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Degenerate Bernoulli polynomials and poly-Cauchy polynomials

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1 Introduction

Carlitz [6, 7] defined the *degenerate Bernoulli polynomials* $\beta_m(\lambda,x)$ by means of the generating function

$$\left(\frac{t}{(1+\lambda t)^{1/\lambda}-1}\right)(1+\lambda t)^{x/\lambda}=\sum_{n=0}^\infty\beta_n(\lambda,x)\frac{t^n}{n!}\,. \tag{1}$$

When $\lambda\to 0$ in (1), $B_n(x)=\beta_n(0,x)$ are the ordinary Bernoulli polynomials because

$$\lim_{\lambda\to 0}\left(\frac{t}{(1+\lambda t)^{1/\lambda}-1}\right)(1+\lambda t)^{x/\lambda}=\frac{te^{xt}}{e^t-1}=\sum_{n=0}^\infty B_n(x)\frac{t^n}{n!}\,.$$

When $\lambda\to 0$ and $x=0$ in (1), $B_n=\beta_n(0,0)$ are the classical Bernoulli numbers defined by

$$\frac{t}{e^t-1}=\sum_{n=0}^\infty B_n\frac{t^n}{n!}\,. \tag{2}$$

The degenerate Bernoulli polynomials in λ and x have rational coefficients. When $x=0$, $\beta_n(\lambda)=\beta(\lambda,0)$ are called *degenerate Bernoulli numbers*. In [17], explicit formulas for the coefficients of the polynomial $\beta_n(\lambda)$ are found. In [27], a general symmetric identity involving the degenerate Bernoulli polynomials and the sums of generalized falling factorials are proved.

On the other direction, *hypergeometric Bernoulli polynomials* $B_{N,n}(z)$ (see, e.g., [19]) are defined by the generating function

$$\frac{e^{tx}}{{}_1F_1(1;N+1;t)}=\sum_{n=0}^\infty B_{N,n}(x)\frac{t^n}{n!}\,, \tag{3}$$

where ${}_1F_1(a;b;z)$ is the confluent hypergeometric function defined by

$$ {}_1F_1(a;b;z)=\sum_{n=0}^{\infty}\frac{(a)^{(n)}}{(b)^{(n)}}\frac{z^n}{n!} $$

with the rising factorial $(x)^{(n)}=x(x+1)\dots(x+n-1)$ $(n\ge 1)$ and $(x)^{(0)}=1$. When $x=0$ in (3), $B_{N,n}=B_{N,n}(0)$ are the hypergeometric Bernoulli numbers ([14, 15, 11, 12, 21]). When $N=1$ in (3), $B_n(x)=B_{1,n}(x)$ are the ordinary Bernoulli polynomials. When $x=0$ and $N=1$ in (3), $B_n=B_{1,n}(0)$ are the classical Bernoulli numbers.

Many kinds of generalizations of the Bernoulli numbers have been considered by many authors. For example, Poly-Bernoulli number, Apostol Bernoulli numbers, various types of q-Bernoulli numbers, Bernoulli Carlitz numbers. One of the advantages of hypergeometric numbers is the natural extension of determinant expressions of the numbers.

The determinant expression of hypergeometric Bernoulli numbers ([2, 20]) are given by

$$ B_{N,n}=(-1)^n n!\begin{vmatrix} \frac{N!}{(N+1)!} & 1 & & & \\ \frac{N!}{(N+2)!} & \frac{N!}{(N+1)!} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{N!}{(N+n-1)!} & \frac{N!}{(N+n-2)!} & \cdots & \frac{N!}{(N+1)!} & 1 \\ \frac{N!}{(N+n)!} & \frac{N!}{(N+n-1)!} & \cdots & \frac{N!}{(N+2)!} & \frac{N!}{(N+1)!} \end{vmatrix}. \tag{4} $$

The determinant expression for the classical Bernoulli numbers $B_n=B_{1,n}$ was discovered by Glaisher ([10, p.52]).

2 Hypergeometric degenerate Bernoulli numbers

Denote the generalized falling factorial by

$$ (x|\alpha)_n=x(x-\alpha)(x-2\alpha)\cdots\big(x-(n-1)\alpha\big)\quad(n\ge 1) $$

with $(x|\alpha)_0=1$. When $\alpha=1$, $(x)_n=(x|1)_n$ is the original falling factorial. Define *hypergeometric degenerate Bernoulli polynomials* $\beta_{N,n}(\lambda,x)$ by

$$ \left({}_2F_1\left(1,N-\frac{1}{\lambda};N+1;-\lambda t\right)\right)^{-1}(1+\lambda t)^{x/\lambda}=\sum_{n=0}^{\infty}\beta_{N,n}(\lambda,x)\frac{t^n}{n!}, \tag{5} $$

where ${}_2F_1(a,b;c;z)$ is the Gauss hypergeometric function defined by

$$ {}_2F_1(a;b;z) = \sum_{n=0}^{\infty} \frac{(a)^{(n)}(b)^{(n)}}{(c)^{(n)}} \frac{z^n}{n!}. $$

When $x = 0$ in (5), $\beta_{N,n}(\lambda) = \beta_{N,n}(\lambda, 0)$ are the *hypergeometric degenerate Bernoulli numbers.* Since

$$ \frac{t}{(1+\lambda t)^{1/\lambda} - 1} = t \left(\sum_{n=1}^{\infty} \frac{(1-\lambda|\lambda)_{n-1}}{n!} t^n \right)^{-1} $$

in (1), we can write

$$ \begin{aligned} & {}_2F_1\left(1, N - \frac{1}{\lambda}; N+1; -\lambda t\right) \\ &= \left(\sum_{n=1}^{\infty} \frac{(1-\lambda|\lambda)_{N-1}}{N!} t^N \right) \left(\sum_{n=N}^{\infty} \frac{(1-\lambda|\lambda)_{n-1}}{n!} t^n \right)^{-1} \\ &= 1 + \sum_{n=1}^{\infty} \frac{(1-\lambda|\lambda)_{N+n-1} N!}{(1-\lambda|\lambda)_{N-1}(N+n)!} t^n \\ &= 1 + \sum_{n=1}^{\infty} \frac{(1-N\lambda|\lambda)_n}{(N+n)_n} t^n . \end{aligned} \tag{6} $$

When $N = 1$, the definition (5) with (6) is reduced to that of degenerate Bernoulli polynomials by

$$ \left(1 + \sum_{n=1}^{\infty} \frac{(1-\lambda|\lambda)_n}{(n+1)!} t^n \right)^{-1} (1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} \beta_n(\lambda, x) \frac{t^n}{n!}. $$

When $N = 1$ and $\lambda \to 0$, the definition (5) with (6) is reduced to that of the classical Bernoulli polynomials by

$$ \left(1 + \sum_{n=1}^{\infty} \frac{t^n}{(n+1)!} \right)^{-1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. $$

We have the following recurrence relation of hypergeometric degenerate Bernoulli numbers $\beta_{N,n}(\lambda)$.

Proposition 1. *For $N \geq 0$, we have*

$$\beta_{N,n}(\lambda) = -\sum_{k=0}^{n-1} \frac{n!(1-N\lambda|\lambda)_{n-k}N!}{(N+n-k)!k!}\beta_{N,k}(\lambda) \quad (n \geq 1)$$

with $\beta_{N,0}(\lambda) = 1$.

We have an explicit expression of $\beta_{N,n}(\lambda)$.

Theorem 1. *For $n \geq 1$,*

$$\beta_{N,n}(\lambda) = n!\sum_{k=1}^{n}(-N!)^k \sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\dots,i_k\geq 1}} \frac{(1-N\lambda|\lambda)_{i_1}}{(N+i_1)!}\cdots\frac{(1-N\lambda|\lambda)_{i_k}}{(N+i_k)!}.$$

There is an alternative form of $\beta_{N,n}(\lambda)$ by using binomial coefficients. The proof is similar to that of Theorem 1 and is omitted.

Theorem 2. *For $n \geq 1$,*

$$\beta_{N,n}(\lambda) = n!\sum_{k=1}^{n}(-N!)^k\binom{n+1}{k+1} \sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\dots,i_k\geq 0}} \frac{(1-N\lambda|\lambda)_{i_1}}{(N+i_1)!}\cdots\frac{(1-N\lambda|\lambda)_{i_k}}{(N+i_k)!}.$$

3 A determinant expression of hypergeometric degenerated Bernoulli numbers

Theorem 3. *For $n \geq 1$, we have*

$$\beta_{N,n}(\lambda)$$
$$= (-1)^n n! \begin{vmatrix} \frac{(1-N\lambda)N!}{(N+1)!} & 1 & & & \\ \frac{(1-N\lambda|\lambda)_2 N!}{(N+2)!} & \frac{(1-N\lambda)N!}{(N+1)!} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{(1-N\lambda|\lambda)_{n-1}N!}{(N+n-1)!} & \frac{(1-N\lambda|\lambda)_{n-2}N!}{(N+n-2)!} & \cdots & \frac{(1-N\lambda)N!}{(N+1)!} & 1 \\ \frac{(1-N\lambda|\lambda)_n N!}{(N+n)!} & \frac{(1-N\lambda|\lambda)_{n-1}N!}{(N+n-1)!} & \cdots & \frac{(1-N\lambda|\lambda)_2 N!}{(N+2)!} & \frac{(1-N\lambda)N!}{(N+1)!} \end{vmatrix}.$$

Remark. When $\lambda \to 0$ in Theorem 3, we have a determinant expression of hypergeometric Bernoulli numbers $B_{N,n}$ in (4). If $\lambda \to 0$ and $N = 1$ in Theorem 3, we recover the classical determinant expression of the Bernoulli numbers B_n ([10, p.52]).

4 Applications by the Trudi's formula

We shall use the Trudi's formula to obtain different explicit expressions and inversion relations for the numbers $\beta_{N,n}(\lambda)$.

Lemma 1. *For a positive integer n, we have*

$$\begin{vmatrix} a_1 & a_0 & 0 & \cdots & \\ a_2 & a_1 & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ a_{n-1} & & \cdots & a_1 & a_0 \\ a_n & a_{n-1} & \cdots & a_2 & a_1 \end{vmatrix}$$
$$= \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\dots,t_n} (-a_0)^{n-t_1-\cdots-t_n} a_1^{t_1} a_2^{t_2} \cdots a_n^{t_n},$$

where $\binom{t_1+\cdots+t_n}{t_1,\dots,t_n} = \frac{(t_1+\cdots+t_n)!}{t_1!\cdots t_n!}$ *are the multinomial coefficients.*

This relation is known as Trudi's formula [25, Vol.3, p.214],[26] and the case $a_0 = 1$ of this formula is known as Brioschi's formula [4],[25, Vol.3, pp.208–209].

In addition, there exists the following inversion formula (see, e.g. [23]), which is based upon the relation

$$\sum_{k=0}^{n} (-1)^{n-k} \alpha_k D(n-k) = 0 \quad (n \ge 1)$$

or Cameron's operator in ([5]).

Lemma 2. *If $\{\alpha_n\}_{n\ge 0}$ is a sequence defined by $\alpha_0 = 1$ and*

$$\alpha_n = \begin{vmatrix} D(1) & 1 & & \\ D(2) & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ D(n) & \cdots & D(2) & D(1) \end{vmatrix}, \text{ then } D(n) = \begin{vmatrix} \alpha_1 & 1 & & \\ \alpha_2 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ \alpha_n & \cdots & \alpha_2 & \alpha_1 \end{vmatrix}.$$

From Trudi's formula, it is possible to give the combinatorial expression

$$\alpha_n = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\dots,t_n} (-1)^{n-t_1-\cdots-t_n} D(1)^{t_1} D(2)^{t_2} \cdots D(n)^{t_n} .$$

By applying these lemmata to Theorem 3, we obtain an explicit expression for the hypergeometric degenerate Bernoulli numbers.

Theorem 4. *For $N, n \ge 1$,*

$$\beta_{N,n}(\lambda) = n! \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\dots,t_n} (-1)^{t_1+\cdots+t_n} \times \left(\frac{(1-N\lambda)N!}{(N+1)!}\right)^{t_1} \left(\frac{(1-N\lambda|\lambda)_2 N!}{(N+2)!}\right)^{t_2} \cdots \left(\frac{(1-N\lambda|\lambda)_n N!}{(N+n)!}\right)^{t_n} .$$

Theorem 5. *For $N, n \ge 1$,*

$$\frac{(-1)^n (1-N\lambda|\lambda)_n N!}{(N+n)!} = \begin{vmatrix} \beta_{N,1}(\lambda) & 1 & & & \\ \frac{\beta_{N,2}(\lambda)}{2!} & \beta_{N,1}(\lambda) & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{\beta_{N,n-1}(\lambda)}{(n-1)!} & \frac{\beta_{N,n-2}(\lambda)}{(n-2)!} & \cdots & \beta_{N,1}(\lambda) & 1 \\ \frac{\beta_{N,n}(\lambda)}{n!} & \frac{\beta_{N,n-1}(\lambda)}{(n-1)!} & \cdots & \frac{\beta_{N,2}(\lambda)}{2!} & \beta_{N,1}(\lambda) \end{vmatrix} .$$

Applying the Trudi's formula in Lemma 1 to Theorem 5, we get the inversion relation of Theorem 4.

Theorem 6. *For $N, n \ge 1$,*

$$\frac{(1-N\lambda|\lambda)_n N!}{(N+n)!} = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\dots,t_n} (-1)^{t_1+\cdots+t_n} \times (\beta_{N,1}(\lambda))^{t_1} \left(\frac{\beta_{N,2}(\lambda)}{2!}\right)^{t_2} \cdots \left(\frac{\beta_{N,n}(\lambda)}{n!}\right)^{t_n} .$$

5 Generalized Stirling numbers

Hsu and Shiue [18] defined generalized Stirling number pairs by the generating function

$$k!\sum_{n=k}^{\infty} S(n,k;\alpha,\beta,r)\frac{t^n}{n!} = (1+\alpha t)^{r/\alpha}\left(\frac{(1+\alpha t)^{\beta/\alpha}-1}{\beta}\right)^k$$

where $(\alpha,\beta)\neq(0,0)$. The usual Stirling numbers of the first and second kinds $s(n,k)$ and $S(n,k)$ are given by the parameters $(\alpha,\beta,r)=(1,0,0)$ and $(\alpha,\beta,r)=(0,1,0)$, respectively. (When $\alpha=0$ or $\beta=0$ the equation is understood to mean the limit as $\alpha\to 0$ or $\beta\to 0$.) The parameters $(1,0,-x)$ and $(0,1,x)$ give Carlitz' weighted Stirling numbers of the first and second kinds, and the parameters $(1,\lambda,0)$ give the degenerate Stirling numbers of Carlitz. Hsu and Shiue demonstrated that there is in general a duality between the generalized Stirling numbers with parameters (α,β,r) and $(\beta,\alpha,-r)$.

Carlitz [7] also defined the *degenerate Bernoulli polynomials of higher order* $\beta_n^{(w)}(\lambda,x)$ for $\lambda\neq 0$ by means of the generating function

$$\left(\frac{t}{(1+\lambda t)^{\mu}-1}\right)^w (1+\lambda t)^{\mu x} = \sum_{n=0}^{\infty}\beta_n^{(w)}(\lambda,x)\frac{t^n}{n!},$$

where $\lambda\mu=1$.

6 Convolution identities

Theorem 7. *If $k\geq w$ we have*

$$\sum_{j=0}^{n-k}\binom{n}{j}S(n-j,k;\alpha,\beta,r)\beta_j^{(w)}(\lambda,x)\beta^j = \frac{\binom{n}{w}}{\binom{k}{w}}S(n-w,k-w;\alpha,\beta,r+\beta x)$$

where $\lambda\beta=\alpha$; and for $k\leq w$ we have

$$\sum_{j=0}^{n-k}\binom{n}{j}S(n-j,k;\alpha,\beta,r)\beta_j^{(w)}(\lambda,x)\beta^j = \binom{n}{k}\beta_{n-k}^{(w-k)}(\lambda,x+\frac{r}{\beta})\beta^{n-k}.$$

6.1 Limiting cases

When $\lambda = 0$ our convolution involves the order w Bernoulli polynomials and weighted Stirling numbers of the second kind, and the result is either a weighted Stirling number of the second kind or a Bernoulli polynomial, depending on whether $k \geq w$.

Corollary 1. *($\lambda = 0$ case) If $k \geq w$ we have*

$$\sum_{j=0}^{n-k} \binom{n}{j} S(n-j, k; 0, 1, r) B_j^{(w)}(x) = \frac{\binom{n}{w}}{\binom{k}{w}} S(n-w, k-w; 0, 1, r+x)$$

and for $k \leq w$ we have

$$\sum_{j=0}^{n-k} \binom{n}{j} S(n-j, k; 0, 1, r) B_j^{(w)}(x) = \binom{n}{k} B_{n-k}^{(w-k)}(x+r).$$

When $\mu = 0$ our convolution involves the order w Bernoulli polynomials of the second kind and weighted Stirling numbers of the first kind, and the result is either a weighted Stirling number of the first kind or a Bernoulli polynomial of the second kind, depending on whether $k \geq w$.

Corollary 2. *($\mu = 0$ case) If $k \geq w$ we have*

$$\sum_{j=0}^{n-k} \binom{n}{j} S(n-j, k; 1, 0, r) j! b_j^{(w)}(x) = \frac{\binom{n}{w}}{\binom{k}{w}} S(n-w, k-w; 1, 0, r+x)$$

and for $k \leq w$ we have

$$\sum_{j=0}^{n-k} \frac{S(n-j, k; 1, 0, r)}{(n-j)!} b_j^{(w)}(x) = \frac{b_{n-k}^{(w-k)}(x+r)}{k!}.$$

6.2 Zero-order cases

In this section we consider the specializations of the main result when either $k = 0$, $w = 0$, or $k - w = 0$. When $k = w$ the sum reduces to a single falling factorial or power; this occurs because $S(n, 0; \alpha, \beta, r) = (r|\alpha)_n$, where

$$(r|\alpha)_n = r(r-\alpha)\cdots(r-(n-1)\alpha)$$

denotes the generalized falling factorial with increment α, with convention $(r|\alpha)_0 = 1$ ([18, eq.(8)]).

Corollary 3. *($k = w$ case) We have*

$$\sum_{j=0}^{n-k} \binom{n}{j} S(n-j,k;\alpha,\beta,r)\beta_j^{(k)}(\lambda,x)\beta^j = \binom{n}{k}(r+\beta x|\alpha)_{n-k}$$

where $\lambda\beta = \alpha$; in particular for $\lambda = 0$ we have

$$\sum_{j=0}^{n-k} \binom{n}{j} S(n-j,k;0,1,r)B_j^{(k)}(x) = \binom{n}{k}(r+x)^{n-k}$$

and for $\mu = 0$ we have

$$\sum_{j=0}^{n-k} \binom{n}{j} S(n-j,k;1,0,r)j!b_j^{(k)}(x) = \binom{n}{k}(r+x|1)_{n-k}.$$

When $r = x = 0$ in the above corollary we obtain the orthogonality relations

$$\sum_{j=0}^{n-k} \binom{n}{j} S(n-j,k;\alpha,\beta,0)\beta_j^{(k)}(\lambda)\beta^j = \delta_{n,k}$$

where $\delta_{n,k}$ is the Kronecker delta; in particular we have

$$\sum_{j=0}^{n-k} \binom{n}{j} S(n-j,k)B_j^{(k)} = \delta_{n,k}$$

and

$$\sum_{j=0}^{n-k} \binom{n}{j} s(n-j,k)j!b_j^{(k)} = \delta_{n,k}$$

in terms of the usual Stirling numbers $s(n,k)$ and $S(n,k)$ of the first and second kinds.

In the case $k = 0$ the generalized Stirling number disappears from the convolution and we obtain a recurrence involving Bernoulli polynomials only.

Corollary 4. *($k = 0$ case) We have*

$$\sum_{j=0}^{n} \binom{n}{j} (r|\alpha)_{n-j}\beta_j^{(w)}(\lambda,x)\beta^j = \beta_n^{(w)}(\lambda, x+(r/\beta))\beta^n$$

where $\lambda\beta = \alpha$; in particular for $\lambda = 0$ we have

$$\sum_{j=0}^{n} \binom{n}{j} r^{n-j} B_j^{(w)}(x) = B_n^{(w)}(x+r)$$

and for $\mu = 0$ we have

$$\sum_{j=0}^{n} \binom{r}{n-j} b_j^{(w)}(x) = b_n^{(w)}(x+r).$$

Note that the second equation ($\lambda = 0$) of this corollary is a well-known recurrence for Bernoulli polynomials, particularly in the case $x = 0$. The third equation ($\mu = 0$) does not appear to be so well known.

In the case $w = 0$ the Bernoulli polynomial disappears from the convolution and we obtain a recurrence involving Stirling numbers only.

Corollary 5. *($w = 0$ case) We have*

$$\sum_{j=0}^{n-k} \binom{n}{j} S(n-j, k; \alpha, \beta, r)(x|\lambda)_j \beta^j = S(n, k; \alpha, \beta, r + \beta x)$$

where $\lambda\beta = \alpha$; in particular for $\lambda = 0$ we have

$$\sum_{j=0}^{n-k} \binom{n}{j} S(n-j, k; 0, 1, r) x^j = S(n, k; 0, 1, r + x)$$

and for $\mu = 0$ we have

$$\sum_{j=0}^{n-k} \binom{n}{j} S(n-j, k; 1, 0, r) j! \binom{x}{j} = S(n, k; 1, 0, r + x)$$

These two special cases ($\lambda = 0$ and $\mu = 0$) are well-known recurrences for weighted Stirling numbers, particularly in the case $r = 0$.

6.3 First-order cases

When either $k = 1$ or $w = 1$ the generalized Stirling number may be simplified to

$$S(n, 1; \alpha, \beta, r) = \beta^{-1} \left((r + \beta|\alpha)_n - (r|\alpha)_n \right)$$

in terms of generalized falling factorials. This may be proven by induction from the recurrence

$$S(n+1,k;\alpha,\beta,r) = S(n,k-1;\alpha,\beta,r) + (k\beta - n\alpha + r)S(n,k;\alpha,\beta,r)$$

[18, eq. (7)]. Taking the limit as $\beta \to 0$ yields

$$S(n,1;1,0,r) = (r|1)_n \left[\frac{1}{r} + \frac{1}{r-1} + \cdots + \frac{1}{r-n+1}\right].$$

Corollary 6. *($k = w = 1$ case) We have*

$$\sum_{j=0}^{n-1} \binom{n}{j} \left((r+\beta|\alpha)_{n-j} - (r|\alpha)_{n-j}\right) \beta_j(\lambda,x)\beta^{j-1} = n(r+\beta x|\alpha)_{n-1}$$

where $\lambda\beta = \alpha$; in particular for $\lambda = 0$ we have

$$\sum_{j=0}^{n-1} \binom{n}{j} \left((r+1)^{n-j} - r^{n-j}\right) B_j(x) = n(r+x)^{n-1}$$

and for $\mu = 0$ we have

$$\sum_{j=0}^{n-1} \binom{r}{n-j} \left[\frac{1}{r} + \frac{1}{r-1} + \cdots + \frac{1}{r-(n-j-1)}\right] b_j(x) = \binom{r+x}{n-1}.$$

In the case $r = 0$, the $\lambda = 0$ case of the above corollary reflects the usual recurrence and difference equation for the Bernoulli polynomials. In the $\mu = 0$ case the weighted Stirling numbers of the first kind reduce to generalized harmonic numbers; in particular taking $r = n$ we obtain

$$\sum_{j=0}^{n} \binom{n}{j} (H_n - H_j) b_j(x) = \binom{n+x}{n-1}$$

where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ denotes the nth harmonic number, and more specifically for $x = 0$ we get

$$\sum_{j=0}^{n} \binom{n}{j} (H_n - H_j) b_j = n.$$

Taking $x = -1$ and using the identity $B_n^{(n)} = n!b_n(-1)$ ([16, eq. (2.10)]) yields the identity

$$\sum_{j=0}^{n} \binom{n}{j} (H_n - H_j) \frac{B_j^{(j)}}{j!} = 1$$

for the Nörlund numbers $B_n^{(n)}$.

Corollary 7. *($k = 1$ case) We have*

$$\sum_{j=0}^{n-1} \binom{n}{j} \left((r+\beta|\alpha)_{n-j} - (r|\alpha)_{n-j}\right) \beta_j^{(w)}(\lambda, x)\beta^{j-1} = n\beta_{n-1}^{(w-1)}(x + (r/\beta))\beta^{n-1}$$

where $\lambda\beta = \alpha$; in particular for $\lambda = 0$ we have

$$\sum_{j=0}^{n-1} \binom{n}{j} \left((r+1)^{n-j} - r^{n-j}\right) B_j^{(w)}(x) = nB_{n-1}^{(w-1)}(x + r)$$

and for $\mu = 0$ we have

$$\sum_{j=0}^{n-1} \binom{r}{n-j} \left[\frac{1}{r} + \frac{1}{r-1} + \cdots + \frac{1}{r-(n-j-1)}\right] b_j^{(w)}(x) = b_{n-1}^{(w-1)}(x + r).$$

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