ASYMPTOTIC EXPANSIONS FOR A CLASS OF GENERALIZED HOLOMORPHIC EISENSTEIN SERIES: APPLICATIONS TO WEIERSTRASS' ELLIPTIC FUNCTION AND RAMANUJAN'S FORMULA FOR $\zeta(2k+1)$

MASANORI KATSURADA AND TAKUMI NODA
DEPT. MATH., FAC. ECON., KEIO UNIV.
AND
DEPT. MATH., COL. ENG., NIHON UNIV.
(慶應義塾大学 経済学部 桂田 昌紀，日本大学 工学部 野田 工)

ABSTRACT. We shall establish complete asymptotic expansions for a class of generalized holomorphic Eisenstein series, when the associated parameter z tends to both 0 and ∞ through the complex upper half-plane $\mathfrak{H}^+$. These expansions are further applied to deduce several variants of classical Euler's and Ramanujan's formula for specific values of the Riemann zeta-function, as well as to show various functional relations for the classical Eisenstein series, and Weierstraß' elliptic and allied functions in terms of generalized Lambert series.

1. INTRODUCTION

Throughout the paper, s denotes a complex variable, z a complex parameter, and a, b, μ and ν real parameters. Let $\mathfrak{H}^\pm$ denote the complex upper and lower half-planes, respectively, where the argument of each branch is chosen as

$$\mathfrak{H}^+ = \{z \in \mathbb{C}^\times \mid 0 < \arg z < \pi\} \qquad \text{and} \qquad \mathfrak{H}^- = \{z \in \mathbb{C}^\times \mid -\pi < \arg z < 0\}.$$

It is frequently used in the sequel the notation $e(s) = e^{2\pi i s}$, and the parameter $\tau = e^{\mp\pi i/2}z$ for $z \in \mathfrak{H}^\pm$, where τ varies within the sector $|\arg\tau| < \pi/2$.

We now define the generalized Eisenstein series $F^\pm_{\mathbb{Z}^2}(s; a, b; \mu, \nu; z)$ by

$$F^\pm_{\mathbb{Z}^2}(s; a, b; \mu, \nu; z) = \sideset{}{'}\sum_{m,n=-\infty}^{\infty} \frac{e((a+m)\mu + (b+n)\nu)}{\{a+m+(b+n)z\}^s} \qquad (\operatorname{Re} s > 2), \tag{1.1}$$

where the primed summation symbols hereafter indicate that the possibly emerging singular terms such as $1/0^s$ are to be omitted, and the branch of each summand is chosen such that $\arg\{(a+m)+(b+n)z\}$ falls within the range $]-\pi,\pi]$ in $F^+_{\mathbb{Z}^2}$, and within $[-\pi,\pi[$ in $F^-_{\mathbb{Z}^2}$. The main object of this paper is the arithmetical mean of $F^\pm_{\mathbb{Z}^2}$ defined by

$$F_{\mathbb{Z}^2}(s; a, b; \mu, \nu; z) = \frac{1}{2}\{F^+_{\mathbb{Z}^2}(s; a, b; \mu, \nu; z) + F^-_{\mathbb{Z}^2}(s; a, b; \mu, \nu; z)\}, \tag{1.2}$$

for which we shall show that complete asymptotic expansions exist when both $\tau \to \infty$ (Theorem 1) and $\tau \to 0$ (Theorems 2 and 3) through the sector $|\arg\tau| < \pi/2$; the combination of Theorems 1–3 can further be applied to obtain several variants of the celebrated

---

2010 *Mathematics Subject Classification.* Primary 11E45; Secondary 11F11, 11M35, 11M41, 33E05.

Key words and phrases. holomorphic Eisenstein series, asymptotic expansion, Lerch zeta-function, Ramanujan's formula, Weierstraß elliptic function.

A portion of the present research was made during the first author's academic stay at Mathematisches Institut, Westfalisch Wilhelms-Universität Münster. He would like to express his sincere gratitude to Professor Christopher Deninger and to the institution for warm hospitality and constant support.

formulae of Euler and of Ramanujan for specific values of the Riemann zeta-function, as well as to deduce various functional relations for the classical Eisenstein series and for Weierstraß' elliptic and allied functions. One can see that a hidden (but crucial) rôle is played by the connection formula (2.23) below for Kummer's confluent hypergeometric functions in producing various functional relations for zeta-functions, Eisenstein series and elliptic functions mentioned above.

We give here a brief overview of the research related to holomorphic and non-holomorphic Eisenstein series of complex variables.

Lewittes [17] first obtained a transformation formula for

$$F(s;z)=F_{\mathbb{Z}^2}(s;0,0;0,0;z) \tag{1.3}$$

(with the notation in (1.0)), which was applied to show a modular relation connecting $F(2;z)$ with $F(2;-1/z)$; this transformation formula can be viewed as a prototype of our Theorem 1 below. He further established in [18] a transformation formula for a more general $F_{\mathbb{Z}^2}(s;a,b;0,0;z)$, which was extensively applied to study its modular relations when the modular group $SL_2(\mathbb{Z})$ acts on the associated parameter $z\in\mathfrak{H}^+$. A subsequent research was made by Berndt [1], who especially treated in this respect a class of generalized Dedekind eta-functions and Dedekind sums. Let $\zeta(s)$ denote the Riemann zeta-function. Berndt [2] then made a further research into this direction in connection with Euler's and Ramanujan's formulae for specific values of $\zeta(s)$.

On the other hand, Matsumoto [22] more recently derived complete asymptotic expansions for $F(s;z)$ when both $z\to 0$ and $z\to\infty$ through $\mathfrak{H}^+$; the latter can be viewed as a prototype of our Theorem 2 below. A transformation formula for a two variable analogue of (1.1) was obtained by Lim [21], while the first author [10] derived complete asymptotic expansions for a generalized *non-holomorphic* Eisenstein series of the form

$$\psi_{\mathbb{Z}^2}(s;a,b;\mu,\nu;z)=\sum_{m,n=-\infty}^{\infty}{}' \frac{e((a+m)\mu+(b+n)\nu)}{|a+m+(b+n)z|^{2s}} \qquad (\operatorname{Re}s>1)$$

both as $z\to 0$ and as $z\to\infty$ through the sector $\mathfrak{H}^+$. It has very recently been shown by the authors [15] that complete asymptotic expansions exist for a two variable analogue of $F(s;z)$, when the associated parameters $\boldsymbol{z}=(z_1,z_2)$ vary within the sectors $\mathfrak{H}^{\pm}$ so as that the distance $|z_2-z_1|$ tends to both 0 and ∞.

2. Main results

Prior to state our main results, we prepare several necessary notations.

Let $\kappa\in\mathbb{R}$ be a parameter. We then introduce the Lerch zeta-function $\phi(s,c,\kappa)$, together with its companion $\psi(s,c,\kappa)$, defined by

$$\phi(s,c,\kappa)=\sum_{k=0}^{\infty}{}' \frac{e(k\kappa)}{(c+k)^s} \qquad (\operatorname{Re}s>1), \tag{2.1}$$

$$\psi(s,c,\kappa)=\sum_{k=0}^{\infty}{}' \frac{e((c+k)\kappa)}{(c+k)^s}=e(c\kappa)\phi(s,c,\kappa), \tag{2.2}$$

which can be continued to entire functions if $\kappa\in\mathbb{R}\setminus\mathbb{Z}$, while for $\kappa\in\mathbb{Z}$ the former (or for $\kappa=0$ the latter) reduces to the Hurwitz zeta-function $\zeta(s,c)$, also for $\kappa\in\mathbb{R}$ and $c=1$ to the exponential zeta-function $\zeta_\kappa(s)=e(\kappa)\psi(s,1,\kappa)=\psi(s,1,\kappa)$, and hence to the Riemann zeta-function $\zeta(s)=\zeta(s,1)=\zeta_\kappa(s)$ if $\kappa\in\mathbb{Z}$. Note that

$$\phi(s,0,\kappa)=e(\kappa)\phi(s,1,\kappa) \qquad \text{and} \qquad \psi(s,0,\kappa)=\psi(s,1,\kappa) \tag{2.3}$$

hold by the convention of primed summation symbols; this implies that $\zeta_\kappa(s)=\phi(s,0,\kappa)=\psi(s,0,\kappa)$. The functional equation for $\phi(s,c,\kappa)$ (see, for e.g., [19][20]) with a slight extension asserts as follows.

Proposition 1 ([16, Lemma 3]). *For any $c,\kappa\in[0,1]$, we have the functional equation*

$$\phi(s,c,\kappa)=\frac{\Gamma(1-s)}{(2\pi)^{1-s}}\{e^{\pi i(1-s)/2}\psi(1-s,\kappa,-c)+e^{-\pi i(1-s)/2}\psi(1-s,1-\kappa,c)\}, \tag{2.4}$$

which reduces if $\kappa\in\{0,1\}$ to

$$\zeta(s,c)=\frac{\Gamma(1-s)}{(2\pi)^{1-s}}\{e^{\pi i(1-s)/2}\zeta_{-c}(1-s)+e^{-\pi i(1-s)/2}\zeta_c(1-s)\}, \tag{2.5}$$

while if $c\in\{0,1\}$ to

$$\zeta_\kappa(s)=\frac{\Gamma(1-s)}{(2\pi)^{1-s}}\{e^{\pi i(1-s)/2}\zeta(1-s,\kappa)+e^{-\pi i(1-s)/2}\zeta(1-s,1-\kappa)\} \tag{2.6}$$

with the convention in (2.3).

Let $\langle x\rangle=x-\lfloor x\rfloor$ for any $x\in\mathbb{R}$ denote the fractional part of x. Then the functional equation (2.4) can be extended to the following form with a satisfactory extension of the domain of parameters.

Proposition 2 ([16, Lemma 4]). *For any $c,\kappa\in\mathbb{R}$, we have the functional equation*

$$\begin{aligned}\psi(s,\langle c\rangle,\kappa)=e(c\kappa)\frac{\Gamma(1-s)}{(2\pi)^{1-s}}\{&e^{\pi i(1-s)/2}\psi(1-s,\langle\kappa\rangle,-c)\\&+e^{-\pi i(1-s)/2}\psi(1-s,\langle-\kappa\rangle,c)\}.\end{aligned} \tag{2.7}$$

Let r be a complex variable, and q a complex (base) parameter with $|q|<1$. We further introduce the generalized Lambert series $\mathcal{S}_r(c,d;\kappa,\lambda;q)$, defined for any real c, d, κ and λ with $c,d\geq 0$ by

$$\mathcal{S}_r(c,d;\kappa,\lambda;q)=e(c'\kappa)\sum_{l=0}^{\infty}{}'\frac{e((d+l)\lambda)q^{c'(d+l)}}{(d+l)^r\{1-e(\kappa)q^{d+l}\}} \tag{2.8}$$

upon the convention (used hereafter) for any $c\in[0,+\infty[$ that

$$c'=\begin{cases}c & \text{if } c>0,\\ 1 & \text{if } c=0.\end{cases}$$

Further let $\delta(x)$ for $x\in\mathbb{R}$ denote the symbol which equals 1 or 0 according to $x\in\mathbb{Z}$ or otherwise, and $\Gamma(s)$ the gamma function and $(s)_n=\Gamma(s+n)/\Gamma(s)$ for any $n\in\mathbb{Z}$ the shifted factorial.

We proceed to state our first main result.

Theorem 1 ([16, Theorem 1]). *Set*

$$\begin{aligned}\mathcal{A}(s,a,\mu)&=\psi(s,\langle-a\rangle,-\mu)\cos(\pi s)+\psi(s,\langle a\rangle,\mu)\\&=e(a\mu)\frac{(2\pi)^s}{2\Gamma(s)}\{e^{-\pi is/2}\psi(1-s,\langle-\mu\rangle,a)+e^{\pi is/2}\psi(1-s,\langle\mu\rangle,-a)\},\end{aligned} \tag{2.9}$$

where the second equality holds by (2.7). Then for any real a, b, μ and ν, and any $z \in \mathfrak{H}^+$, we have the formula

$$F_{\mathbb{Z}^2}(s;a,b;\mu,\nu;z) = \delta(b)\mathcal{A}(s,a,\mu) + e(a\mu)\frac{(2\pi)^s}{\Gamma(s)}\{e^{-\pi is/2}\mathcal{S}_{1-s}(\langle b\rangle,\langle -\mu\rangle;\nu,a;q) + e^{\pi is/2}\mathcal{S}_{1-s}(\langle -b\rangle,\langle \mu\rangle;-\nu,-a;q)\}, \tag{2.10}$$

which is valid in the whole s-plane.

Remark. The formula (2.10) can be viewed as a transformation formula, and at the same time as a convergent asymptotic expansion when $\tau \to \infty$ through the sector $|\arg\tau| < \pi/2$, where the asymptotic series are given by $\mathcal{S}_{1-s}(\langle \pm b\rangle,\langle \mp\mu\rangle;\pm\nu,\pm a;q)$ on the right side, since each term of $\mathcal{S}_r(c,d;\kappa,\lambda;q)$ in (2.8) is of order $O\{e^{-2\pi\tau c'(d'+l)}/(d'+l)^r\}$ when $\tau \to \infty$ $(l = 0,1,\ldots)$.

Let $\widetilde{\mathbb{C}^\times}$ denote the universal covering of the punctured complex plane $\mathbb{C}^\times = \mathbb{C}\setminus\{0\}$, where the mapping $\widetilde{\mathbb{C}^\times} \ni \widetilde{Y} \mapsto \log\widetilde{Y} = \log|\widetilde{Y}| + i\arg\widetilde{Y} \in \mathbb{C}$ is bijective (with the range of $\arg\widetilde{Y}$ being extended over $\mathbb{R}$). We define for any $X \in \mathbb{C}$ and $\widetilde{Y} \in \widetilde{\mathbb{C}^\times}$ the operation

$$\widetilde{\mathbb{C}^\times} \ni \widetilde{Y} \longmapsto \widetilde{Y}^X = \exp(X\log\widetilde{Y}) = \exp\{X(\log|\widetilde{Y}| + i\arg\widetilde{Y})\} = |\widetilde{Y}|^X\exp(iX\arg\widetilde{Y}) \in \mathbb{C}. \tag{2.11}$$

Let $\widetilde{e}(\kappa) \in \widetilde{\mathbb{C}^\times}$ for any $\kappa \in \mathbb{R}$ denote the point defined by $\log\widetilde{e}(\kappa) = 2\pi i\kappa$, and write $\widetilde{e}(0) = \widetilde{1}$. Then $\widetilde{e}(\kappa)^c = e(c\kappa)$ holds for all $c \in \mathbb{R}$ by (2.11).

It is convenient for describing specific values of $\psi(s,c,\kappa)$ to introduce the sequence of functions $\mathcal{C}_k : \mathbb{C}\times\widetilde{\mathbb{C}^\times} \ni (X,\widetilde{Y}) \longmapsto \mathcal{C}_k(X,\widetilde{Y}) \in \mathbb{C}$ $(k = 0,1,\ldots)$, defined by the Taylor series expansion (with the variable Z in $\mathbb{C}$)

$$\frac{Z\widetilde{Y}^X e^{XZ}}{\widetilde{Y}^1 e^Z - 1} = \sum_{k=0}^{\infty}\frac{\mathcal{C}_k(X,\widetilde{Y})}{k!}Z^k \tag{2.12}$$

near $Z = 0$ (notice that $\widetilde{Y}^1 = |\widetilde{Y}|\exp(\log\widetilde{Y})$); this in particular implies that

$$\mathcal{C}_0(X,\widetilde{Y}) = \begin{cases} \widetilde{Y}^X & \text{if } \widetilde{Y}^1 = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.13}$$

Note that $\mathcal{C}_k(X,\widetilde{Y})$ reduces if $\widetilde{Y} = \widetilde{1}$ (and so $\widetilde{Y}^X = 1$) to the usual Bernoulli polynomial $B_k(X)$, and also to the rational function $A_k(Y)$ if $X = 0$, defined by the Taylor series expansion

$$\frac{Z}{Ye^Z - 1} = \sum_{k=0}^{\infty}\frac{A_k(Y)}{k!}Z^k$$

centered at $Z = 0$. Professor Andrzej Schinzel kindly informed me (in a private communication [27]) about the explicit form of $A_k(Y)$ involving Eulerian (not Euler's) numbers (cf. [26, p.215]) in its coefficients. We have further shown in [10] the following properties of $\mathcal{C}_k(X,\widetilde{Y})$.

Proposition 3 ([10, Lemma 3]). *For any integer $k \geq 0$ and any $(X, \widetilde{Y}) \in \mathbb{C} \times \widetilde{\mathbb{C}^\times}$, the relations*

$$\mathcal{C}_k(1 - X, \widetilde{1}/\widetilde{Y}) = (-1)^k \mathcal{C}_k(X, \widetilde{Y}), \tag{2.14}$$

$$\mathcal{C}_k(0, \widetilde{1}/\widetilde{Y}) = \begin{cases} (-1)^k \mathcal{C}_k(0, \widetilde{Y}) & \text{if } k \neq 1, \\ -\mathcal{C}_1(0, \widetilde{Y}) - 1 & \text{if } k = 1 \end{cases} \tag{2.15}$$

hold, where $\widetilde{1}/\widetilde{Y} \in \widetilde{\mathbb{C}^\times}$ is the point defined by $|\widetilde{1}/\widetilde{Y}| = 1/|\widetilde{Y}|$ and by $\arg(\widetilde{1}/\widetilde{Y}) = -\arg \widetilde{Y}$.

We proceed to state our second main result.

Theorem 2 ([16, Theorem 2]). *Let a, b, μ and ν be arbitrary real parameter, and $z \in \mathfrak{H}^+$, write $q = e(z) = e^{2\pi i z}$ and $\widehat{q} = e(-1/z) = e^{-2\pi i/z}$ for any $z \in \mathfrak{H}^+$, and set*

$$\begin{aligned} \mathcal{B}_1(s, a, \mu) &= \sin(\pi s)\psi(s, \langle -a \rangle, -\mu) \\ &= e(a\mu)\frac{(2\pi)^s}{2\Gamma(s)}\{e^{\pi i(1-s)/2}\psi(1 - s, \langle -\mu \rangle, a) \\ &\quad + e^{-\pi i(1-s)/2}\psi(1 - s, \langle \mu \rangle, -a)\}, \end{aligned} \tag{2.17}$$

$$\begin{aligned} \mathcal{B}_2(s, b, \nu) &= e^{\pi i s/2}\psi(s, \langle -b \rangle, -\nu) + e^{-\pi i s/2}\psi(s, \langle b \rangle, \nu) \\ &= e(b\nu)\frac{(2\pi/\tau)^s}{\Gamma(s)}\psi(1 - s, \langle \nu \rangle, -b), \end{aligned} \tag{2.18}$$

where the second equalities in (2.17) and (2.18) hold by (2.7). Then for any integer $J \geq 0$, in the region $\operatorname{Re} s > 1 - J$, we have the formula

$$\begin{aligned} &F_{\mathbb{Z}^2}(s; a, b; \mu, \nu; z) \\ &\quad = i\delta(b)\mathcal{B}_1(s, a, \mu) + \delta(a)\mathcal{B}_2(s, b, \nu)\tau^{-s} \\ &\quad + 2\sin(\pi s)\sum_{j=-1}^{J-1}\frac{i^{j+1}(s)_j}{(j+1)!}\psi(s + j, \langle -a \rangle, -\mu)\mathcal{C}_{j+1}(\langle b \rangle, \widetilde{e}(\nu))\tau^j \\ &\quad + R_J(s; a, b; \mu, \nu; z), \end{aligned} \tag{2.19}$$

where $R_J(s; a, b; \mu, \nu; z)$ is the remainder term satisfying the estimate

$$R_J(s; a, b; \mu, \nu; z) = O(|\tau|^J) \tag{2.20}$$

as $\tau \to 0$ through the sector $|\arg \tau| \leq \pi/2 - \eta$ with any small $\eta > 0$. Here the constant implied in the O-symbol depends at most on s a, b, μ, ν, J and η.

We use the symbol $\varepsilon(Z) = \operatorname{sgn}(\arg Z)$ for $|\arg Z| > 0$, and let ${}_1F_1({\alpha \atop \gamma}; Z)$ and $U(\alpha; \gamma; Z)$ denote Kummer's confluent hypergeometric functions of the first and second kind, respectively, defined for any complex α and γ by

$${}_1F_1\left({\alpha \atop \gamma}; Z\right) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\gamma)_k k!} Z^k \tag{2.21}$$

with $\gamma \neq -k$ ($k = 0, -1, \ldots$) and for $|Z| < +\infty$ (cf. [5, 6.1 (1)]), and

$$U(\alpha; \gamma; Z) = \frac{1}{\Gamma(\alpha)\{e(\alpha) - 1\}} \int_{\infty}^{(0+)} e^{-Zw} w^{\alpha-1}(1 + w)^{\gamma-\alpha-1} dw \tag{2.22}$$

for $|\arg Z| < \pi/2$, where the latter can be continued to the whole sector $|\arg Z| < 3\pi/2$ by rotating appropriately the path of integration. An application of the connection formula

$$(2.23)\qquad {}_1F_1\binom{\alpha}{\gamma}; Z\Big) = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)}e^{\varepsilon(Z)\pi i\alpha}U(\alpha;\gamma;Z) + \frac{\Gamma(\gamma)}{\Gamma(\alpha)}e^{\varepsilon(Z)\pi i(\alpha-\gamma)}e^{Z}U(\gamma-\alpha;\gamma;e^{-\varepsilon(Z)\pi i}Z),$$

valid in the sectors $0 < |\arg Z| < 3\pi/2$ (cf. [5, 6.7 (7)]), allows us to extract the exponentially small order terms of the form $\mathcal{S}_{1-s}(c,d;\kappa,\lambda;\widehat{q})$ with $\widehat{q} = e^{-2\pi/\tau}$ as $\tau \to 0$ from the remainder in (2.19).

Theorem 3 ([16, Theorem 3]). *In the region $\sigma > 1-J$ with any $J \geq 1$ and in the sectors $0 < |\arg\tau| < \pi/2$, we have the formula*

$$(2.24)\qquad R_J(s;a,b;\mu,\nu;z) = e(b\nu)\frac{(2\pi/\tau)^s}{\Gamma(s)}\{\mathcal{S}_{1-s}(\langle a\rangle,\langle\nu\rangle;\mu,-b;\widehat{q}) + e^{\varepsilon(\tau)\pi is}\mathcal{S}_{1-s}(\langle -a\rangle,\langle -\nu\rangle;-\mu,b;\widehat{q})\} + (-1)^J e(b\nu)(2\pi/\tau)^s\frac{\sin(\pi s)}{\pi}(s)_J S_J^*(s;a,b;\mu,\nu;z),$$

where the expression

$$(2.25)\qquad S_J^*(s;a,b;\mu,\nu;z) = {\sum_{m,n=0}^{\infty}}' \frac{e(-(\langle -a\rangle+m)\mu-(\langle\nu\rangle+n)b)}{(\langle\nu\rangle+n)^{1-s}} \times f_{s,J}(2\pi(\langle -a\rangle+m)(\langle\nu\rangle+n)/\tau) - e^{\varepsilon(\tau)\pi is}{\sum_{m,n=0}^{\infty}}' \frac{e(-(\langle -a\rangle+m)\mu+(\langle -\nu\rangle+n)b)}{(\langle -\nu\rangle+n)^{1-s}} \times f_{s,J}(2\pi e^{\varepsilon(\tau)\pi i}(\langle -a\rangle+m)(\langle -\nu\rangle+n)/\tau)$$

holds with

$$(2.26)\qquad f_{s,J}(Z) = U(s+J;s+J;Z).$$

Furthermore, for any integers J and K with $J \geq 1$ and $K \geq 0$, in the region $\mathrm{Re}\,s > 1-J-K$, *we have the formula*

$$(2.27)\qquad S_J^*(s;a,b;\mu,\nu;z) = \frac{(-1)^J e(-b\nu)}{(2\pi)^{s-1}}\sum_{k=0}^{K-1}\frac{i^{J+k+1}(s+J)_k}{(J+k+1)!} \times \psi(s+J+k,\langle -a\rangle,-\mu)\mathcal{C}_{J+k+1}(\langle b\rangle,\widetilde{e}(\nu))\tau^{s+J+k} \times R_{J,K}^*(s;a,b;\mu,\nu;z),$$

valid in the sectors $0 < |\arg\tau| < \pi/2$, where $R_{J,K}^*(s;a,b;\mu,\nu;z)$ is the remainder term satisfying the estimate

$$(2.28)\qquad R_{J,K}^*(s;a,b;\mu,\nu;z) = O(|\tau|^{\mathrm{Re}\,s+J+K})$$

as $\tau \to 0$ through $\eta \leq |\arg\tau| \leq \pi/2-\eta$ with any small $\eta > 0$. Here the constant implied in the O-symbol depends at most on s, a, b, μ, ν, J, K and η.

3. Variants of Euler's and Ramanujan's formula for $\zeta(s)$

It is in fact possible to deduce from the combination of Theorems 1–3 the celebrated formulae of Euler and Ramanujan for specific values of the Riemann zeta-function as well as their several variants. One can observe that the connection formula (2.23) works as a key ingredient in the background to generate various Ramanujan's type formulae for specific values of zeta-functions.

Theorem 4 ([16, Theorem 4]). *Let $q = e(i\tau) = e^{-2\pi\tau}$ and $\widehat{q} = e(i/\tau) = e^{-2\pi/\tau}$ for any complex τ in the sector $|\arg\tau| < \pi/2$. Then for any real a, b, μ and ν, and any integer $k \neq 0$, we have the formula*

$$\begin{aligned}(3.1)\qquad & e(a\mu)\bigl\{\delta(b)\psi(k,\langle -\mu\rangle,a)\\ &+\mathcal{S}_k(\langle b\rangle,\langle -\mu\rangle;\nu,a;q)+(-1)^{k-1}\mathcal{S}_k(\langle -b\rangle,\langle \mu\rangle;-\nu,-a;q)\bigr\}\\ &-(-2\pi)^k\sum_{j=0}^{k+1}\frac{(-i)^j\mathcal{C}_{k+1-j}(\langle b\rangle,\widetilde{e}(\nu))\mathcal{C}_j(\langle a\rangle,\widetilde{e}(\mu))}{(k+1-j)!j!}\tau^{k-j}\\ &=e(b\nu)(-i\tau)^{k-1}\bigl\{\delta(a)\psi(k,\langle\nu\rangle,-b)\\ &+\mathcal{S}_k(\langle a\rangle,\langle\nu\rangle;\mu,-b;\widehat{q})+(-1)^{k-1}\mathcal{S}_k(\langle -a\rangle,\langle -\nu\rangle;-\mu,b;\widehat{q})\bigr\},\end{aligned}$$

whose variant asserts upon replacing $(\tau,q)\mapsto(1/\tau,\widehat{q})$ that

$$\begin{aligned}(3.2)\qquad & e(b\nu)\bigl\{\delta(a)\psi(k,\langle\nu\rangle,-b)\\ &+\mathcal{S}_k(\langle a\rangle,\langle\nu\rangle;\mu,-b;q)+(-1)^{k-1}\mathcal{S}_k(\langle -a\rangle,\langle -\nu\rangle;-\mu,b;q)\bigr\}\\ &-(-2\pi)^k\sum_{j=0}^{k+1}\frac{i^j\mathcal{C}_{k+1-j}(\langle a\rangle,\widetilde{e}(\mu))\mathcal{C}_j(\langle b\rangle,\widetilde{e}(\nu))}{(k+1-j)!j!}\tau^{k-j}\\ &=e(a\mu)(i\tau)^{k-1}\bigl\{\delta(b)\psi(k,\langle -\mu\rangle,a)\\ &+\mathcal{S}_k(\langle b\rangle,\langle -\mu\rangle;\nu,a;\widehat{q})+(-1)^{k-1}\mathcal{S}_k(\langle -b\rangle,\langle\mu\rangle;-\nu,-a;\widehat{q})\bigr\}.\end{aligned}$$

The particular case $(\mu,\nu)=(0,0)$ of Theorem 4 reduces to the following formula for the pairing of $\zeta_a(k)$ and $\zeta_{-b}(k)$.

Corollary 4.1 ([16, Corollary 4.1]). *For any real a and b, and any integer $k \neq 0$, we have*

$$\begin{aligned}(3.3)\qquad & \delta(b)\zeta_a(k)+\mathcal{S}_k(\langle b\rangle,0;0,a;q)+(-1)^{k-1}\mathcal{S}_k(\langle -b\rangle,0;0,-a;q)\\ &-(-2\pi)^k\sum_{j=0}^{k+1}\frac{(-i)^jB_{k+1-j}(\langle b\rangle)B_j(\langle a\rangle)}{(k+1-j)!j!}\tau^{k-j}\\ &=(-i\tau)^{k-1}\bigl\{\delta(a)\zeta_{-b}(k)+\mathcal{S}_k(\langle a\rangle,0;0,-b;\widehat{q})+(-1)^{k-1}\mathcal{S}_k(\langle -a\rangle,0;0,b;\widehat{q})\bigr\},\end{aligned}$$

whose variant asserts that

$$\begin{aligned}(3.4)\qquad & \delta(a)\zeta_{-b}(k)+\mathcal{S}_k(\langle a\rangle,0;0,-b;q)+(-1)^{k-1}\mathcal{S}_k(\langle -a\rangle,0;0,b;q)\\ &-(-2\pi)^k\sum_{j=0}^{k+1}\frac{i^jB_{k+1-j}(\langle a\rangle)B_j(\langle b\rangle)}{(k+1-j)!j!}\tau^{k-j}\\ &=(i\tau)^{k-1}\bigl\{\delta(b)\zeta_a(k)+\mathcal{S}_k(\langle b\rangle,0;0,a;\widehat{q})+(-1)^{k-1}\mathcal{S}_k(\langle -b\rangle,0;0,-a;\widehat{q})\bigr\}.\end{aligned}$$

The particular case $(a,b)=(0,0)$ of Theorem 4 reduces to the following formula for the pairing of $\zeta(k,\langle -\mu\rangle)$ and $\zeta(k,\langle\nu\rangle)$.

Corollary 4.2 ([16, Corollary 4.2]). *For any real μ and ν, and any integer $k \neq 1$, we have*

$$
\begin{aligned}
(3.5) \qquad & \zeta(k, \langle -\mu\rangle) + \mathcal{S}_k(0, \langle -\mu\rangle; \nu, 0; q) + (-1)^{k-1}\mathcal{S}_k(0, \langle \mu\rangle; -\nu, 0; q)\} \\
& - (-2\pi)^k \sum_{j=0}^{k+1} \frac{(-i)^j A_{k+1-j}(e(\nu))A_j(e(\mu))}{(k+1-j)!j!}\tau^{k-j} \\
& = (-i\tau)^{k-1}\{\zeta(k, \langle \nu\rangle) + \mathcal{S}_k(0, \langle \nu\rangle; \mu, 0; \widehat{q}) + (-1)^{k-1}\mathcal{S}_k(0, \langle -\nu\rangle; -\mu, 0; \widehat{q})\},
\end{aligned}
$$

whose variant asserts that

$$
\begin{aligned}
(3.6) \qquad & \zeta(k, \langle \nu\rangle) + \mathcal{S}_k(0, \langle \nu\rangle; \mu, 0; q) + (-1)^{k-1}\mathcal{S}_k(0, \langle -\nu\rangle; -\mu, 0; q)\} \\
& - (-2\pi)^k \sum_{j=0}^{k+1} \frac{i^j A_{k+1-j}(e(\mu))A_j(e(\nu))}{(k+1-j)!j!}\tau^{k-j} \\
& = (i\tau)^{k-1}\{\zeta(k, \langle -\mu\rangle) + \mathcal{S}_k(0, \langle -\mu\rangle; \nu, 0; \widehat{q}) + (-1)^{k-1}\mathcal{S}_k(0, \langle \mu\rangle; -\nu, 0; \widehat{q})\}.
\end{aligned}
$$

The particular case $(b, \nu) = (0, 0)$ of Theorem 4 reduces to the following formula for the paring of $\psi(k, \langle -\mu\rangle, a)$ and $\zeta(k)$.

Corollary 4.3 ([16, Corollary 4.3]). *For any real a and μ, and any integer $k \neq 1$, we have*

$$
\begin{aligned}
(3.7) \qquad & e(a\mu)\{\psi(k, \langle -\mu\rangle, a) + \mathcal{S}_k(0, \langle -\mu\rangle; 0, a; q) + (-1)^{k-1}\mathcal{S}_k(0, \langle \mu\rangle; 0, -a; q)\} \\
& - (-2\pi)^k \sum_{j=0}^{k+1} \frac{(-i)^j B_{k+1-j}\mathcal{C}_j(\langle a\rangle, \widetilde{e}(\mu))}{(k+1-j)!j!}\tau^{k-j} \\
& = (-i\tau)^{k-1}\{\delta(a)\zeta(k) + \mathcal{S}_k(\langle a\rangle, 0; \mu, 0; \widehat{q}) + (-1)^{k-1}\mathcal{S}_k(\langle -a\rangle, 0; \mu, 0; \widehat{q})\},
\end{aligned}
$$

whose variant asserts that

$$
\begin{aligned}
(3.8) \qquad & \delta(a)\zeta(k) + \mathcal{S}_k(\langle a\rangle, 0; \mu; 0, a; q) + (-1)^{k-1}\mathcal{S}_k(\langle -a\rangle, 0, -\mu; 0; q)\} \\
& - (-2\pi)^k \sum_{j=0}^{k+1} \frac{i^j \mathcal{C}_{k+1-j}(\langle a\rangle, \widetilde{e}(\mu))B_j}{(k+1-j)!j!}\tau^{k-j} \\
& = e(a\mu)(i\tau)^{k-1}\{\psi(k, \langle -\mu\rangle, a) \\
& \quad + \mathcal{S}_k(0, \langle -\mu\rangle, 0; a, 0; \widehat{q}) + (-1)^{k-1}\mathcal{S}_k(0, \langle \mu\rangle, 0; 0, -a; \widehat{q})\}.
\end{aligned}
$$

The particular case $(a, \nu) = (0, 0)$ of Theorem 4 reduces to the following formula for the pairing of $\zeta(k, \langle -\mu\rangle)$ and $\zeta_{-b}(k)$.

Corollary 4.4 ([16, Corollary 4.4]). *For any real b and μ, and any integers $k \neq 1$, we have*

$$
\begin{aligned}
(3.9) \qquad & \delta(b)\zeta(k, \langle -\mu\rangle) + \mathcal{S}_k(b, \langle -\mu\rangle; 0, 0; q) + (-1)^{k-1}\mathcal{S}_k(\langle -b\rangle, \langle \mu\rangle; 0, 0; q) \\
& - (-2\pi)^k \sum_{j=0}^{k+1} \frac{(-i)^j B_{k+1-j}(\langle b\rangle)A_j(e(\mu))}{(k+1-j)!j!}\tau^{k-j} \\
& = e(b\nu)(-i\tau)^{k-1}\{\zeta_{-b}(k) + \mathcal{S}_k(0, 0; \mu, -b; \widehat{q}) + (-1)^{k-1}\mathcal{S}_k(0, 0; -\mu, b; \widehat{q}))\},
\end{aligned}
$$

whose variant asserts that

$$\begin{aligned}(3.10)\qquad &\zeta_{-b}(k) + \mathcal{S}_k(0,0;\mu,-b;q) + (-1)^{k-1}\mathcal{S}_k(0,0;-\mu,b;q)\\ &- (-2\pi)^k \sum_{j=0}^{k+1} \frac{i^j A_{k+1-j}(e(\mu))B_j(\langle b\rangle)}{(k+1-j)!j!}\tau^{k-j}\\ &= (i\tau)^{k-1}\big\{\delta(b)\zeta(k,\langle -\mu\rangle)\\ &+ \mathcal{S}_k(\langle b\rangle,\langle -\mu\rangle;0,0;\widehat{q}) + (-1)^{k-1}\mathcal{S}_k(\langle -b\rangle,\langle \mu\rangle;0,0;\widehat{q})\big\}.\end{aligned}$$

The simplest case $(a,b,\mu,\nu) = (0,0,0,0)$ of Theorem 4 reduces to the celebrated formulae of Euler and Ramanujan, respectively, for specific values of $\zeta(s)$.

Corollary 4.5 ([16, Corollary 4.5]). *We have the the following formulae:*

i) *for any integer* $k \geq 1$,

$$(3.11)\qquad \zeta(2k) = \frac{(-1)^{k+1}(2\pi)^{2k}}{2(2k)!}B_{2k};$$

ii) *for any integer* $k \neq 0$,

$$\begin{aligned}(3.12)\qquad &\zeta(2k+1) + 2\mathcal{S}_{2k+1}(0,0;0,0;q) + (2\pi)^{2k+1}\sum_{j=0}^{k+1}\frac{(-1)^j B_{2k+2-2j}B_{2j}}{(2k+2-2j)!(2j)!}\tau^{2k+1-2j}\\ &= (i\tau)^{2k}\{\zeta(2k+1) + 2\mathcal{S}_{2k+1}(0,0;0,0;\widehat{q})\}.\end{aligned}$$

4. Classical Eisenstein series

We present in this section several applications of Theorems 1–3 to the classical Eisenstein series. Let $E_{2k}(z)$ denote the classical holomorphic Eisenstein series defined for $k \geq 1$ by

$$(4.1)\qquad E_{2k}(z) = 1 - \frac{4k}{B_{2k}}\sum_{l=1}^{\infty}\frac{l^{2k-1}q^l}{1-q^l}$$

with $q = e(z)$ (cf. [4, Chap.4, 4.5 (4.5.1)]). Theorem 1 in fact shows that

$$(4.2)\qquad E_{2k}(z) = \frac{(-1)^{k-1}(2k)!}{(2\pi)^{2k}B_{2k}}F_{\mathbb{Z}^2}(2k;0,0;0,0;z).$$

for any integer $k \geq 1$. We shall treat in what follows the cases $k = 1$ and $k \geq 2$ separately.

Consider first the case $k = 1$. The combination of Theorems 2 and 3 reduces in this case to

$$F_{\mathbb{Z}^2}(2;0,0;0,0;z) = \frac{\pi^2}{3z^2} + \frac{2\pi i}{z} - \frac{8\pi^2}{z^2}\mathcal{S}_{-1}(0,0;0,0;\widehat{q}),$$

while Theorem 1 applied with $-1/z$ instead of z implies that

$$F_{\mathbb{Z}^2}\Big(2;0,0;0,0;-\frac{1}{z}\Big) = \frac{\pi^2}{3} - 8\pi^2\mathcal{S}_{-1}(0,0;0,0;\widehat{q}),$$

and hence the relation between $F_{\mathbb{Z}^2}(2;0,0;0,0;z)$ and $F_{\mathbb{Z}^2}(2;0,0;0,0;-1/z)$ asserts

$$(4.3)\qquad F_{\mathbb{Z}^2}(2;0,0;0,0;z) = \frac{2\pi i}{z} + \frac{1}{z^2}F_{\mathbb{Z}^2}\Big(2;0,0;0,0;-\frac{1}{z}\Big),$$

which gives the following transformation formula (cf. [30, Chap.2, 2.4 (2.58)]):

Corollary 4.6 ([16, Corollary 4.6]). *For any $z \in \mathfrak{H}^+$, we have*

$$E_2\Big(-\frac{1}{z}\Big) = \frac{6z}{\pi i} + z^2 E_2(z). \tag{4.4}$$

One can see that the procedure of derivation above gives a zeta-function theoretic or asymptotic methodological proof of the modular relation for $E_2(z)$.

We next treat the case $k \geq 2$. The combination of Theorems 2 and 3 in this case reduces to

$$F_{\mathbb{Z}^2}(2k;0,0;0,0;z) = \frac{(-1)^k 2(2\pi/z)^{2k}}{(2k-1)!}\Big\{-\frac{B_{2k}}{4k} + \mathcal{S}_{1-2k}(0,0;0,0;\widehat{q})\Big\},$$

while Theorem 1 applied with $-1/z$ instead of z implies that

$$F_{\mathbb{Z}^2}\Big(2k;0,0;0,0;-\frac{1}{z}\Big) = \frac{(-1)^k 2(2\pi)^{2k}}{(2k-1)!}\Big\{-\frac{B_{2k}}{4k} + \mathcal{S}_{1-2k}(0,0;0,0;\widehat{q})\Big\},$$

and hence the relation between $F_{\mathbb{Z}^2}(2;0,0;0,0;z)$ and $F_{\mathbb{Z}^2}(2;0,0;0,0;-1/z)$ asserts

$$F_{\mathbb{Z}^2}(2k;0,0;0,0;z) = \frac{1}{z^{2k}} F_{\mathbb{Z}^2}\Big(2k;0,0;0,0;-\frac{1}{z}\Big), \tag{4.5}$$

which gives the transformation formula:

Corollary 4.7 ([16, Corollary 4.7]). *For any $z \in \mathfrak{H}^+$, we have*

$$E_{2k}\Big(-\frac{1}{z}\Big) = z^{2k} E_{2k}(z) \qquad (k \geq 2). \tag{4.6}$$

It is known for $k \geq 2$ that the double series expression

$$E_{2k}(z) = \frac{1}{2} \sum_{\substack{c,d=-\infty \\ (c,d)=1}}^{\infty} \frac{1}{(cz+d)^{2k}}$$

is valid (cf. [4, Chap.4, 4.5 (4.5.1)]). One can therefore see that the procedure of derivation above successfully(?) gives a stupidly lengthy proof(!) of the modular relation for $E_{2k}(z)$ with $k \geq 2$.

5. Weierstrass' elliptic and allied functions

We present in this section several applications of Theorems 1–3 to Weierstraß' elliptic and allied functions. Let $\boldsymbol{\omega} = (\omega_1, \omega_2) \in \mathbb{C}^2$ be a fundamental parallelogram with $\mathrm{Im}(\omega_2/\omega_1) > 0$. Set $\omega_2/\omega_1 = z$, and choose the branch with $\arg z \in]0, \pi[$. Weierstraß' elliptic function with the periods $\boldsymbol{\omega} = (\omega_1, \omega_2)$ is defined by

$$\wp(w \mid \boldsymbol{\omega}) = \frac{1}{w^2} + \sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \Big\{\frac{1}{(w - m\omega_1 - n\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2}\Big\} \tag{5.1}$$

(cf. [6, 13.12 (4)]), while (allied) Weierstraß' zeta and sigma functions by

$$\zeta(w \mid \boldsymbol{\omega}) = \frac{1}{w} + \sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \Big\{\frac{1}{w - m\omega_1 - n\omega_2} + \frac{1}{m\omega_1 + n\omega_2} + \frac{w}{(m\omega_1 + n\omega_2)^2}\Big\}, \tag{5.2}$$

$$(5.3)\quad \sigma(w \mid \boldsymbol{\omega}) = w \prod_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \left(1 - \frac{w}{m\omega_1 + n\omega_2}\right) \exp\left\{\frac{w}{m\omega_1 + n\omega_2} + \frac{1}{2}\left(\frac{w}{m\omega_1 + n\omega_2}\right)^2\right\},$$

respectively (cf. [6, 13.12 (6) and (11)]). It suffices in fact to study the elliptic and allied functions defined with the normalized periods $\boldsymbol{z} = (1, z)$, in view of the relations

$$\wp(cw \mid c\boldsymbol{\omega}) = c^{-2}\wp(w \mid \boldsymbol{\omega}), \qquad \zeta(cw \mid c\boldsymbol{\omega}) = c^{-1}\zeta(w \mid \boldsymbol{\omega}), \qquad \sigma(cw \mid c\boldsymbol{\omega}) = c\sigma(w \mid \boldsymbol{\omega})$$

for any $c \in \mathbb{C}^{\times}$. One can then see that the limiting relation

$$(5.4)\qquad \wp(w \mid \boldsymbol{z}) = \lim_{\substack{s\to 2 \\ \operatorname{Re} s>2}} \left\{F_{\mathbb{Z}^2}(s; a, b; 0, 0; z) - F_{\mathbb{Z}^2}(s; 0, 0; 0, 0; z)\right\}$$

is valid for any $w = a + bz \in \mathbb{C}$ with $(a, b) \in \mathbb{R}^2$, since the limiting point $s = 2$ is located in the boundary of the region where the defining series in (1.1) converges absolutely. Theorem 1 can therefore be applied on the right side of (5.4) to show the following expression of $\wp(w \mid \boldsymbol{z})$.

Corollary 4.8 ([16, Corollary 4.8]). *For any $w = a + bz \in \mathbb{C}$ with $(a, b) \in \mathbb{R}^2 \setminus \mathbb{Z}^2$, we have*

$$(5.5)\qquad \begin{aligned} \wp(w \mid \boldsymbol{z}) = &-\frac{\pi^2}{3}E_2(z) + \frac{\delta(b)\pi^2}{\sin^2 \pi a} \\ &- 4\pi^2\left\{\mathcal{S}_{-1}(\langle b\rangle, 0; 0, a; q) + \mathcal{S}_{-1}(\langle -b\rangle, 0; 0, -a; q)\right\}. \end{aligned}$$

Combining Theorems 2, 3 and Corollary 4.8, we obtain the period change formula for $\wp(w \mid \boldsymbol{z})$ in the form

$$(5.6)\qquad \wp(w \mid \boldsymbol{z}) = \frac{1}{z^2}\wp\left(\frac{w}{z} \;\Big|\; \widehat{\boldsymbol{z}}\right),$$

where $\widehat{\boldsymbol{z}} = (1, -1/z)$ are the dual periods (cf. [30, Chap.2, 2.4]). We next write the (base) parameter corresponding to the half period as $p = e(z/2) = e^{-\pi\tau}$ (i.e. $q = p^2$), and then define the Weierstrassian invariants by

$$(5.7)\qquad e_1(\boldsymbol{z}) = \wp\left(\frac{1}{2} \;\Big|\; \boldsymbol{z}\right), \qquad e_2(\boldsymbol{z}) = \wp\left(\frac{z}{2} \;\Big|\; \boldsymbol{z}\right), \qquad e_3(\boldsymbol{z}) = \wp\left(\frac{1+z}{2} \;\Big|\; \boldsymbol{z}\right).$$

Then Corollary 4.8 in fact implies the following Lambert series expressions for Weierstrassian invariants (cf. [30, Chap.4, 4.2 (4.46)–(4.48)])

$$(5.8)\qquad \begin{aligned} e_1(\boldsymbol{z}) &= 4\pi^2\left\{\frac{1}{6} + 4\sum_{l=1}^{\infty}\frac{(2l-1)p^{4l-2}}{1-p^{4l-2}}\right\}, \\ e_2(\boldsymbol{z}) &= 4\pi^2\left\{-\frac{1}{12} - 2\sum_{l=1}^{\infty}\frac{(2l-1)p^{2l-2}}{1-p^{2l-2}}\right\}, \\ e_3(\boldsymbol{z}) &= 4\pi^2\left\{-\frac{1}{12} + 2\sum_{l=1}^{\infty}\frac{(2l-1)p^{2l-1}}{1+p^{2l-1}}\right\}, \end{aligned}$$

which further yield a significant relation (cf. [30, Chap.4, 4.2 (4.49)]):

$$(5.9)\qquad e_1(\boldsymbol{z}) + e_2(\boldsymbol{z}) + e_3(\boldsymbol{z}) = 0.$$

Furthermore, combining Theorems 2, 3 and Corollary 4.8, we obtain the period change formulae for Weierstrassian invariants:

$$(5.10)\qquad e_j(\boldsymbol{z}) = e_j(\widehat{\boldsymbol{z}}) \qquad (j = 1, 2, 3).$$

We next consider Weierstraß' zeta function. It is misleading to validate that $\zeta(w \mid \boldsymbol{z})$ is defined to be the limit

$$\lim_{\substack{s\to 1\\ \operatorname{Re} s>1}} \big\{F_{\mathbb{Z}^2}(s;a,b;0,0;z) - F_{\mathbb{Z}^2}(s;0,0;0,0;z) + swF_{\mathbb{Z}^2}(s+1;0,0;0,0;z)\big\},$$

since the limiting point $s=1$ is located in the exterior of the region where the defining series in (1.1) converges absolutely. We rather take another route for defining Weierstraß' zeta function in terms of $\wp(w \mid \boldsymbol{z})$, which asserts

$$\zeta(w \mid \boldsymbol{z}) = \frac{1}{w} - \int_0^w \Big\{\wp(u \mid \boldsymbol{z}) - \frac{1}{u^2}\Big\}du \tag{5.11}$$

(cf. [6, 13.12 (7)]). The expression in (5.5) can therefore be integrated to show the following formula for $\zeta(w \mid \boldsymbol{z})$.

Corollary 4.9 ([16, Corollary 4.9]). *For any $w = a + bz \in \mathbb{C}$ with $(a,b) \in\,]-1,1[^2 \setminus \{(0,0)\}$, we have*

$$\begin{aligned}\zeta(w \mid \boldsymbol{z}) = {}& \frac{\pi^2}{3}E_2(z)w + \delta(b)\pi\cot\pi a - (\operatorname{sgn} b)\pi i\\ & - 2\pi i\big\{\mathcal{S}_0(\langle b\rangle,0;0,a;q) - \mathcal{S}_0(\langle -b\rangle,0;0,-a;q)\big\}.\end{aligned} \tag{5.12}$$

Weierstraß' eta invariants are defined by

$$\eta_1(\boldsymbol{z}) = \zeta\Big(\frac{1}{2}\ \Big|\ \boldsymbol{z}\Big), \qquad \eta_2(\boldsymbol{z}) = \zeta\Big(\frac{z}{2}\ \Big|\ \boldsymbol{z}\Big), \qquad \eta_3(\boldsymbol{z}) = \zeta\Big(-\frac{1+z}{2}\ \Big|\ \boldsymbol{z}\Big). \tag{5.13}$$

Corollary 4.9 therefore gives the evaluations

$$\begin{aligned}\eta_1(\boldsymbol{z}) &= \frac{\pi^2}{6}E_2(z),\\ \eta_2(\boldsymbol{z}) &= \frac{\pi^2}{6}E_2(z)z - \pi i,\\ \eta_3(\boldsymbol{z}) &= -\frac{\pi^2}{6}E_2(z)(1+z) + \pi i,\end{aligned} \tag{5.14}$$

which imply the classical Legendre relations (cf. [6, 13.12 (10)])

$$\begin{aligned}\eta_1(\boldsymbol{z})\cdot\frac{z}{2} - \eta_2(\boldsymbol{z})\cdot\frac{1}{2} &= \frac{\pi i}{2},\\ \eta_2(\boldsymbol{z})\cdot\Big(-\frac{1+z}{2}\Big) - \eta_3(\boldsymbol{z})\cdot\frac{z}{2} &= \frac{\pi i}{2},\\ \eta_3(\boldsymbol{z})\cdot\frac{1}{2} - \eta_1(\boldsymbol{z})\cdot\Big(-\frac{1+z}{2}\Big) &= \frac{\pi i}{2}.\end{aligned} \tag{5.15}$$

We finally consider Weierstraß' sigma function. It is misleading again to validate that $\log\sigma(w \mid \boldsymbol{z})$ is defined to be the limit

$$\begin{aligned}\lim_{\substack{s\to 0\\ \operatorname{Re} s>0}} \Big\{&-\frac{\partial}{\partial s}F_{\mathbb{Z}^2}(s;a,b;0,0;z) + \frac{\partial}{\partial s}F_{\mathbb{Z}^2}(s;0,0;0,0;z) - wF_{\mathbb{Z}^2}(s+1;0,0;0,0;z)\\ &+ \frac{1}{2}w^2F_{\mathbb{Z}^2}(s+2;0,0;0,0;z)\Big\},\end{aligned}$$

since the limiting point $s = 0$ is again located in the exterior of the region where the defining series in (1.1) converges absolutely. We rather take another route for defining Weierstraß' sigma function in terms of $\zeta(w \mid \boldsymbol{z})$, which asserts

$$\log\sigma(w \mid \boldsymbol{z}) = \log w + \int_0^w \left\{\zeta(u \mid \boldsymbol{z}) - \frac{1}{u}\right\} du \tag{5.16}$$

(cf. [6, 13.12 (12)]). We use the customary notation $(z;q) = \prod_{l=0}^{\infty}(1-zq^l)$ for any $z \in \mathbb{C}$ in the sequel. Then the expression in (5.12) can therefore be integrated to show the following formula for $\log\sigma(w \mid \boldsymbol{z})$.

Corollary 4.10 ([16, Corollary 4.10]). *For any $w = a + bz \in \mathbb{C}$ with $(a,b) \in]-1,1[^2 \setminus \{(0,0)\}$, we have*

$$\begin{aligned}\log\sigma(w \mid \boldsymbol{z}) = {} & \frac{\pi^2}{6}E_2(z)w^2 + (\operatorname{sgn} b)\pi i\Bigl(\frac{1}{2} - w\Bigr) + \delta(b)\log(2\sin\pi a) \\ & - \mathcal{S}_1(\langle b\rangle, 0; 0, a; q) - \mathcal{S}_1(\langle -b\rangle, 0; 0, -a; q) \\ & + 2\mathcal{S}_1(0,0;0,0;q) - \log 2\pi,\end{aligned} \tag{5.17}$$

whose exponential form asserts

$$\begin{aligned}\sigma(w \mid \boldsymbol{z}) = {} & \exp\Bigl\{\frac{\pi^2}{6}E_2(z)w^2 + (\operatorname{sgn} b)\pi i\Bigl(\frac{1}{2} - w\Bigr)\Bigr\}(2\sin\pi a)^{\delta(b)} \\ & \times \frac{(e(a)q^{\langle b\rangle'}; q)_\infty (e(-a)q^{\langle -b\rangle'}; q)_\infty}{2\pi(q;q)_\infty^2},\end{aligned} \tag{5.18}$$

where (only) the many valued term $\log(2\sin\pi a)$ on the right side of (5.17) becomes one valued after its exponentiation on the right side of (5.18).

References

[1] B. Berndt, *Generalized Dedekind eta-functions and generalized Dedekind sums*, Trans. Amer. Math. Soc. **178** (1973), 495–508.

[2] B. Berndt, *Modular transformations and generalizations of several formulae of Ramanujan*, Rocky Mountain J. Math. **7** (1977), 182–193.

[3] B. Berndt, *Ramanujan's theory of theta functions*, in "Theta Functions: From the Classical to the Modern," CRM Proceedings & Lecture Notes, vol. 1, pp. 1–63, Amer. Math. Soc., Providence, 1992.

[4] B. Berndt, *Number Theory in the Spirit of Ramanujan*, Amer. Math. Soc., Providence, 2006.

[5] A. Eldélyi (ed.), W. Magnus, F. Oberhettinger, F. G. Tricomi, *Higher Transcendental Functions*, Vol. I, McGraw-Hill, New York, 1953.

[6] A. Eldélyi (ed.), W. Magnus, F. Oberhettinger, F. G. Tricomi, *Higher Transcendental Functions*, Vol. II, McGraw-Hill, New York, 1953.

[7] A. Eldélyi (ed.), W. Magnus, F. Oberhettinger, F. G. Tricomi, *Tables of Integral Transforms*, Vol. I, McGraw-Hill, New York, 1954.

[8] A. Ivić, *The Riemann Zeta-Function*, Wiley, New York, 1985.

[9] M. Katsurada, *Complete asymptotic expansions associated with Epstein zeta-functions*, Ramanujan J. **14** (2007), 249–275.

[10] M. Katsurada, *Complete asymptotic expansions associated with Epstein zeta-functions II*, Ramanujan J. **36** (2015), 403–437.

[11] M. Katsurada and T. Noda, *Differential actions on the asymptotic expansions of non-holomorphic Eisenstein series*, Int. J. Number Theory **5** (2009), 1061–1088.

[12] M. Katsurada and T. Noda, *On generalized Lipschitz type formulae and applications*, in "Diophantine Analysis and Related Fields 2010", AIP Conf. Proc., No. 1264, pp. 123–138, Amer. Inst. Phys., Melville, NY, 2010.

[13] M. Katsurada and T. Noda, *On generalized Lipschitz type formulae and applications II*, in "Diophantine Analysis and Related Fields 2010", AIP Conf. Proc., No. 1385, pp. 73–86, Amer. Inst. Phys., Melville, NY, 2011.

[14] M. Katsurada and T. Noda, *Transformation formulae and asymptotic expansions for double holomorphic Eisenstein series of two complex variables (summarized version)*, in "Kôkyûroku", (to appear).

[15] M. Katsurada and T. Noda, *Transformation formulae and asymptotic expansions for double non-holomorphic Eisenstein series of two variables*, Ramanujan J. **44** (2017), 237–280.

[16] M. Katsurada and T. Noda, *Asymptotic expansions for a class of generalized holomorphic Eisenstein series: applications to Ramanujan's formula for* $\zeta(2k+1)$ *and Weierstraß elliptic function*, (submitted for publication).

[17] J. Lewittes, *Analytic continuation of the series* $\sum(m+nz)^{-s}$, Trans. Amer. Math. Soc. **159** (1971), 505–509.

[18] J. Lewittes, *Analytic continuation of Eisenstein series*, Trans. Amer. Math. Soc. **171** (1971), 469–490.

[19] M. Lerch, *Note sur la fonction* $K(w,x,s) = \sum_{n\geq 0} \exp\{2\pi inx\}(n+x)^{-s}$, Acta Math. **11** (1887), 19–24.

[20] R. Lipschitz, *Untersuchung einer aus vier Elementen gebildeten Riehe*, J. Reine Angew. Math. **105** (1889), 127–156.

[21] S.-G. Lim, *On generalized two variable Eisenstein series*, Honam Math. J. **36** (2014), 895–899.

[22] K. Matsumoto, *Asymptotic expansions of double zeta-functions of Barnes, of Shintani, and Eisenstein series*, Nagoya Math. J. **172** (2003), 59–102.

[23] T. Noda, *Asymptotic expansions of the non-holomorphic Eisenstein series*, in "Kôkyûroku" R.I.M.S., No. 1319, pp. 29–32, 2003.

[24] ———, *A transformation formula for Maass-type Eisenstein series of two variables*, in "Kôkyûroku R.I.M.S., No. 1806, pp. 210–218, 2012.

[25] S. Ramanujan, *On certain arithmetical functions*, Trans. Camb. Philos. Soc. **22** (1916), 159–184.

[26] J. Riordan, *Combinatorial Identities*, John Wiley & Sons, New York, London, Sydney, 1968.

[27] A. Schinzel, *Private communication with the author, dated Feb. 8, 2015*

[28] C. L. Siegel, *Advanced Analytic Number Theory*, Tata Institute of Fundamental Research, Bombay, 1980.

[29] L. J. Slater, *Confluent Hypergeometric Functions*, Cambridge University Press, Cambridge, 1960.

[30] K. Venkatachaliengar, (edited and revised by S. Cooper), *Development of Elliptic Functions According to Ramanujan*, World Scientific, New Jersey, London, Singapore, 2012.

[31] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 44th ed., Cambridge University Press, Cambridge, 1927.

(Katsurada) Department of Mathematics, Faculty of Economics, Keio University, 4–1–1 Hiyoshi, Kouhoku-ku, Yokohama 223–8521, Japan; (Noda) Department of Mathematics, College of Engineering, Nihon University, 1 Nakagawara, Tokusada, Tamuramachi, Koriyama, Fukushima 963–8642, Japan

E-mail address: katsurad@z3.keio.jp; takumi@ge.ce.nihon-u.ac.jp