attempts to perform an additional step failed, whence the question: how large a $D(n)$–m–set can be? A general answer is given in [13]: any $D(n)$–m–set satisfies $m \le 31$ if $1 \le |n| \le 400$ and $m < 15.476 \log |n|$ if $|n| > 400$.

Much better bounds are known in particular cases. Thus, a short elementary argument is sufficient to verify that no $D(4k+2)$–quadruple exists. It is also known that if an integer n does not have the form $4k + 2$ and $n \notin \mathcal{S} = \{-4, -3, -1, 3, 5, 8, 12, 20\}$, then there exists at least one Diophantine quadruple with the property $D(n)$, see [7]. In fact, it is conjectured that for $n \in \mathcal{S}$ there does not exist a $D(n)$–quadruple. An absolute bound $m < 3 \cdot 2^{168}$ is known for $D(n)$–m–sets with n prime, see [14].

For the class of Diophantine sets, Dujella showed in [12] there are no $D(1)$–sextuples and only finitely many $D(1)$–quintuples. The same results hold for $D(4)$–sets [18]. In both cases it is expected no quintuple exists. This is just an instance of a well-documented strong similarity between $D(1)$– and $D(4)$–sets. The close relationship is not at all surprising, since doubling all elements of an arbitrary $D(1)$–m–set results in a $D(4)$–m–tuple. In a recent preprint [25], He, Togbé, and Ziegler introduced a strategy aiming to a proof for the non-existence of $D(1)$–quintuples. Bliznac Trebješanin and Filipin [4] have different ideas leading to the proof that no $D(4)$–quintuple exists.

A peculiar result of Dujella and Fuchs [16] refers to $D(-1)$–sets: There is no $D(-1)$–quadruple whose smallest element is at least 2. Hence, there is no $D(-1)$–quintuple. The same authors together with Filipin proved in [15] that there are only finitely many $D(-1)$–quadruples. However, the dominant feeling is that no $D(-1)$–quadruple exists.

An exhaustive bibliography is found on the web page [32].

In the next section we sketch the classical approach to the study of hypothetical $D(-1)$–quadruples. In the final section we point out a different point of view on which work in progress with N. C. Bonciocat (Bucharest, Romania) and M. Mignote (Strasbourg, France) is based.

2. Standard recipe for dealing with $D(-1)$–quadruples

For the rest of the paper, $(1, b, c, d)$ is a $D(-1)$–quadruple with $1 < b < c < d$. According to Definition 1, there are positive integers r, s, t, x, y, z satisfying

$$b - 1 = r^2, \ c - 1 = s^2, \ bc - 1 = t^2, \tag{2}$$

$$d - 1 = x^2, \ bd - 1 = y^2, \ cd - 1 = z^2. \tag{3}$$

Elimination of d in Eq. (3) results in a system of three Pellian equations:

$$z^2 - cx^2 = c - 1, \tag{4}$$

$$bz^2 - cy^2 = c - b, \tag{5}$$

$$y^2 - bx^2 = b - 1. \tag{6}$$

If a Pellian equation is solvable in positive integers, then it has infinitely many positive solutions, obtained from a finite set of 'fundamental solutions' by multiplication with powers of solutions to the associated Pell equation (whose right hand side is 1). It is not difficult to see that the positive integer solutions of each of the above Pellian equations are respectively given by

$$z + x\sqrt{c} = s(s + \sqrt{c})^{2m}, \quad m \geq 0,$$
$$z\sqrt{b} + y\sqrt{c} = (s\sqrt{b} + \rho r\sqrt{c})(t + \sqrt{bc})^{2n}, \quad n \geq 0,$$
$$y + x\sqrt{b} = (y_2 + x_2\sqrt{b})(r + \sqrt{b})^{2l}, \quad l \geq 0,$$

for suitable integers $0 < y_2 < b$, $|x_2| < r$, and fixed $\rho \in \{-1, 1\}$. Therefore, the triples (x, y, z) of positive integers that simultaneously satisfy Eqs. (4)–(5) are such that

$$z = v_m = w_n, \tag{7}$$

where the integer sequences $(v_p)_{p \geq 0}$, $(w_p)_{p \geq 0}$ are given by explicit formulæ

$$v_p = \frac{s}{2}\left(\alpha^{2p} + \overline{\alpha}^{2p}\right) \tag{8}$$

and respectively

$$w_p = \frac{s\sqrt{b} + \rho r\sqrt{c}}{2\sqrt{b}}\beta^{2p} + \frac{s\sqrt{b} - \rho r\sqrt{c}}{2\sqrt{b}}\overline{\beta}^{2p}, \tag{9}$$

with

$$\alpha = s + \sqrt{c}, \quad \overline{\alpha} = s - \sqrt{c}, \quad \beta = t + \sqrt{bc}, \quad \overline{\beta} = t - \sqrt{bc}.$$

There are at least two possibilities to extract information from an equality of the type (7). One of them is based on the explicit formulæ mentioned above, another exploits the fact that the integer sequences $(v_p)_{p \geq 0}$, $(w_p)_{p \geq 0}$ are given by second order linear recurrences.

More precisely, the first consists in dividing both sides of Eq. (7) by $s\alpha^{2m}/2$, isolating the maximal term, and taking the logarithm in both sides of the resulting equality. This procedure gives rise to a linear form in the logarithms of three algebraic numbers, for which a strong enough upper bound is obtained directly, while useful lower bounds are given by

Baker theory. Comparison of these bounds results in inequalities for indices m and n in terms of elementary functions in b and c.

The alternative way to exploit Eq. (7) has been introduced by Dujella and Pethő in [32] and is usually referred to as 'the congruence method'. Their idea is to consider the recurrent sequences modulo c^2 and prove that suitable hypotheses entail that these congruences are actually equalities. Consequently, another set of inequalities relating the same variables m, n, b, c pops up. Juggling simultaneously with these relations and those obtained via Baker theory of bounds for linear forms in logarithms of algebraic numbers, some absolute bounds for the entries of any $D(-1)$–quadruple are derived.

Following the approach just sketched, Dujella, Filipin and Fuchs proved the following result.

Theorem 1. ([15]) *Let* $(1, b, c, d)$ *with* $1 < b < c < d$ *be a* $D(-1)$*–quadruple. Then* $b > 100$ *and* $c < \min\{11\,b^6, 10^{491}\}$. *More precisely:*

a) *If* $b^3 \leq c < 11\,b^6$ *then* $c < 10^{238}$.
b) *If* $b^{1.1} \leq c < b^3$ *then* $c < 10^{491}$.
c) *If* $3b \leq c < b^{1.1}$ *then* $c < 10^{94}$.
d) *If* $b < c < 3b$ *then* $c < 10^{74}$.

A variant of the congruence method has been introduced in [5]. The alternative idea on which this work is based is to interpret an equivalence $L \equiv R \pmod{c}$ as an equality $L - R = jc$ for a suitable integer j. The crux of the original congruence method is to find hypotheses under which one can conclude $j = 0$. Instead of striving to get $j = 0$, in [5] all possibilities for the sign of j have been analysed. Clearly, this approach requires less stringent conditions on the parameters. As a result of this study, inequalities of the form $n > f(b, c)^{\gamma(j)}$ have been established. It has been noticed that the consequences of such inequalities have been strengthened by applying them to shorter intervals of variation for c. This idea is called 'smoothification' in [5]. The approach has been implemented in the package PARI/GP [28]. The outcome of very long computations is summarised below.

Theorem 2. ([5]) *Let* $(1, b, c, d)$ *with* $1 < b < c < d$ *be a* $D(-1)$*–quadruple. Then* $b > 1.024 \cdot 10^{13}$ *and* $\max\{10^{14}b, b^{1.16}\} < c < \min\{2.5\,b^6, 10^{148}\}$. *More precisely:*

i) *If* $b^5 \leq c < 2.5\,b^6$ *then* $c < 10^{100}$.
ii) *If* $b^4 \leq c < b^5$ *then* $c < 10^{82}$.
iii) *If* $b^{3.5} \leq c < b^4$ *then* $c < 10^{66}$.
iv) *If* $b^3 \leq c < b^{3.5}$ *then* $c < 10^{57}$.
v) *If* $b^2 \leq c < b^3$ *then* $c < 10^{111}$.
vi) *If* $b^{1.5} \leq c < b^2$ *then* $c < 10^{109}$.
vii) *If* $b^{1.4} \leq c < b^{1.5}$ *then* $c < 10^{128}$.
viii) *If* $b^{1.3} \leq c < b^{1.4}$ *then* $c < 10^{148}$.
ix) *If* $b^{1.2} \leq c < b^{1.3}$ *then* $c < 10^{133}$.
x) *If* $b^{1.16} \leq c < b^{1.2}$ *then* $c < 10^{107}$.

Sometimes it is possible to replace use of Baker theory by the so-called hypergeometric method of Thue and Siegel. Roughly speaking, the former is always applicable, while the second approach requires some hypotheses to be met. The downside of universality is a comparatively decreased quality of the output. More precisely, the bounds derived by applying any result of the former type are weaker than those produced with the latter technique. Another striking difference: while theorems of the first kind are directly applicable, those based on the hypergeometric method need to be adjusted to the specific problem one wants to solve.

A suitable specialisation of a general theorem given in [29] is the main ingredient in the work of Filipin and Fujita [19] in which they obtained an even better relative bound for the third element of a hypothetical $D(-1)$-quadruple.

Theorem 3. ([19]) *Any $D(-1)$-quadruple $(1, b, c, d)$ with $1 < b < c < d$ satisfies $c < 9.6\, b^4$.*

As expected, in this survey many details of the proofs have been deliberately ignored. Some of the parameters involved in the published work and which have been disregarded in our presentation will be mentioned in the next section, devoted to an overview of work in progress on the existence of $D(-1)$-quadruples.

3. Alternative approach

In arguments leading to the results stated above appear, among others, the integers $f = t - rs$, $g = bs - rt$, $h = st - cr$, and $e = 2bc - 2rst - c - b - 1$. It is easy to verify that f, g, h are positive while e is nonnegative, and they are related by several simple equations:

$$e + 1 = f^2, \quad be + 1 = g^2, \quad ce + 1 = h^2, \quad c = 1 + b + (2b - 1)e + 2rfg.$$

All this has been used in various ways, cf. [11, 20, 21, 24]. However, it seems that usefulness has not been exhausted. Our starting point was an elementary observation: one can reduce the number of variables appearing in the equality $f + rs = t$ by squaring it. Thus, one gets $f^2 + 2frs + r^2s^2 = (r^2 + 1)(s^2 + 1) - 1$, so that

$$r^2 + s^2 = 2frs + f^2. \tag{10}$$

Our approach is essentially a study of solutions in positive integers to equation (10) in its various disguises, starting with

$$(s - rf)^2 - (f^2 - 1)r^2 = f^2. \tag{11}$$

The change in center of interest is accompanied by higher priority bestowed on f. While the focus in the published papers is on b, c, m, and n, we pay more attention to the neglected parameter f.

In the first instance we revised the known results keeping in mind to pinpoint the locations where the shift of priorities allows us to uncover hidden phenomena, overlooked prospects, disregarded arguments. This phase of our research has provided noticeable results. In order to obtain the next theorem, we used, besides computations already employed and reported in the literature, a new experimental result, according to which there are no $D(-1)$-quadruples $(1, b, c, d)$ with the corresponding f less than or equal to 10^7.

Theorem 4. *There are no $D(-1)$-quadruples with $f = \gcd(r, s)$. In particular, there exists no $D(-1)$-quadruple for which the corresponding f has no prime divisor congruent to 1 modulo 4.*

The next step was to determine to what extent inequalities relating b, c, m, and n have analogue relations in which f appears. One particularly neat result of the kind is $f > 2r^2$. Our quest for analogy permitted us to bring to light the surprising existence of a gap in the values taken by $\log_b c = \log c / \log b$.

Theorem 5. *If $c \geq b^2$ then $c > 16b^3$.*

From Theorems 1–3 it is clear that the strategy for proving the non-existence of $D(-1)$–quadruples consists of increasing the lower bound and decreasing the upper bound for $\log_b c$. The best published results give that no $D(-1)$–quadruple has $\log_b c < 1.16$ or $\log_b c > 4.1$ and might give impression that $\log_b c$ covers an interval.

Theorem 5 is of a completely different nature since it implies that the set of values taken by $\log_b c$ is not connected, and suggests a new basic idea, viz. to split the interval $[1.16, 4.1]$ into several subintervals and proceed with computations in parallel. The endpoints are selected as suggested by the statement of Theorem 2. Implementation of the splitting requires heavy computations for which we rely on a so-called 'kit' for linear forms in three logarithms [27]. This is a result somewhat difficult to use, but whose output is at least an order of magnitude better than other theorems belonging to the area known as Baker theory for linear forms in logarithms.

Results available so far are encouraging, the existence of additional gaps in the set of values for $\log_b c$ is certified. Presently we are in the process of sorting the available data and performing computations which hopefully will confirm that no $D(-1)$–quadruple exists.

Acknowledgements

It is my pleasure to acknowledge the excellent work done by Yasutsugu Fujita and Hidehiko Misho in organising the Workshop on Analytic Number Theory and Related Fields 2017. I am grateful to RIMS for providing financial support to attend this meeting.

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