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THE DISTRIBUTION OF RELATIVELY R-PRIME LATTICE POINTS

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1. Introduction

In this article we consider the error term of the distribution of relatively r-prime lattice points over number fields. Let K be a number field and let $\mathcal{O}_K$ be its ring of integers. We regard an m-tuple of ideals $(\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_m)$ of $\mathcal{O}_K$ as a lattice point in K^m. We say that a lattice point $(\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_m)$ is *relatively r-prime*, if there exists no prime ideal $\mathfrak{p}$ such that $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_m \subset \mathfrak{p}^r$. Let $V_m^r(x, K)$ denote the number of relatively r-prime lattice points $(\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_m)$ such that their ideal norm $\mathfrak{N}\mathfrak{a}_i \leq x$. One can show that for all $x \geq 1$ and for all number fields K the number of relatively 1-prime lattice points $V_1^1(x, K) = 1$, so in this paper we assume $rm \geq 2$.

B. D. Sittinger shows that

Theorem 1.1 (cf. [Si10]). *Let $n = [K : \mathbf{Q}]$ then*

$$V_m^r(x, K) = \frac{\rho_K^m}{\zeta_K(rm)} x^m + \begin{cases} O_K\left(x^{m-\frac{1}{n}}\right) & \text{if } m \geq 3, \text{ or } m = 2 \text{ and } r \geq 2, \\ O_K\left(x^{2-\frac{1}{n}} \log x\right) & \text{if } m = 2 \text{ and } r = 1, \\ O_K\left(x^{1-\frac{1}{n}} \log x\right) & \text{if } m = 1 \text{ and } \frac{n(r-2)}{r-1} = 1, \\ O_K\left(x^{1-\frac{1}{n}}\right) & \text{if } m = 1 \text{ and } \frac{n(r-2)}{r-1} > 1, \\ O_K\left(x^{\frac{1}{r}\left(2-\frac{1}{n}\right)}\right) & \text{if } m = 1 \text{ and } \frac{n(r-2)}{r-1} < 1, \end{cases}$$

where ζ_K is the Dedekind zeta function over K and ρ_K is the residue of $\zeta_K(s)$ at $s = 1$.

It is well known that

$$\rho_K = \frac{2^{r_1}(2\pi)^{r_2} hR}{w\sqrt{D_K}}, \tag{1.2}$$

where h is the class number of K, r_1 is the number of real embeddings of K, r_2 is the number of pairs of complex embeddings, R is the regulator of K, w is the number of roots of unity in $\mathcal{O}_K^*$ and D_K is absolute value of the discriminant of K.

Let $I_K(x)$ be the number of ideals of $\mathcal{O}_K$ with their ideal norm less than or equal to x. Then it is known that

$$I_K(x) \sim \rho_K x. \tag{1.3}$$

We denote $\Delta_K(x)$ be the error term of $I_K(x)$, that is, $I_K(x) - \rho_K x$. And we know that the number of relatively r-prime lattice points $V_m^r(x, K)$ can be expressed

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as the sum of product of $I_K(x)$ and $\mu_K(\mathfrak{a})$ and the relation between $\Delta_K(x)$ and $E_m^r(x,K)$. Thus we consider $\Delta_K(x)$ throughout this paper.

In the next section, we introduce and show some auxiliary theorems to consider an uniform upper bound of the error term $\Delta_K(x)$. In Section 3, we prove the following theorem about the error term of relatively r-prime lattice points $E_m^r(x,K)$, where K runs through number fields with $[K:\mathbf{Q}] \le n$.

Theorem. *For all $\varepsilon > 0$ and $C > 0$ the followings hold.*

(1) *When K runs through cubic extensions field with $D_K \le Cx^{\frac{1}{4}}$, then*

$$E_m^r(x,K) = \begin{cases} O_{C,\varepsilon}\left(x^{\frac{19}{13r}+\varepsilon} D_K^{\frac{4}{13}-\frac{m-1}{2}+\varepsilon}\right) & \text{if } rm = 2, \\ O_{C,\varepsilon}\left(x^{m-\frac{7}{13}+\varepsilon} D_K^{\frac{2}{13}-\frac{m-1}{2}+\varepsilon}\right) & \text{otherwise.} \end{cases}$$

(2) *If K runs through number fields with $[K:\mathbf{Q}] \le n$ and $D_K \le Cx^{\frac{8}{2n+5}-\varepsilon}$, then*

$$E_m^r(x,K) = \begin{cases} O_{C,n,\varepsilon}\left(x^{\frac{4n-2}{r(2n+1)}+\varepsilon} D_K^{\frac{4}{2n+1}-\frac{m-1}{2}+\varepsilon}\right) & \text{if } rm = 2, \\ O_{C,n,\varepsilon}\left(x^{m-\frac{4}{2n+1}+\varepsilon} D_K^{\frac{2}{2n+1}-\frac{m-1}{2}+\varepsilon}\right) & \text{otherwise.} \end{cases}$$

2. Auxiliary Theorem

In this section, we show some important lemmas for our argument. Let $s = \sigma + it$ and $n = [K:\mathbf{Q}]$. We use the convexity bound of the Dedekind zeta function:

$$\zeta_K(\sigma+it) = O_{n,\varepsilon}\left(|t|^{\frac{n(1-\sigma)}{2}+\varepsilon} D_K^{\frac{1-\sigma}{2}+\varepsilon}\right) \tag{2.1}$$

as $|t|^n D_K \to \infty$ on $0 \le \sigma \le 1$, where the constant implied in O depends on ε and extension degree.

It is also well-known fact that Dedekind zeta function satisfies the following functional equation

$$Z_K(1-s) = Z_K(s), \tag{2.2}$$

where $Z_K(s) = D_K^{\frac{s}{2}} 2^{-(s-1)r_2} \pi^{-\frac{ns}{2}} \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} \zeta_K(s)$.

In the previous papers, we used upper bound of Dedekind zeta function to estimate the distribution of ideals. In the following sections, we show some estimate for $\Delta_K(x)$ in the similar way to our previous papers [Ta17].

Lemma 2.3 states the growth of the product of Gamma function and trigonometric functions in the functional equation of Dedekind zeta function.

Lemma 2.3. *Let $f \in \{cos, sin\}$ and n be a positive integer*

$$\begin{aligned} &\frac{\Gamma(s)^n}{1-s}\left(\cos\frac{\pi s}{2}\right)^{r_1+r_2}\left(\sin\frac{\pi s}{2}\right)^{r_2} \\ &= Cn^{-ns}\Gamma\left(ns - \frac{n+1}{2}\right) f\left(\frac{n\pi s}{2}\right) + O_{n,\varepsilon}\left(|t|^{-2+\frac{n}{2}+\varepsilon}\right), \end{aligned}$$

where C is a constant.

Proof. This lemma is shown from the Stirling formula and estimate for trigonometric function. □

Next we introduce the generalized Atkinson's lemma:

Lemma 2.4. *Let* $y > 0,\ 1 < A \le B$ *and* $f \in \{\cos, \sin\}$, *and we define*

$$I = \frac{1}{2\pi i}\int_{A-iB}^{A+iB} \Gamma(s) f\left(\frac{\pi s}{2}\right) y^{-s}\, ds.$$

If $y \le B$, *then*

$$I = f(y) + O\left(y^{-\frac{1}{2}} \min\left(\left(\log \frac{B}{y}\right), B^{\frac{1}{2}}\right) + y^{-A}B^{A-\frac{1}{2}} + y^{-\frac{1}{2}}\right).$$

If $y > B$, *then*

$$I = O\left(y^{-A}\left(B^{A-\frac{1}{2}} \min\left(\left(\log \frac{y}{B}\right), B^{\frac{1}{2}}\right) + A^{A-\frac{1}{2}}\right)\right).$$

This lemma is quite useful for calculating the integral of the Dedekind zeta function. Many authors used this generalized lemma but as far as I know there is no proof. But we give proof of this lemma following the Atkinson's original proof in my preprint paper.

Finally we introduce the following lemma to reduce the ideal counting problem to an exponential sum problem.

Lemma 2.5 (Proposition 3.1 of [Bo15]). *Let* $1 \le L \le R$ *be a real number and* φ *be an arithmetical function satisfying* $\varphi(m) = O(m^{\varepsilon})$, *and let* $\mathbf{e}(x) = exp(2\pi i x)$ *and* $F = \varphi * \mu$, *where* $*$ *is the Dirichlet product symbol. For* $a \in \mathbf{R} - 1$, $b, x \in \mathbf{R}$ *and for every* $\varepsilon > 0$ *the following estimate holds.*

$$\sum_{m \le R} \frac{\varphi(m)}{m^a} f\left(2\pi x m^b\right)$$
$$= O_{n,\varepsilon}\left(\begin{array}{l} L^{1-a} + R^{\varepsilon} \max\limits_{L<S\le R} S^{-a} \times \\ \times \max\limits_{S<S_1\le 2S} \max\limits_{\substack{M,N\le S_1 \\ MN\asymp S}} \max\limits_{\substack{M\le M_1\le 2M \\ N\le N_1\le 2N}} \left| \sum\limits_{M<m\le M_1} F(m) \sum\limits_{N<n\le N_1} \mathbf{e}\left(x(mn)^b\right)\right| \end{array}\right).$$

Next proposition plays a crucial role in our computing $I_K(x)$. We consider the distribution of ideals of $\mathcal{O}_K$, where K runs through extensions with $[K : \mathbf{Q}] = n$ and some conditions. The detail of the conditions will be determined later, but they state the relation of main term and error term. Let $a_K(x)$ be the number of ideals of $\mathcal{O}_K$ with their ideal norm equal to x.

Proposition 2.6. *Let* $F_K = a_K * \mu$, *where* $*$ *is the Dirichlet product symbol, and let* $\mathbf{e}(x) = exp(2\pi i x)$. *For every* $\varepsilon > 0$ *the following estimate holds.*

$$\Delta_K(x)$$
$$= O_{n,\varepsilon}\left(\begin{array}{l} x^{\frac{n-1}{2n}} D_K^{\frac{1}{2n}} R^{\varepsilon} \max\limits_{S\le R} S^{-\frac{n+1}{2n}} \times \\ \times \max\limits_{S<S_1\le 2S} \max\limits_{\substack{M,N\le S_1 \\ MN\asymp S}} \max\limits_{\substack{M\le M_1\le 2M \\ N\le N_1\le 2N}} \left| \sum\limits_{M<m\le M_1} F_K(m) \sum\limits_{N<k\le N_1} \mathbf{e}\left(n\left(\frac{xmk}{D_K}\right)^{\frac{1}{n}}\right)\right| \\ + x^{\frac{n-2}{2n}+\varepsilon} D_K^{\frac{1}{n}+\varepsilon} R^{\frac{n-2}{2n}+\varepsilon} + x^{\frac{n-1}{n}+\varepsilon} D_K^{\frac{1}{n}+\varepsilon} R^{-\frac{1}{n}+\varepsilon} \end{array}\right),$$

where K *runs through number fields with* $[K : \mathbf{Q}] = n$ *and some conditions.*

Proof. We consider the integral

$$\frac{1}{2\pi i}\int_C \zeta_K(s)\frac{x^s}{s}\,ds,$$

where C is the contour $C_1 \cup C_2 \cup C_3 \cup C_4$ shown in the following Figure 1.

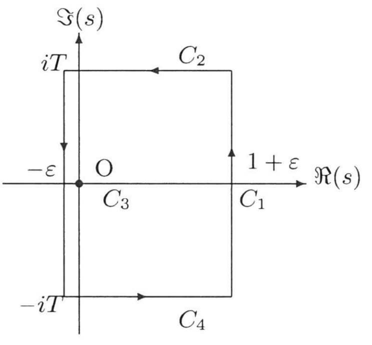

FIGURE 1. path of integral

In a way similar to the well-known proof of Perron's formula, we estimate

$$\frac{1}{2\pi i}\int_{C_1} \zeta_K(s)\frac{x^s}{s}\,ds = I_K(x) + O_\varepsilon\left(\frac{x^{1+\varepsilon}}{T}\right).$$

We can select the large T, so that the O-term in the right hand side is sufficiently small. For estimating the left hand side by using estimate (2.1), we divide it into the integrals over C_2, C_3 and C_4.

First we consider the integrals over C_2 and C_4 as

$$\left|\frac{1}{2\pi i}\int_{C_2\cup C_4} \zeta_K(s)\frac{x^s}{s}\,ds\right| \leq \frac{1}{2\pi}\int_{-\varepsilon}^{1+\varepsilon} |\zeta_K(\sigma+iT)|\frac{x^\sigma}{T}\,d\sigma + \frac{1}{2\pi}\int_{-\varepsilon}^{1+\varepsilon} |\zeta_K(\sigma-iT)|\frac{x^\sigma}{T}\,d\sigma.$$

It holds by the convexity bound of Dedekind zeta function (2.1) that their sum is estimated as

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{C_2\cup C_4} \zeta_K(s)\frac{x^s}{s}\,ds\right| &= O_{n,\varepsilon}\left(\int_{-\varepsilon}^{1+\varepsilon} (T^n D_K)^{\frac{1-\sigma}{2}+\varepsilon}\frac{x^\sigma}{T}\,d\sigma\right) \\ &= O_{n,\varepsilon}\left(\frac{x^{1+\varepsilon}D_K^\varepsilon}{T^{1-n\varepsilon}} + T^{\frac{n}{2}-1+\varepsilon}D_K^{\frac{1}{2}+\varepsilon}x^{-\varepsilon}\right).\end{aligned} \tag{2.7}$$

By Cauchy's residue theorem we get

$$\frac{1}{2\pi i}\int_C \zeta_K(s)\frac{x^s}{s}\,ds = \rho_K x.$$

This leads to

$$I_K(x) = \rho_K x + \int_{C_3} \zeta_K(s)\frac{x^s}{s}\,ds + O_{n,\varepsilon}\left(\frac{x^{1+\varepsilon}D_K^\varepsilon}{T^{1-n\varepsilon}} + T^{\frac{n}{2}-1+\varepsilon}D_K^{\frac{1}{2}+\varepsilon}x^{-\varepsilon}\right). \tag{2.8}$$

Thus it suffices to consider the integral over C_3 as

$$\frac{1}{2\pi i}\int_{C_3}\zeta_K(s)\frac{x^s}{s}\,ds=\frac{1}{2\pi i}\int_{-\varepsilon-iT}^{-\varepsilon+iT}\zeta_K(s)\frac{x^s}{s}\,ds.$$

Changing the variable s to $1-s$, we have

$$\frac{1}{2\pi i}\int_{C_3}\zeta_K(s)\frac{x^s}{s}\,ds=\frac{1}{2\pi i}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT}\zeta_K(1-s)\frac{x^{1-s}}{1-s}\,ds.$$

From the functional equation (2.2), it holds that

$$\begin{aligned}&\frac{1}{2\pi i}\int_{C_3}\zeta_K(s)\frac{x^s}{s}\,ds\\&=\frac{1}{2\pi i}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT}D_K^{s-\frac{1}{2}}2^{n(1-s)}\pi^{-ns}\Gamma(s)^n\left(\cos\frac{\pi s}{2}\right)^{r_1+r_2}\left(\sin\frac{\pi s}{2}\right)^{r_2}\zeta_K(s)\frac{x^{1-s}}{1-s}\,ds.\end{aligned}$$

By lemma 2.3 the integral over C_3 can be expressed as

$$\begin{aligned}&\frac{1}{2\pi i}\int_{C_3}\zeta_K(s)\frac{x^s}{s}\,ds\\&=\frac{Cx}{2\pi i}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT}D_K^{-\frac{1}{2}}\left(\frac{(2n)^n\pi^n x}{D_K}\right)^{-s}\Gamma\left(ns-\frac{n+1}{2}\right)f\left(\frac{n\pi s}{2}\right)\zeta_K(s)\,ds\\&+O_{n,\varepsilon}\left(D_K^{\frac{1}{2}+\varepsilon}T^{\frac{n}{2}-1+\varepsilon}x^{-\varepsilon}\right).\end{aligned}$$

Changing the variable $ns-\frac{n+1}{2}$ to s, we have

$$\begin{aligned}&\frac{1}{2\pi i}\int_{C_3}\zeta_K(s)\frac{x^s}{s}\,ds\\&=\frac{Cx^{\frac{n-1}{2n}}D_K^{\frac{1}{2n}}}{2\pi i}\int_{\frac{n-1}{2}+n\varepsilon-niT}^{\frac{n-1}{2}+n\varepsilon+niT}\left(2n\pi\left(\frac{x}{D_K}\right)^{\frac{1}{n}}\right)^{-s}\Gamma(s)f\left(\frac{\pi s}{2}+\frac{(n+1)\pi}{4}\right)\\&\times\zeta_K\left(\frac{1}{n}s+\frac{n+1}{2n}\right)\,ds+O_{n,\varepsilon}\left(D_K^{\frac{1}{2}+\varepsilon}T^{\frac{n}{2}-1+\varepsilon}x^{-\varepsilon}\right).\end{aligned}$$

The Dedekind zeta function $\zeta_K(s)$ can be expressed as

$$\zeta_K(s)=\sum_{n=1}^{\infty}\frac{a_K(n)}{n^s}\quad\text{for }\Re s>1. \tag{2.9}$$

This Dirichlet series is absolutely and uniformly convergent on compact subsets on $\Re(s)>1$. Therefore we can interchange the order of summation and integral. Thus we obtain

$$\begin{aligned}&\int_{\frac{n-1}{2}+n\varepsilon-niT}^{\frac{n-1}{2}+n\varepsilon+niT}\left(2n\pi\left(\frac{x}{D_K}\right)^{\frac{1}{n}}\right)^{-s}\Gamma(s)f\left(\frac{\pi s}{2}+\frac{(n+1)\pi}{4}\right)\zeta_K\left(\frac{s}{n}+\frac{n+1}{2n}\right)\,ds\\&=\sum_{m=1}^{\infty}\frac{a_K(m)}{m^{\frac{n+1}{2n}}}\int_{\frac{n-1}{2}+n\varepsilon-niT}^{\frac{n-1}{2}+n\varepsilon+niT}\left(2n\pi\left(\frac{mx}{D_K}\right)^{\frac{1}{n}}\right)^{-s}\Gamma(s)f\left(\frac{\pi s}{2}+\frac{(n+1)\pi}{4}\right)\,ds.\end{aligned}$$

Properties of trigonometric function lead to

$$f\left(\frac{\pi s}{2}+\frac{(n+1)\pi}{4}\right)=\pm\begin{cases} f\left(\frac{\pi s}{2}\right) & \text{if } n \text{ is odd,}\\ \frac{1}{\sqrt{2}}\left(f\left(\frac{\pi s}{2}\right)\pm g\left(\frac{\pi s}{2}\right)\right) & \text{if } n \text{ is even,}\end{cases}$$

where $\{f,g\}=\{\sin,\cos\}$. Hence it holds that

$$\begin{aligned}&\frac{1}{2\pi i}\int_{C_3}\zeta_K(s)\frac{x^s}{s}\,ds\\ &=\frac{Cx^{\frac{n-1}{2n}}D_K^{\frac{1}{2n}}}{2\pi i}\sum_{m=1}^{\infty}\frac{a_K(m)}{m^{\frac{n+1}{2n}}}\int_{\frac{n-1}{2}+n\varepsilon-niT}^{\frac{n-1}{2}+n\varepsilon+niT}\left(2n\pi\left(\frac{mx}{D_K}\right)^{\frac{1}{n}}\right)^{-s}\Gamma(s)f\left(\frac{\pi s}{2}\right)\,ds\\ &\quad+O_{n,\varepsilon}\left(D_K^{\frac{1}{2}+\varepsilon}T^{\frac{n}{2}-1+\varepsilon}x^{-\varepsilon}\right).\end{aligned}$$

Now we apply lemma 2.4 to this integral with $y=2n\pi\left(\frac{mx}{D_K}\right)^{\frac{1}{n}}$, $A=1+n\varepsilon$, $B=nT$ and $T=2\pi\left(\frac{xR}{D_K}\right)^{\frac{1}{n}}$, this becomes

$$\begin{aligned}&\frac{1}{2\pi i}\int_{C_3}\zeta_K(s)\frac{x^s}{s}\,ds\\ &=\frac{Cx^{\frac{n-1}{2n}}D_K^{\frac{1}{2n}}}{2\pi i}\sum_{m\le R}\frac{a_K(m)}{m^{\frac{n+1}{2n}}}f\left(2n\pi\left(\frac{mx}{D_K}\right)^{\frac{1}{n}}\right)\\ &\quad+O_{n,\varepsilon}\left(x^{\frac{n-2}{2n}}D_K^{\frac{1}{n}}\sum_{m\le R}\frac{a_K(m)}{m^{\frac{n+2}{2n}}}\min\left\{\left(\log\frac{R}{m}\right)^{-1},\left(\frac{Rx}{D_K}\right)^{\frac{1}{2n}}\right\}\right)\\ &\quad+O_{n,\varepsilon}\left(x^{\frac{n-2}{2n}}D_K^{\frac{1}{n}}\sum_{m\le R}\frac{a_K(m)}{m^{\frac{n+2}{2n}}}\left(\left(\frac{R}{m}\right)^{\frac{n-2}{2n}}+1\right)\right)\\ &\quad+O_{n,\varepsilon}\left(x^{\frac{n-2}{2n}}D_K^{\frac{1}{n}}R^{\frac{n-2}{2n}+\varepsilon}\sum_{m>R}\frac{a_K(m)}{m^{1+\varepsilon}}\min\left\{\left(\log\frac{m}{R}\right)^{-1},\left(\frac{Rx}{D_K}\right)^{\frac{1}{2n}}\right\}\right)\\ &\quad+O_{n,\varepsilon}\left(x^{\frac{n-2}{2n}+\varepsilon}D_K^{\frac{1}{n}+\varepsilon}R^{\frac{n-2}{2n}+\varepsilon}\right).\end{aligned}$$

We evaluate three O-terms as follows.

First we consider the first O-term. One can estimate $\left(\log \frac{R}{m}\right)^{-1} = O\left(\frac{R}{R-m}\right)$, so we obtain

$$O_{n,\varepsilon}\left(x^{\frac{n-2}{2n}} D_K^{\frac{1}{n}} \sum_{m\le R} \frac{a_K(m)}{m^{\frac{n+2}{2n}}} \min\left\{\left(\log \frac{R}{m}\right)^{-1}, \left(\frac{Rx}{D_K}\right)^{\frac{1}{2n}}\right\}\right)$$
$$= O_{n,\varepsilon}\left(x^{\frac{n-2}{2n}} D_K^{\frac{1}{n}} \sum_{m\le [R]-1} \frac{a_K(m)}{m^{\frac{n+2}{2n}}} \left(\log \frac{R}{m}\right)^{-1} + x^{\frac{n-2}{2n}} D_K^{\frac{1}{n}} \sum_{[R]\le m\le R} \frac{a_K(m)}{m^{\frac{n+2}{2n}}} \left(\frac{Rx}{D_K}\right)^{\frac{1}{2n}}\right)$$
$$= O_{n,\varepsilon}\left(x^{\frac{n-2}{2n}} D_K^{\frac{1}{n}} \sum_{m\le [R]-1} \frac{a_K(m)}{m^{\frac{n+2}{2n}}} \frac{R}{R-m} + x^{\frac{n-1}{2n}} D_K^{\frac{1}{2n}} R^{\frac{1}{2n}} \sum_{[R]\le m\le R} \frac{a_K(m)}{m^{\frac{n+2}{2n}}}\right)$$
$$= O_{n,\varepsilon}\left(x^{\frac{n-2}{2n}} D_K^{\frac{1}{n}} R^{\frac{n-2}{2n}+\varepsilon} + x^{\frac{n-1}{2n}} D_K^{\frac{1}{2n}} R^{-\frac{n+1}{2n}}\right).$$

Next we calculate the second O-term.

$$O_{n,\varepsilon}\left(x^{\frac{n-2}{2n}} D_K^{\frac{1}{n}} \sum_{m\le R} \frac{a_K(m)}{m^{\frac{n+2}{2n}}} \left(\left(\frac{R}{m}\right)^{\frac{n-2}{2n}} + 1\right)\right) = O_{n,\varepsilon}\left(x^{\frac{n-2}{2n}} D_K^{\frac{1}{n}} R^{\frac{n-2}{2n}} \sum_{m\le R} \frac{a_K(m)}{m}\right).$$

Since it is well-known that $a_K(m) = O(m^{\varepsilon})$, we get

$$O_{n,\varepsilon}\left(x^{\frac{n-2}{2n}} D_K^{\frac{1}{n}} \sum_{m\le R} \frac{a_K(m)}{m^{\frac{n+2}{2n}}} \left(\left(\frac{R}{m}\right)^{\frac{n-2}{2n}} + 1\right)\right) = O_{n,\varepsilon}\left(x^{\frac{n-2}{2n}} D_K^{\frac{1}{n}} R^{\frac{n-2}{2n}} \int_1^R \frac{t^{\varepsilon}}{t}\, dt\right)$$
$$= O_{n,\varepsilon}\left(x^{\frac{n-2}{2n}} D_K^{\frac{1}{n}} R^{\frac{n-2}{2n}+\varepsilon}\right).$$

Finally we estimate the third O-term in a similar way to calculate the first O-term. One can estimate $\left(\log \frac{m}{R}\right)^{-1} = O\left(\frac{R}{m-R}\right)$, so we obtain

$$O_{n,\varepsilon}\left(x^{\frac{n-2}{2n}} D_K^{\frac{1}{n}} R^{\frac{n-2}{2n}+\varepsilon} \sum_{m>R} \frac{a_K(m)}{m^{1+\varepsilon}} \min\left\{\left(\log \frac{m}{R}\right)^{-1}, \left(\frac{Rx}{D_K}\right)^{\frac{1}{2n}}\right\}\right)$$
$$= O_{n,\varepsilon}\left(x^{\frac{n-2}{2n}} D_K^{\frac{1}{n}} R^{\frac{n-2}{2n}+\varepsilon}\left(\sum_{R<m\le [R]+1} \frac{a_K(m)}{m^{1+\varepsilon}} \left(\frac{Rx}{D_K}\right)^{\frac{1}{2n}} + \sum_{[R]+2\le m} \frac{a_K(m)}{m^{1+\varepsilon}} \left(\log \frac{m}{R}\right)^{-1}\right)\right)$$
$$= O_{n,\varepsilon}\left(x^{\frac{n-1}{2n}} D_K^{\frac{1}{2n}} R^{\frac{n-1}{2n}+\varepsilon} \sum_{R<m\le [R]+1} \frac{a_K(m)}{m^{1+\varepsilon}} + x^{\frac{n-2}{2n}} D_K^{\frac{1}{n}} R^{\frac{n-2}{2n}+\varepsilon} \sum_{[R]+2\le m} \frac{a_K(m)}{m^{1+\varepsilon}} \frac{R}{m-R}\right)$$
$$= O_{n,\varepsilon}\left(x^{\frac{n-1}{2n}} D_K^{\frac{1}{2n}} R^{-\frac{n+1}{2n}+\varepsilon} + x^{\frac{n-2}{2n}} D_K^{\frac{1}{n}} R^{\frac{n-2}{2n}+\varepsilon}\right).$$

From above results, we obtain

$$\frac{1}{2\pi i}\int_{C_3}\zeta_K(s)\frac{x^s}{s}\,ds = \frac{Cx^{\frac{n-1}{2n}}D_K^{\frac{1}{2n}}}{2\pi i}\sum_{m\le R}\frac{a_K(m)}{m^{\frac{n+1}{2n}}}f\left(2n\pi\left(\frac{mx}{D_K}\right)^{\frac{1}{n}}\right) + O_{n,\varepsilon}\left(x^{\frac{n-1}{2n}}D_K^{\frac{1}{2n}}R^{-\frac{n+1}{2n}+\varepsilon}+x^{\frac{n-2}{2n}}D_K^{\frac{1}{n}}R^{\frac{n-2}{2n}+\varepsilon}\right). \tag{2.10}$$

From estimate (2.8) and (2.10), it is obtained that

$$\Delta_K(x) = \frac{Cx^{\frac{n-1}{2n}}D_K^{\frac{1}{2n}}}{2\pi i}\sum_{m\le R}\frac{a_K(m)}{m^{\frac{n+1}{2n}}}f\left(2n\pi\left(\frac{mx}{D_K}\right)^{\frac{1}{n}}\right) + O_{n,\varepsilon}\left(x^{\frac{n-2}{2n}+\varepsilon}D_K^{\frac{1}{n}+\varepsilon}R^{\frac{n-2}{2n}+\varepsilon}+x^{\frac{n-1}{n}+\varepsilon}D_K^{\frac{1}{n}+\varepsilon}R^{-\frac{1}{n}+\varepsilon}\right).$$

Next we consider the above sum. Let $F_K = a_K * \mu$, where $*$ is the Dirichlet product symbol. From lemma 2.5 with $L = 1$ this becomes

$$\Delta_K(x) = O_{n,\varepsilon}\left(\begin{array}{l} x^{\frac{n-1}{2n}}D_K^{\frac{1}{2n}}R^{\varepsilon}\max\limits_{S\le R}S^{-\frac{n+1}{2n}}\times \\ \times \max\limits_{S<S_1\le 2SM,N\le S_1}\ \max\limits_{MN\asymp S}\ \max\limits_{\substack{M\le M_1\le 2M\\ N\le N_1\le 2N}}\left|\sum\limits_{M<m\le M_1}F_K(m)\sum\limits_{N<k\le N_1}\boldsymbol{e}\left(n\left(\frac{xmk}{D_K}\right)^{\frac{1}{n}}\right)\right| \\ +x^{\frac{n-2}{2n}+\varepsilon}D_K^{\frac{1}{n}+\varepsilon}R^{\frac{n-2}{2n}+\varepsilon}+x^{\frac{n-1}{n}+\varepsilon}D_K^{\frac{1}{n}+\varepsilon}R^{-\frac{1}{n}+\varepsilon}\end{array}\right).$$

This proves this proposition. □

Let $\mathcal{S}_K(x,S)$ be the sum in the O-term, that is,

$$\max_{S\le R}S^{-\frac{n+1}{2n}}\max_{S<S_1\le 2SM,N\le S_1}\ \max_{MN\asymp S}\ \max_{\substack{M\le M_1\le 2M\\ N\le N_1\le 2N}}\left|\sum_{M<m\le M_1}F_K(m)\sum_{N<k\le N_1}\boldsymbol{e}\left(n\left(\frac{xmk}{D_K}\right)^{\frac{1}{n}}\right)\right|.$$

This proposition reduces the initial problem to an exponential sums problem. There are many results to estimate exponential sum. In the next section, we estimate ideal counting function by using some results for exponential sum established by many authors.

3. Ideal counting function

In this section, we consider the distribution of ideals of $\mathcal{O}_K$, where K runs through extensions with $[K:\mathbf{Q}] = n$. In the last section, we show that the error term of ideal counting function can be expressed as a exponential sum. Let $X > 1$ be a real number, $1 \le M < M_1 \le 2M$ and $1 \le N < N_1 \le 2N$ be integers and $(a_m), (b_n) \subset \mathbf{C}$ be sequence of complex numbers, and let $\alpha, \beta \in \mathbf{R}$ and we define

$$\mathcal{S} = \sum_{M<m\le M_1}a_m\sum_{N<n\le N_1}b_n\boldsymbol{e}\left(X\left(\frac{m}{M}\right)^{\alpha}\left(\frac{n}{N}\right)^{\beta}\right). \tag{3.1}$$

In 1998 Wu shows this lemma.

Lemma 3.2 (Theorem 2 of [Wu98]). *Let $\alpha, \beta \in \mathbf{R}$ such that $\alpha\beta(\alpha-1)(\beta-1) \neq 0$, and $|a_m| \leq 1$ and $|b_n| \leq 1$ and $\mathcal{L} = log(XMN+2)$. Then*

$$\mathcal{L}^{-2}\mathcal{S} = O\begin{pmatrix} (XM^3N^4)^{\frac{1}{5}} + (X^4M^{10}N^{11})^{\frac{1}{16}} + (XM^7N^{10})^{\frac{1}{11}} \\ +MN^{\frac{1}{2}} + (X^{-1}M^{14}N^{23})^{\frac{1}{22}} + X^{-\frac{1}{2}}MN \end{pmatrix}.$$

Next Bordellès also shows this lemma by using estimate for triple exponential sums by Robert and Sargos.

Lemma 3.3 (Proposition 3.5 of [Bo15]). *Let $\alpha, \beta \in \mathbf{R}$ such that $\alpha\beta(\alpha-1)(\beta-1) \neq 0$, and $|a_m| \leq 1$ and $|b_n| \leq 1$. If $X = O(M)$ then*

$$(MN)^{-\varepsilon}\mathcal{S}$$
$$= O\left((XM^5N^7)^{\frac{1}{8}} + N(X^{-2}M^{11})^{\frac{1}{12}} + (X^{-3}M^{21}N^{23})^{\frac{1}{24}} + M^{\frac{3}{4}}N + X^{-\frac{1}{4}}MN\right).$$

In the following Sriaivasan's result is important for our estimating.

Lemma 3.4 (Lemma 4 of [Sr62]). *Let N and P be positive integers and $u_n \geq 0$, $v_p > 0$, A_n and B_p denote constants for $1 \leq n \leq N$ and $1 \leq p \leq P$. Then there exists a q with properties*

$$Q_1 \leq q \leq Q_2$$

and

$$\sum_{n=1}^{N} A_n q^{u_n} + \sum_{p=1}^{P} B_p q^{-v_p} = O\left(\sum_{n=1}^{N}\sum_{p=1}^{P} \sqrt[u_n+v_p]{A_n^{v_p} B_p^{u_n}} + \sum_{n=1}^{N} A_n Q_1^{u_n} + \sum_{p=1}^{P} B_p Q_2^{-v_p}\right).$$

The constant involved in O-symbol is less than $N+P$.

Sriaivasan remarked that the inequality in lemma 3.4 corresponds to the 'best possible' choice of q in the range $Q_1 \leq q \leq Q_2$ [Sr62].

Theorem 3.5. *For all $\varepsilon > 0$ and $C > 0$ the followings hold.*

(1) *If K runs through cubic fields with $D_K \leq Cx^{\frac{1}{4}}$*

$$\Delta_K(x) = O_{C,\varepsilon}\left(x^{\frac{6}{13}+\varepsilon} D_K^{\frac{2}{13}+\varepsilon}\right).$$

(2) *If K runs through number fields with $[K:\mathbf{Q}] \leq n$ and $D_K \leq Cx^{\frac{8}{2n+5}-\varepsilon}$ then*

$$\Delta_K(x) = O_{C,n,\varepsilon}\left(x^{\frac{2n-3}{2n+1}+\varepsilon} D_K^{\frac{2}{2n+1}+\varepsilon}\right).$$

Proof. We note that

$$\left|\sum_{M<m\leq M_1} F_K(m) \sum_{N<k\leq N_1} \mathbf{e}\left(n\left(\frac{xmk}{D_K}\right)^{\frac{1}{n}}\right)\right|$$
$$= \left|\sum_{M<m\leq M_1} F_K(m) \sum_{N<k\leq N_1} \mathbf{e}\left(n\left(\frac{xMN}{D_K}\right)^{\frac{1}{n}} \left(\frac{m}{M}\right)^{\frac{1}{n}} \left(\frac{k}{N}\right)^{\frac{1}{n}}\right)\right|.$$

We use the above lemmas with $X = n\left(\frac{xMN}{D_K}\right)^{\frac{1}{n}} > 0$. We consider four cases:

Case 1.	$S^{\alpha} \ll N \ll S^{\frac{1}{2}}$
Case 2.	$S^{\frac{1}{2}} \ll N \ll S^{1-\alpha}$
Case 3.	$S^{1-\alpha} \ll N$
Case 4.	$N \ll S^{\alpha}$

When $S^{\alpha} \ll N \ll S^{\frac{1}{2}}$, we apply lemma 3.2 and this gives

$$S^{-\varepsilon} x^{\frac{n-1}{2n}} D_K^{\frac{1}{2n}} S_K(x,S) = O_{n,\varepsilon}\left(\begin{array}{l} x^{\frac{5n-3}{10n}} D_K^{\frac{3}{10n}} R^{\frac{2n-3}{10n}} + x^{\frac{2n-1}{4n}} D_K^{\frac{1}{4n}} R^{\frac{5n-8}{32n}} \\ + x^{\frac{11n-9}{22n}} D_K^{\frac{9}{22n}} R^{\frac{6n-9}{22n}} + x^{\frac{n-1}{2n}} D_K^{\frac{1}{2n}} R^{\frac{n-1}{2n}-\frac{1}{2}\alpha} \\ + x^{\frac{11n-12}{22n}} D_K^{\frac{6}{11n}} R^{\frac{15n-24}{44n}} + x^{\frac{n-2}{2n}} D_K^{\frac{1}{n}} R^{\frac{n-2}{2n}} \end{array}\right). \tag{3.6}$$

When $S^{\frac{1}{2}} \ll N \ll S^{1-\alpha}$ we use lemma 3.2 again reversing the role of M and N. We obtain the same estimate for the case that $S^{\alpha} \ll N \ll S^{\frac{1}{2}}$.
For the case 3, we use lemma 3.3

$$S^{-\varepsilon} x^{\frac{n-1}{2n}} D_K^{\frac{1}{2n}} S_K(x,S) = O_{n,\varepsilon}\left(\begin{array}{l} x^{\frac{4n-3}{8n}} D_K^{\frac{3}{8n}} R^{\frac{n-3}{8n}+\frac{1}{4}\alpha} + x^{\frac{3n-4}{6n}} D_K^{\frac{2}{3n}} R^{\frac{5n-8}{12n}+\frac{1}{12}\alpha} \\ + x^{\frac{4n-5}{8n}} D_K^{\frac{5}{8n}} R^{\frac{3n-5}{8n}-\frac{1}{12}\alpha} \\ + x^{\frac{n-1}{2n}} D_K^{\frac{1}{2n}} R^{\frac{n-2}{4n}+\frac{1}{4}\alpha} + x^{\frac{2n-3}{4n}} D_K^{\frac{3}{4n}} R^{\frac{2n-3}{4n}} \end{array}\right). \tag{3.7}$$

For the case 4, we use lemma 3.3 again reversing the role of M and N. We obtain the same estimate for the case that $S^{1-\alpha} \ll N$. Combining (3.6) and (3.7) with proposition 2.6, we obtain

$$I_K(x) = \rho_K x + O_{n,\varepsilon}\left(\begin{array}{l} x^{\frac{5n-3}{10n}} D_K^{\frac{3}{10n}} R^{\frac{2n-3}{10n}+\varepsilon} + x^{\frac{2n-1}{4n}} D_K^{\frac{1}{4n}} R^{\frac{5n-8}{32n}+\varepsilon} \\ + x^{\frac{11n-9}{22n}} D_K^{\frac{9}{22n}} R^{\frac{6n-9}{22n}+\varepsilon} + x^{\frac{n-1}{2n}} D_K^{\frac{1}{2n}} R^{\frac{n-1}{2n}-\frac{1}{2}\alpha+\varepsilon} \\ + x^{\frac{11n-12}{22n}} D_K^{\frac{6}{11n}} R^{\frac{15n-24}{44n}} + x^{\frac{n-2}{2n}} D_K^{\frac{1}{n}} R^{\frac{n-2}{2n}+\varepsilon} \\ + x^{\frac{4n-3}{8n}} D_K^{\frac{3}{8n}} R^{\frac{n-3}{8n}+\frac{1}{4}\alpha+\varepsilon} + x^{\frac{3n-4}{6n}} D_K^{\frac{2}{3n}} R^{\frac{5n-8}{12n}+\frac{1}{12}\alpha+\varepsilon} \\ + x^{\frac{4n-5}{8n}} D_K^{\frac{5}{8n}} R^{\frac{3n-5}{8n}+\frac{1}{12}\alpha+\varepsilon} + x^{\frac{2n-3}{4n}} D_K^{\frac{3}{4n}} R^{\frac{2n-3}{4n}+\varepsilon} \\ + x^{\frac{n-1}{n}+\varepsilon} D_K^{\frac{1}{n}+\varepsilon} R^{-\frac{1}{n}+\varepsilon} \end{array}\right). \tag{3.8}$$

By lemma 3.4 with $1 \leq R \leq xD_K$ there exists an R such that the error term of estimate (3.8) is much less than

$$\begin{array}{l} x^{\frac{2n}{2n+7}+\varepsilon} D_K^{\frac{2}{2n+7}+\varepsilon} + x^{\frac{5n+3}{5n+24}+\varepsilon} D_K^{\frac{5}{5n+24}+\varepsilon} + x^{\frac{6n-4}{6n+13}+\varepsilon} D_K^{\frac{6}{6n+13}+\varepsilon} \\ + x^{\frac{(1-\alpha)n+\alpha-1}{(1-\alpha)n+1}+\varepsilon} D_K^{\frac{1-\alpha}{(1-\alpha)n+1}+\varepsilon} + x^{\frac{15n-17}{15n+20}+\varepsilon} D_K^{\frac{3}{3n+4}+\varepsilon} + x^{\frac{n-2}{n}+\varepsilon} D_K^{\frac{1}{n}+\varepsilon} \\ + x^{\frac{(2\alpha+1)n-2\alpha}{(2\alpha+1)n+5}+\varepsilon} D_K^{\frac{2\alpha+1}{(2\alpha+1)n+5}+\varepsilon} + x^{\frac{(\alpha+5)n-\alpha-7}{(\alpha+5)n+4}+\varepsilon} D_K^{\frac{\alpha+5}{(\alpha+5)n+4}+\varepsilon} \\ + x^{\frac{(2\alpha+9)n-2\alpha-12}{(2\alpha+9)n+9}+\varepsilon} D_K^{\frac{2\alpha+9}{(2\alpha+9)n+9}+\varepsilon} + x^{\frac{2n-3}{2n+1}+\varepsilon} D_K^{\frac{2}{2n+1}+\varepsilon}. \end{array}$$

In the following we choose this R. When $n = 3$ and $\alpha = \frac{1}{3}$, then we have

$$\Delta_K(x) = O_{C,\varepsilon}\left(\begin{array}{l} x^{\frac{6}{13}+\varepsilon} D_K^{\frac{2}{13}+\varepsilon} + x^{\frac{14}{31}+\varepsilon} D_K^{\frac{6}{31}+\varepsilon} + x^{\frac{4}{9}+\varepsilon} D_K^{\frac{2}{9}+\varepsilon} \\ + x^{\frac{1}{3}+\varepsilon} D_K^{\frac{1}{3}+\varepsilon} + x^{\frac{13}{30}+\varepsilon} D_K^{\frac{4}{15}+\varepsilon} + x^{\frac{25}{57}+\varepsilon} D_K^{\frac{29}{114}+\varepsilon} + x^{\frac{3}{7}+\varepsilon} D_K^{\frac{2}{7}+\varepsilon} \end{array}\right).$$

If K runs through cubic extension fields with$D_K \le Cx^{\frac{1}{4}}$, we obtain

$$\Delta_K(x) = O_{C,\varepsilon}\left(x^{\frac{6}{13}+\varepsilon} D_K^{\frac{2}{13}+\varepsilon}\right).$$

When $n \ge 4$ and $\alpha = \frac{n+3}{7n-5}$, then we have

$$\Delta_K(x) = O_{C,n,\varepsilon}\left(x^{\frac{n-2}{n}+\varepsilon} D_K^{\frac{1}{n}+\varepsilon} + x^{\frac{2n-3}{2n+1}+\varepsilon} D_K^{\frac{2}{2n+1}+\varepsilon}\right).$$

From this estimate we obtain the following result. If K runs through number fields with $[K : \mathbf{Q}] \le n$ and $D_K \le Cx^{\frac{8}{2n+5}-\varepsilon}$

$$\Delta_K(x) = O_{C,n,\varepsilon}\left(x^{\frac{2n-3}{2n+1}+\varepsilon} D_K^{\frac{2}{2n+1}+\varepsilon}\right).$$

This proves our theorem. □

Theorem 3.5 leads to the following corollary.

Corollary 3.9. *For all $\varepsilon > 0$ and $C > 0$ the followings hold.*

(1) *When K runs through cubic extensions field with $D_K \le Cx^{\frac{1}{4}}$, then*

$$E_m^r(x, K) = \begin{cases} O_{C,\varepsilon}\left(x^{\frac{19}{13r}+\varepsilon} D_K^{\frac{4}{13}-\frac{m-1}{2}+\varepsilon}\right) & \text{if } rm = 2, \\ O_{C,\varepsilon}\left(x^{m-\frac{7}{13}+\varepsilon} D_K^{\frac{2}{13}-\frac{m-1}{2}+\varepsilon}\right) & \text{otherwise.} \end{cases}$$

(2) *If K runs through number fields with $[K : \mathbf{Q}] \le n$ and $D_K \le Cx^{\frac{8}{2n+5}-\varepsilon}$, then*

$$E_m^r(x, K) = \begin{cases} O_{C,n,\varepsilon}\left(x^{\frac{4n-2}{r(2n+1)}+\varepsilon} D_K^{\frac{4}{2n+1}-\frac{m-1}{2}+\varepsilon}\right) & \text{if } rm = 2, \\ O_{C,n,\varepsilon}\left(x^{m-\frac{4}{2n+1}+\varepsilon} D_K^{\frac{2}{2n+1}-\frac{m-1}{2}+\varepsilon}\right) & \text{otherwise.} \end{cases}$$

Proof. We show the first case. The Inclusion–Exclusion Principle shows that

$$V_m^r(x, K) = \sum_{\mathfrak{N}\mathfrak{a} \le x^{\frac{1}{r}}} \mu(\mathfrak{a}) I_K\left(\frac{x}{\mathfrak{N}\mathfrak{a}^r}\right)^m. \tag{3.10}$$

Theorem 3.5 and the binomial theorem lead to

$$\begin{aligned} V_m^r(x, K) &= \sum_{\mathfrak{N}\mathfrak{a} \le x^{1/r}} \mu(\mathfrak{a}) \left(\frac{\rho_K x}{\mathfrak{N}\mathfrak{a}^r} + O_{C,\varepsilon}\left(\left(\frac{x}{\mathfrak{N}\mathfrak{a}^r}\right)^{\frac{6}{13}+\varepsilon} D_K^{\frac{2}{13}+\varepsilon}\right)\right)^m \\ &= (\rho_K x)^m \sum_{\mathfrak{N}\mathfrak{a} \le x^{1/r}} \frac{\mu(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}^{rm}} + O_{C,\varepsilon}\left(\sum_{\mathfrak{N}\mathfrak{a} \le x^{1/r}} \left(\frac{x}{\mathfrak{N}\mathfrak{a}^r}\right)^{m-\frac{7}{13}+\varepsilon} D_K^{\frac{2}{13}-\frac{m-1}{2}+\varepsilon}\right). \end{aligned}$$

By using the fact $\displaystyle\sum_{\mathfrak{a}} \frac{\mu(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}^{rm}} = \frac{1}{\zeta_K(rm)}$, we get

$$V_m^r(x,K) = \frac{(\rho_K x)^m}{\zeta_K(rm)} - (\rho_K x)^m \sum_{\mathfrak{N}\mathfrak{a} > x^{1/r}} \frac{\mu(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}^{rm}} + O_{C,\varepsilon}\left(\sum_{\mathfrak{N}\mathfrak{a} \le x^{1/r}} \left(\frac{x}{\mathfrak{N}\mathfrak{a}^r}\right)^{m-\frac{7}{13}+\varepsilon} D_K^{\frac{2}{13}-\frac{m-1}{2}+\varepsilon} \right).$$

Theorem 3.5 states $I_K(x) - I_K(x-1) = O_{C,\varepsilon}\left(x^{\frac{6}{13}+\varepsilon} D_K^{\frac{2}{13}+\varepsilon}\right)$, so we have

$$(\rho_K x)^m \sum_{\mathfrak{N}\mathfrak{a} > x^{1/r}} \frac{\mu(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}^{rm}} = O_{C,\varepsilon}\left(x^m \int_{x^{1/r}}^{\infty} \frac{y^{\frac{6}{13}+\varepsilon} D_K^{\frac{2}{13}-\frac{m}{2}+\varepsilon}}{y^{rm}} \, dy \right)$$
$$= O_{C,\varepsilon}(x^{\frac{19}{13r}+\varepsilon} D_K^{\frac{2}{13}-\frac{m}{2}+\varepsilon}),$$

and

$$\sum_{\mathfrak{N}\mathfrak{a} \le x^{1/r}} \left(\frac{x}{\mathfrak{N}\mathfrak{a}^r}\right)^{m-\frac{7}{13}+\varepsilon} D_K^{\frac{2}{13}-\frac{m-1}{2}+\varepsilon}$$
$$= O_{C,\varepsilon}\left(x^{m-\frac{7}{13}+\varepsilon} D_K^{\frac{2}{13}-\frac{m-1}{2}+\varepsilon} \left(1 + \int_1^{x^{1/r}} \frac{y^{\frac{6}{13}+\varepsilon} D_K^{\frac{2}{13}+\varepsilon}}{y^{r(m-\frac{7}{13}+\varepsilon)}} \, dy\right)\right)$$
$$= O_{C,\varepsilon}\left(x^{m-\frac{7}{13}+\varepsilon} D_K^{\frac{2}{13}-\frac{m-1}{2}+\varepsilon} + x^{\frac{19}{13r}+\varepsilon} D_K^{\frac{4}{13}-\frac{m-1}{2}+\varepsilon} \right).$$

Hence we get

$$V_m^r(x,K) = \frac{\rho_K^m}{\zeta_K(rm)} x^m + O_{C,\varepsilon}\left(x^{m-\frac{7}{13}+\varepsilon} D_K^{\frac{2}{13}-\frac{m-1}{2}+\varepsilon} + x^{\frac{19}{13r}+\varepsilon} D_K^{\frac{4}{13}-\frac{m-1}{2}+\varepsilon} \right)$$
$$= \frac{\rho_K^m}{\zeta_K(rm)} x^m + \begin{cases} O_{C,\varepsilon}\left(x^{\frac{19}{13r}+\varepsilon} D_K^{\frac{4}{13}-\frac{m-1}{2}+\varepsilon} \right) & \text{if } rm = 2, \\ O_{C,\varepsilon}\left(x^{m-\frac{7}{13}+\varepsilon} D_K^{\frac{2}{13}-\frac{m-1}{2}+\varepsilon} \right) & \text{otherwise.} \end{cases}$$

This proves the first case. We can also show second case by the same argument of first case with $\frac{6}{13}$ and $\frac{2}{13}$ replaced by $\frac{2n-3}{2n+1}$ and $\frac{2}{2n+1}$, respectively.

This proves the theorem. □

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