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On the effects of spatial expansion and contraction on several semilinear partial differential equations

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Abstract

The derivation of several second order partial differential equations is considered based on the scalar-field equation and its non-relativistic limit in the uniform and isotropic space. The field equation is derived as the Euler-Lagrange equation for a Lagrangian given in the spacetime which is a solution of the Einstein equation for non-Hermitian line elements. Some results on the Cauchy problem of the limit equation are introduced. The derivation of some equations for vectors and their energy estimates are also introduced. A dissipative property of the spatial expansion is remarked.

1 Introduction

In this paper, we first consider the derivation of several semilinear second order partial differential equations based on the field equation and its nonrelativistic limit in the spacetime generated by the Einstein equation for complex-valued line elements. Second, we consider the effect of the spatial variation (expansion or contraction) on the Cauchy problem of the limit equation.

For m, c, λ, $\hbar(\neq 0)\in\mathbb{R}$ and $1\le p<\infty$, let us consider the semilinear Klein-Gordon equation

$$\partial_t^2\phi - c^2\Delta_x\phi + \frac{m^2c^4}{\hbar^2}\phi - c^2\lambda|\phi|^{p-1}\phi = 0, \tag{1.1}$$

the semilinear Schrödinger equation

$$\pm i\frac{2m}{\hbar}\partial_t u + \Delta_x u + \lambda|u|^{p-1}u = 0, \tag{1.2}$$

the semilinear elliptic equation

$$\partial_t^2\phi + c^2\Delta_x\phi + \frac{m^2c^4}{\hbar^2}\phi - c^2\lambda|\phi|^{p-1}\phi = 0, \tag{1.3}$$

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and the semilinear parabolic equation

$$\frac{2m}{\hbar}\partial_t u - \Delta_x u - i\lambda|u|^{p-1}u = 0, \tag{1.4}$$

where we have put $\Delta_x := \sum_{j=1}^{n} \partial^2/(\partial x^j)^2$. For the elliptic equation (1.3), the variable ct can be naturally regarded as one of spatial variables. The terms $\lambda|\phi|^{p-1}\phi$ and $\lambda|u|^{p-1}u$ are fundamental semilinear terms in the nonlinear theory to describe the self-interaction of the solution. For the last parabolic equation (1.4), we note that the dimension of $\hbar/m$ in the SI units is $\mathsf{M}^2\mathsf{S}^{-1}$ (M: meter, S: second), which is equivalent to the dimension of the thermal diffusivity K_1 of the heat equation $\partial_t u - K_1\Delta_x u = 0$, and also to the dimension of the diffusion coefficient K_2 of the diffusion equation $\partial_t u - K_2\Delta_x u = 0$.

To consider the derivation of the above equations, let us consider the following line element. For any natural number n and any fixed real numbers $\omega = (\omega^0, \cdots, \omega^n) \in (-\pi/2, \pi/2]^{1+n}$, we consider a $(1+n)$-dimensional space $\mathbb{M}^{1+n}$ defined by

$$\mathbb{M}^{1+n} := \{z \in \mathbb{C}^{1+n} \mid z^\alpha = x^\alpha e^{i\omega^\alpha},\ x^\alpha \in \mathbb{R},\ 0 \le \alpha \le n\},$$

where $\mathbb{C}$ denotes the set of complex numbers. We consider a generalization of the Einstein equation for non-Hermitian complex line elements of the form $g_{\alpha\beta}(z)dz^\alpha dz^\beta$, where $\{g_{\alpha\beta}\}_{0\le\alpha,\beta\le n}$ are complex-valued functions for $z = (z^0, \cdots, z^n) \in \mathbb{M}^{1+n}$. Under the cosmological principle, we give the solution of the generalized Einstein equation as

$$g_{\alpha\beta}dz^\alpha dz^\beta = -c^2(dz^0)^2 + a(z^0)^2 q^2\left(1 + \frac{k^2 r^2}{4}\right)^{-2} \sum_{j=1}^{n}(dz^j)^2, \tag{1.5}$$

where $c > 0$ is the speed of light, $q(\neq 0), k \in \mathbb{C}$ are constants, $r := \left\{\sum_{\alpha=1}^{n}(z^\alpha)^2\right\}^{1/2}$, and $a(\cdot)$ is a complex-valued function which denotes the scale-function of the space. There is a large body of literature on the generalization of the Einstein equation for Hermitian line elements and general dimensions (see e.g. [1, 2, 3, 4, 5, 6]).

For any function f on $\mathbb{M}^{1+n}$, we consider the derivative $\partial_\alpha f(z)$ for $z \in \mathbb{M}^{1+n}$ by

$$\partial_\alpha f(z) := \lim_{\substack{h\to 0\\ h\in\mathbb{R}\setminus\{0\}}} \frac{f(z^0, \cdots, z^{\alpha-1}, z^\alpha + he^{i\omega^\alpha}, z^{\alpha+1}, \cdots, z^n) - f(z)}{he^{i\omega^\alpha}}. \tag{1.6}$$

Since $z = (z^0, \cdots, z^n) \in \mathbb{M}^{1+n}$ is parametrized by $x = (x^0, \cdots, x^n) \in \mathbb{R}^{1+n}$ by the relation

$$z^\alpha = x^\alpha e^{i\omega^\alpha}, \tag{1.7}$$

if we put $f_*(x) := f(z)$ with (1.7), then we have $\partial_\alpha f(z) = e^{-i\omega^\alpha}\partial f_*(x)/\partial x^\alpha$. Let us consider the background spacetime $\mathbb{M}^{1+n}$ with the line element (1.5). We put $q = 1$ and $k = 0$ in (1.5). As the equation of motion of the massive scalar field described by a complex-valued function $\phi = \phi(z^0, \cdots, z^n)$ with the mass m and a potential

$\lambda|\phi|^{p-1}\phi^2/(p+1)$ for $\lambda\in\mathbb{C}$ and $1\le p<\infty$, we derive the second order differential equation

$$-\frac{1}{c^2}\left(\partial_0^2+\frac{n\partial_0 a}{a}\partial_0+\frac{m^2c^4}{\hbar^2}\right)\phi+\frac{1}{a^2}\Delta_z\phi+\lambda|\phi|^{p-1}\phi=0, \tag{1.8}$$

where $\hbar$ is the Planck constant, $\partial_0:=\partial/\partial z^0$ and $\Delta_z:=\sum_{j=1}^n\partial^2/(\partial z^j)^2$. We also show that the nonrelativistic limit of (1.8) yields the equation

$$\pm i\frac{2m}{\hbar}\partial_0 u+\frac{1}{a^2}\Delta_z u+\lambda|uw|^{p-1}u=0 \tag{1.9}$$

with a suitable transform from ϕ to $u=u(z^0,\cdots,z^n)$ (see (2.18), below), where $i:=(-1)^{1/2}$ and w is a weight function defined by $w(z^0):=b_0(a(0)/a(z^0))^{n/2}$ for a constant $b_0\in\mathbb{C}$. By the transform (1.7), the equations (1.8) and (1.9) give the equations (1.1), (1.2), (1.3) and (1.4) when $a(\cdot)=1$.

2 Derivation of the field equation

In this section, let us derive the line element (1.5). In the following, Greek letters $\alpha,\beta,\gamma,\cdots$ run from 0 to n, Latin letters $j,k,\ell,\cdots$ run from 1 to n. We use the Einstein rule for the sum of indices of tensors, for example, $T^\alpha{}_\alpha:=\sum_{\alpha=0}^n T^\alpha{}_\alpha$ and $T^j{}_j:=\sum_{j=1}^n T^j{}_j$. For any function f on $\mathbb{M}^{1+n}$, we put $f_*(x):=f(z)$ with (1.7). We define the integral $\int_{\mathbb{M}^{1+n}}f(z)dz$ by

$$\int_{\mathbb{M}^{1+n}}f(z)dz:=e^{i\sum_{\alpha=0}^n\omega^\alpha}\int_{\mathbb{R}^{1+n}}f_*(x)dx. \tag{2.1}$$

We consider a bilinear symmetric complex-valued functional $\langle\cdot,\cdot\rangle$ on the vector space spanned by the vectors $\{\partial_\alpha\}_{0\le\alpha\le n}$. We put $g_{\alpha\beta}(z):=\langle\partial_\alpha,\partial_\beta\rangle$. We denote by $(g_{\alpha\beta}(z))$ the $(1+n)\times(1+n)$-matrix whose components are given by $\{g_{\alpha\beta}(z)\}_{0\le\alpha,\beta\le n}$. Put $g(z):=\det(g_{\alpha\beta}(z))$. Let $(g^{\alpha\beta}(z))$ be the inverse matrix of $(g_{\alpha\beta}(z))$. We consider a line element

$$-(cd\tau)^2=(d\ell)^2:=g_{\alpha\beta}(z)dz^\alpha dz^\beta, \tag{2.2}$$

where τ denotes the proper time and we take the square root of $(cd\tau)^2$ as $-\pi<\arg(cd\tau)\le\pi$. We define dz by

$$dz=dz^0\wedge\cdots\wedge dz^n:=\sum_\sigma\operatorname{sgn}(\sigma)\,dz^{\sigma(0)}\cdots dz^{\sigma(n)},$$

where σ denotes the permutation of $\{0,\cdots,n\}$.

We define the Christoffel symbol by

$$\Gamma^\alpha{}_{\beta\gamma}:=\frac{1}{2}g^{\alpha\delta}\left(\partial_\beta g_{\delta\gamma}+\partial_\gamma g_{\beta\delta}-\partial_\delta g_{\beta\gamma}\right). \tag{2.3}$$

We define the covariant derivative ∇_β for T^α by

$$\nabla_\beta T^\alpha(z):=\partial_\beta T^\alpha(z)+\Gamma^\alpha{}_{\beta\gamma}(z)T^\gamma(z).$$

In general, we define

$$\nabla_\delta T^{\alpha\beta\cdots}{}_{\mu\nu\cdots} := \partial_\delta T^{\alpha\beta\cdots}{}_{\mu\nu\cdots} + \Gamma^\alpha{}_{\delta\varepsilon} T^{\varepsilon\beta\cdots}{}_{\mu\nu\cdots} + \Gamma^\beta{}_{\delta\varepsilon} T^{\alpha\varepsilon\cdots}{}_{\mu\nu\cdots} + \cdots \\ - \Gamma^\varepsilon{}_{\delta\mu} T^{\alpha\beta\cdots}{}_{\varepsilon\nu\cdots} - \Gamma^\varepsilon{}_{\delta\nu} T^{\alpha\beta\cdots}{}_{\mu\varepsilon\cdots} - \cdots$$

for any tensor $T^{\alpha\beta\cdots}{}_{\mu\nu\cdots}$.

We define the Riemann curvature tensor

$$R^\delta{}_{\alpha\beta\gamma} := \partial_\beta \Gamma^\delta{}_{\alpha\gamma} - \partial_\gamma \Gamma^\delta{}_{\alpha\beta} + \Gamma^\delta{}_{\varepsilon\beta}\Gamma^\varepsilon{}_{\alpha\gamma} - \Gamma^\delta{}_{\varepsilon\gamma}\Gamma^\varepsilon{}_{\alpha\beta}$$

which is derived from $R^\delta{}_{\alpha\beta\gamma} T^\alpha = (\nabla_\beta\nabla_\gamma - \nabla_\gamma\nabla_\beta)T^\delta$. We define the Ricci tensor $R_{\alpha\beta} := R^\gamma{}_{\alpha\beta\gamma}$, and the scalar curvature $R := g^{\alpha\beta}R_{\alpha\beta}$. We define the Einstein tensor by $G_{\alpha\beta} := R_{\alpha\beta} - g_{\alpha\beta}R/2$. The change of upper and lower indices is done by $g_{\alpha\beta}$ and $g^{\alpha\beta}$, for example, $G^\alpha{}_\beta := g^{\alpha\gamma}G_{\gamma\beta}$.

Let $\Lambda \in \mathbb{C}$ be a constant, which is called the cosmological constant. Let us consider the variation by $g_{\alpha\beta}$ of the Einstein-Hilbert action $\int_{\mathbb{M}^{1+n}} (R + 2\Lambda)\,(-g)^{1/2}\,dz$. Then the Euler-Lagrange equation for the action is given by the Einstein equation $G_{\alpha\beta} - \Lambda g_{\alpha\beta} = 0$ in the vacuum. For a stress-energy tensor $T^\alpha{}_\beta$, we define the $(1+n)$-dimensional Einstein equation

$$G^\alpha{}_\beta - \Lambda g^\alpha{}_\beta = \kappa\, T^\alpha{}_\beta, \tag{2.4}$$

where κ is a constant and we assume that κ is written as $\kappa = \kappa_0/c^4$ for some constant κ_0 which is independent of c. For the case $n = 3$ and real line elements, the constant κ is called the Einstein gravitational constant which is given by $\kappa = 8\pi\mathcal{G}/c^4$, where $\mathcal{G}$ is the Newton gravitational constant. For the case $n \ge 3$ and complex line elements, we are able to generalize the constant κ to

$$\kappa = \frac{2(n-1)\pi^{n/2}\mathcal{G}}{(n-2)\Gamma(n/2)c^4}, \tag{2.5}$$

where Γ denotes the gamma function. We have obtained the generalized Einstein equation (2.4) with (2.5) for complex line elements.

Let us derive the line element (1.5) as the solution of the Einstein equation (2.4). We assume that the space is uniform and isotropic, and we consider the line element

$$g_{\alpha\beta}dz^\alpha dz^\beta := -c^2(dz^0)^2 + e^{h(z^0)}e^{f(r)}\sum_{j=1}^n (dz^j)^2, \tag{2.6}$$

where h and f are complex-valued functions. This line element is uniform in the sense that for any two points P and Q in $\mathbb{C}^n$, the ratio of the coefficients $e^{h(z^0)}e^{f(r(P))}/e^{h(z^0)}e^{f(r(Q))}$ is independent of z^0.

By direct calculations, we have $G^0{}_j = G^j{}_0 = 0$,

$$G^0{}_0 := \frac{n-1}{2c^2}\left\{\frac{n}{4}(\partial_0 h)^2 - c^2 e^{-h-f}\left(f'' + (n-1)\frac{f'}{r} + \frac{n-2}{4}(f')^2\right)\right\},$$

and

$$G^j{}_k := g^j{}_k \left\{ \frac{n-1}{2c^2} \left(\partial_0^2 h + \frac{n}{4}(\partial_0 h)^2 \right) \right. \\ \left. - \frac{n-2}{2} e^{-h-f} \left(f'' + (n-2)\frac{f'}{r} + \frac{n-3}{4}(f')^2 \right) \right\} \\ + \frac{n-2}{2} e^{-h-f} \left(f'' - \frac{f'}{r} - \frac{(f')^2}{2} \right) \frac{z^j z^k}{r^2},$$

where $f' := df/dr$. Since the space is isotropic, the coefficient of $z^j z^k$ must vanish. So that, f must satisfy $f'' - f'/r - (f')^2/2 = 0$, by which we obtain

$$e^{f(r)} = q^2 \left(1 + \frac{k^2 r^2}{4} \right)^{-2} \tag{2.7}$$

for constants $q(\neq 0), k \in \mathbb{C}$. We define a function

$$a(z^0) := e^{h(z^0)/2}. \tag{2.8}$$

Let us consider the stress-energy tensor $T^\alpha{}_\beta$ of the perfect fluid

$$T^\alpha{}_\beta := \mathrm{diag}(\rho c^2, -p, \cdots, -p)$$

for constant density ρ and pressure p. We put $\widetilde{\rho} := \rho + \Lambda/\kappa c^2$ and $\widetilde{p} := p - \Lambda/\kappa$. Then (2.4) is rewritten as $G^\alpha{}_\beta = \kappa \cdot \mathrm{diag}(\widetilde{\rho} c^2, -\widetilde{p}, \cdots, -\widetilde{p})$. This equation shows that the cosmological constant $\Lambda > 0$ is regarded as the energy which has positive density and negative pressure in the vacuum $\rho = p = 0$ for $\kappa > 0$, by which we regard the cosmological constant Λ as "the dark energy." The equation $G^0{}_0 = \kappa \widetilde{\rho} c^2 g^0{}_0$ is rewritten as

$$\frac{n-1}{2} \left\{ \left(\frac{\partial_0 a}{ca} \right)^2 + \frac{k^2}{q^2 a^2} \right\} = \frac{\kappa c^2}{n} \cdot \widetilde{\rho}. \tag{2.9}$$

The equation $G^j{}_k = -\kappa \widetilde{p} g^j{}_k$ is rewritten as

$$\frac{n-1}{2} \left\{ \frac{2}{n-2} \cdot \frac{\partial_0^2 a}{c^2 a} + \left(\frac{\partial_0 a}{ca} \right)^2 + \frac{k^2}{q^2 a^2} \right\} = -\frac{\kappa}{n-2} \cdot \widetilde{p}, \tag{2.10}$$

which is rewritten as the Raychaudhuri equation

$$\frac{\partial_0^2 a}{c^2 a} = -\frac{n-2}{n-1} \cdot \kappa \left(\frac{\widetilde{\rho} c^2}{n} + \frac{\widetilde{p}}{n-2} \right) \tag{2.11}$$

by (2.9). Multiplying a^n to the both sides in (2.9), taking the derivative by z^0 variable, and using (2.10), we have the conservation of the mass

$$\partial_0(\widetilde{\rho} c^2 a^n) + \widetilde{p} \partial_0 a^n = 0. \tag{2.12}$$

For any number σ, we assume the equation of state

$$\widetilde{p} = \sigma \widetilde{\rho} c^2. \tag{2.13}$$

Then $a(z^0)$ must satisfy

$$\frac{\partial_0^2 a(z^0)}{c^2 a(z^0)} = -\frac{n-2+n\sigma}{n(n-1)} \cdot \kappa \widetilde{\rho} c^2$$

with

$$\widetilde{\rho} = \frac{n-1}{2} \cdot \frac{n}{\kappa c^4} \cdot \left(\frac{\partial_0 a(0)}{a(0)} \right)^2 \cdot \left(\frac{a(0)}{a(z^0)} \right)^{n(1+\sigma)} \tag{2.14}$$

by (2.11) and (2.12). We consider the solution which has the curvature $k = 0$ given by

$$a(z^0) := \begin{cases} a(0) \left(1 + \frac{n(1+\sigma)\partial_0 a(0) z^0}{2a(0)} \right)^{2/n(1+\sigma)} & \text{if } \sigma \neq -1, \\ a(0) \exp\left(\frac{\partial_0 a(0) z^0}{a(0)} \right) & \text{if } \sigma = -1. \end{cases} \tag{2.15}$$

Let us derive the equations (1.8) and (1.9). For any $\lambda \in \mathbb{C}$ and any complex-valued C^2 function ϕ on $\mathbb{M}^{1+n}$, we define the Lagrangian

$$L(\phi) := -\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \, \partial_\beta \phi - \frac{1}{2} \left(\frac{mc}{\hbar} \right)^2 \phi^2 + \frac{\lambda}{p+1} |\phi|^{p-1} \phi^2.$$

Then with the constraint condition $\arg \delta\phi = \arg \phi$, the Euler-Lagrange equation for the action $\int_{\mathbb{M}^{1+n}} L(\phi)(-g)^{1/2} dz$ is given by

$$\frac{1}{(-g)^{1/2}} \partial_\alpha ((-g)^{1/2} g^{\alpha\beta} \partial_\beta \phi) - \left(\frac{mc}{\hbar} \right)^2 \phi + \lambda |\phi|^{p-1} \phi = 0, \tag{2.16}$$

which is rewritten as the equation (1.8). For any constant $b_0 \in \mathbb{C}$, we define a weight function $w(z^0)$ and a function $b(z^0)$ by

$$w(z^0) := b_0 \left(\frac{a(0)}{a(z^0)} \right)^{n/2}, \qquad b(z^0) := w(z^0) \exp\left(\mp i \frac{mc^2}{\hbar} z^0 \right), \tag{2.17}$$

where we note $b(0) = b_0$. We transform ϕ to u by the equation

$$\phi(z^0, \cdots, z^n) = u(z^0, \cdots, z^n) b(z^0). \tag{2.18}$$

We assume $mz^0/\hbar \in \mathbb{R}$. Then the nonrelativistic limit ($c \to \infty$) of this equation yields (1.9).

3 The Cauchy problem

In this section, we introduce some results on the Cauchy problem of the equation (1.9) without proofs. The equation (1.9) is rewritten as

$$\pm i \frac{2m}{\hbar} \partial_t u + \frac{1}{a^2 e^{2i\omega}} \Delta u - \frac{\lambda}{e^{2i\omega}} |uw|^{p-1} u = 0 \tag{3.1}$$

by a transformation (1.7) with $t := x^0$, $\omega^0 = 0$, $\omega := \omega^1 = \cdots = \omega^n$. We consider how the spatial variance affects the existence of the solutions. Let T_0 be the maximal existence time of the scale-function $a(\cdot)$ defined by (2.15). Since the equation (3.1) has a variable coefficient, we use a change of variable $s = s(t) := \int_0^t a(\tau)^{-2} d\tau$. We put $S_0 := s(T_0)$. We use conventions $a(s) := a(t(s))$ and $w(s) := w(t(s))$ for $s \in [0, S_0)$ as far as there is no fear of confusion. A direct computation shows

$$S_0 = \begin{cases} \frac{2}{a_0 a_1 (4 - n(1+\sigma))} & \text{if } a_1 (4 - n(1+\sigma)) > 0, \\ \infty & \text{if } a_1 (4 - n(1+\sigma)) \leq 0. \end{cases}$$

For $0 \leq \mu_0 < n/2$ and $0 < S \leq S_0$, we consider the Cauchy problem given by

$$\begin{cases} \pm i \dfrac{2m}{\hbar} \partial_s u(s,x) + \dfrac{1}{e^{2i\omega}} \Delta u(s,x) - \dfrac{\lambda a(s)^2}{e^{2i\omega}} \left(|uw|^{p-1} u\right)(s,x) = 0, \\ u(0,\cdot) = u_0(\cdot) \in H^{\mu_0}(\mathbb{R}^n) \end{cases} \tag{3.2}$$

for $(s,x) \in [0,S) \times \mathbb{R}^n$, where $H^{\mu_0}(\mathbb{R}^n)$ denotes the Sobolev space of order $\mu_0 \geq 0$. Since $u = u(t,\cdot)$ is a global solution of (3.1) if it exists on $[0,T_0)$, we say $u = u(s,\cdot) = u(t(s),\cdot)$ is a global solution of (3.2) if it exists on $[0, S_0)$.

Let us consider the well-posedness of (3.2). For any real numbers $2 \leq q \leq \infty$ and $2 \leq r < \infty$, we say that the pair (q,r) is admissible if it satisfies $1/r + 2/nq = 1/2$. For $\mu_0 \geq 0$ and two admissible pairs $\{(q_j, r_j)\}_{j=1,2}$, we define a function space

$$X^{\mu_0}([0,S)) := \{u \in C([0,S), H^{\mu_0}(\mathbb{R}^n));\ \max_{\mu=0,\mu_0} \|u\|_{X^\mu([0,S))} < \infty\}$$

with a metric $d(u,v) := \|u - v\|_{X^0([0,S))}$ for u, $v \in X^{\mu_0}([0,S))$, where

$$\|u\|_{X^\mu([0,S))} := \begin{cases} \|u\|_{L^\infty((0,S),L^2(\mathbb{R}^n)) \cap \bigcap_{j=1,2} L^{q_j}((0,S),L^{r_j}(\mathbb{R}^n))} & \text{if } \mu = 0, \\ \|u\|_{L^\infty((0,S),\dot{H}^\mu(\mathbb{R}^n)) \cap \bigcap_{j=1,2} L^{q_j}((0,S),\dot{B}^\mu_{r_j 2}(\mathbb{R}^n))} & \text{if } \mu > 0. \end{cases}$$

Here, $\dot{H}^\mu(\mathbb{R}^n)$ and $\dot{B}^\mu_{r_j 2}(\mathbb{R}^n)$ denote the homogeneous Sobolev and Besov spaces, respectively. Since the propagator of the linear part of the first equation in (3.2) is written as $\exp(\pm i\hbar \exp(-2i\omega) s\Delta/2m)$, we assume $0 \leq \pm\omega \leq \pi/2$ to define it as a pseudo-differential operator. We note that the scaling critical number of p for (3.2) is $p(\mu_0) := 1 + 4/(n - 2\mu_0)$ when $a(\cdot) = 1$. We put

$$p_1(\mu_0) := 1 + \frac{4}{n - 2\mu_0} \cdot \left(1 + \frac{4}{n - 2\mu_0} \cdot \frac{2\mu_0}{n(1+\sigma)}\right)^{-1}$$

for $\sigma \neq -1$.

We have the following results for time-local, time-global and blowing-up solutions for the problem (3.2).

Theorem 3.1. *Let $n \geq 1$, $\lambda \in \mathbb{C}$, $0 \leq \mu_0 < n/2$, and $1 \leq p \leq p(\mu_0)$. Let ω satisfies $0 \leq \pm\omega \leq \pi/2$ and $\omega \neq -\pi/2$. Assume $\mu_0 < p$ if p is not an odd number. There exist two admissible pairs $\{(q_j, r_j)\}_{j=1,2}$ with the following properties.*

(1) (Local solutions.) For any $u_0 \in H^{\mu_0}(\mathbb{R}^n)$, there exist $S > 0$ with $S \le S_0$ and a unique local solution u of (3.2) in $X^{\mu_0}([0,S))$. Here, S depends only on the norm $\|u_0\|_{\dot{H}^{\mu_0}(\mathbb{R}^n)}$ when $p < p(\mu_0)$, while S depends on the profile of u_0 when $p = p(\mu_0)$. The solutions depend on the initial data continuously.

(2) (Small global solutions.) Assume that one of the following conditions from (i) to (vi) holds: (i) $\mu_0 = 0$, $p = p(0)$, (ii) $\mu_0 > 0$, $p = p(\mu_0)$, $a_1 \ge 0$, (iii) $1 < p < p(\mu_0)$, $a_1 > 0$, $\sigma < -1$, (iv) $1 < p < p_1(\mu_0)$, $a_1 < 0$, $\sigma > -1$, (v) $p_1(\mu_0) < p < p(\mu_0)$, $a_1 > 0$, $\sigma > -1$, (vi) $\mu_0 > 0$, $1 < p < p(\mu_0)$, $a_1 > 0$, $\sigma = -1$. If $\|u_0\|_{\dot{H}^{\mu_0}(\mathbb{R}^n)}$ is sufficiently small, then the solution u obtained in (1) is a global solution, namely, $S = S_0$.

Corollary 3.2. *Let $\mu_0 = 0$ or $\mu_0 = 1$. Let $\lambda > 0$. Let $1 \le p < 1 + 4/n$ when $\mu_0 = 0$. Let $1 \le p < 1 + 4/(n-2)$ and $a_1(p - 1 - 4/n) \ge 0$ when $\mu_0 = 1$. For any $u_0 \in H^{\mu_0}(\mathbb{R}^n)$, the local solution u given by (1) in Theorem 3.1 is a global solution.*

Corollary 3.3. *Let $\mu_0 = 1$, $\lambda < 0$, $a_1 \ge 0$ and $1 \le p < 1 + 4/n$. Let $\omega = 0$ or $\omega = \pi/2$. For any $u_0 \in H^1(\mathbb{R}^n)$, the local solution u given by (1) in Theorem 3.1 is a global solution.*

Corollary 3.4. *Let $\mu_0 = 1$ and $\lambda < 0$. Let $\omega \ne 0, \pi/2$. Put $p_0 := 2/(\sin 2\omega)^2 - 1$. Let $p_0 < p \le 1 + 4/(n-2)$. Let $a_1(p - 1 - 4/n) \le 0$ and $S_0 = \infty$. For any $u_0 \in H^1(\mathbb{R}^n)$ with negative energy*

$$\int_{\mathbb{R}^n} \frac{1}{2}|\nabla u_0(x)|^2 + \frac{\lambda a_0^2 |u_0(x)|^{p+1}}{p+1} dx < 0, \tag{3.3}$$

the solution u given by (1) in Theorem 3.1 blows up in finite time.

Corollary 3.5. *Let $\mu_0 = 1$ and $\lambda < 0$. Let $\omega = 0$ or $\omega = \pi/2$. Let $1 + 4/n \le p \le 1 + 4/(n-2)$. Let $a_1 \le 0$ and $S_0 = \infty$. For any $u_0 \in H^1(\mathbb{R}^n)$ which satisfies $\||x|u_0(x)\|_{L^2_x(\mathbb{R}^n)} < \infty$ and (3.3), the solution u given by (1) in Theorem 3.1 blows up in finite time.*

4 Equations for vectors

So far, we have considered the partial differential equations for scalars. Our equations (1.8) and (1.9) are obtained as the Euler-Lagrange equation for a Lagrangian and its non-relativistic limit. To consider the equations for vectors such as Navier-Stokes equations and elastic wave equations, we are based on the classical method by Landau and Eckart. We note that the stress-energy tensor $T^\alpha{}_\beta$ must satisfy the conservation law

$$\nabla_\alpha T^\alpha{}_\beta = 0$$

in the Einstein equation (2.4). We introduce the stress tensor $P^{\alpha\beta}$. Let λ, μ be two constants. Let p be the pressure. Let v^α be a contravariant tensor which satisfies

$\lim_{c\to\infty} \partial_j v^0 = 0$. Put

$$P^{\alpha\beta} := -pg^{\alpha\beta} + \lambda g^{\alpha\beta}\nabla_\gamma v^\gamma + \mu(\nabla^\alpha v^\beta + \nabla^\beta v^\alpha). \tag{4.1}$$

Then the nonrelativistic limit yields the equation

$$\lim_{c\to\infty} \nabla_\alpha P^{\alpha\beta} = -\partial^\beta p + \mu\partial_j\partial^j v^\beta + (\mu+\lambda)\partial^\beta\partial_j v^j,$$

where we regard $\partial^0 := -\frac{1}{c^2}\partial_0 = 0$ in the RHS.

4.1 Navier-Stokes equations

Let us consider the Navier-Stokes equation. Since any velocity tensor u^α must satisfy

$$-c^2 = -c^2(u^0)^2 + a(x^0)^2\sum_{j=1}^n (u^j)^2,$$

we have $\lim_{c\to\infty} u^0 = \pm 1$. Based on this, we assume

$$\lim_{c\to\infty} u^0 = 1 \quad\text{and}\quad \lim_{c\to\infty} \partial_\alpha u^0 = 0 \quad\text{for}\quad 0 \le \alpha \le n. \tag{4.2}$$

Let $P^{\alpha\beta}$ be the stress tensor with $v^\alpha := u^\alpha$. Put

$$T^{\alpha\beta} := \left(\rho + \frac{p}{c^2}\right) u^\alpha u^\beta - P^{\alpha\beta}. \tag{4.3}$$

Put $f^k := u^j\partial_j u^k$. Let $\rho(x^0) := C/a(x^0)^n$ (the density of mass). Then

$$\lim_{c\to\infty} \nabla_\alpha T^{\alpha\beta} = 0 \tag{4.4}$$

are equivalent to

$$\partial_j u^j = 0 \tag{4.5}$$

and

$$\partial_0 u^k + f^k + \frac{\partial_0 a^2}{a^2}u^k + \frac{1}{\rho}\partial^k p - \frac{\mu}{\rho}\partial_j\partial^j u^k = 0. \tag{4.6}$$

For the equation (4.6) with (4.5), the energy estimate

$$\partial_0 e^0 + \partial_j e^j + e_* = 0 \tag{4.7}$$

holds, where

$$e^0 := \frac{1}{2}u_k u^k, \qquad J^j := \frac{1}{2}u^j u_k u^k - u^j\partial_k\frac{1}{\partial^\ell\partial_\ell}f^k + \partial_k u^k\frac{1}{\partial^\ell\partial_\ell}f^j,$$

$$e^j := J^j - \frac{\mu}{\rho}u_k\partial^j u^k, \qquad K := \frac{1}{2}\left(\frac{\partial_0 a^2}{a^2} - \partial_j u^j\right),$$

$$e_* := K u_k u^k + \frac{\mu}{\rho}\partial^\ell u_k\partial_\ell u^k.$$

Here, K satisfies

$$K = \frac{\partial_0 a^2}{2a^2}. \tag{4.8}$$

4.2 The elastic wave equations

Let us consider the elastic wave equations. For the displacement tensor r^α, and its relativistic velocity $u^\alpha := dr^\alpha/d\tau$, we put

$$dr^\alpha := r^\alpha(\tau + d\tau) - r^\alpha(\tau).$$

Since dr must satisfy

$$-c^2(d\tau)^2 = -c^2(dr^0)^2 + a(x^0)^2 \sum_{j=1}^{n} (dr^j)^2,$$

we have $\lim_{c\to\infty} dr^0/d\tau = \pm 1$. Based on this, we assume

$$\lim_{c\to\infty} r^0 = x^0, \quad \text{and} \quad \lim_{c\to\infty} \partial_j r^0 = 0 \quad \text{for} \quad 1 \le j \le n. \tag{4.9}$$

Let $P^{\alpha\beta}$ be the stress tensor with $v^\alpha := r^\alpha$. We put $u^\alpha := dr^\alpha/d\tau$,

$$T^{\alpha\beta} := \left(\rho + \frac{p}{c^2}\right) u^\alpha u^\beta - P^{\alpha\beta}, \tag{4.10}$$

and $h^k := \partial_0 r^j \partial_j \partial_0 r^k$. Put $\rho(x^0) := C/a(x^0)^n$ (density of mass). Then

$$\lim_{c\to\infty} \nabla_\alpha T^{\alpha\beta} = 0 \tag{4.11}$$

are equivalent to

$$\partial_0 \partial_j r^j = 0 \tag{4.12}$$

and

$$\partial_0^2 r^k + h^k + \frac{\partial_0 a^2}{a^2} \partial_0 r^k + \frac{1}{\rho} \partial^k p - \frac{\mu}{\rho} \partial_j \partial^j r^k - \frac{\mu + \lambda}{\rho} \partial^k \partial_j r^j = 0. \tag{4.13}$$

We put

$$(\partial_\tau r)_\alpha := g_{\alpha\beta} \partial_\tau r^\beta, \quad K := \frac{\partial_0 a^2}{2a^2}. \tag{4.14}$$

For the equation (4.13) with (4.12), the energy estimate

$$\partial_0 e^0 + \partial_j e^j + e_* = 0 \tag{4.15}$$

holds, where

$$e^0 := \frac{1}{2} (\partial_0 r)_k \partial_0 r^k + \frac{\mu}{2\rho} \partial_j r_k \partial^j r^k - \frac{\mu}{2\rho} (\partial_j r^j)^2,$$

$$e^j := \frac{1}{2} (\partial_0 r^j)(\partial_0 r)_k \partial_0 r^k - \partial_0 r^j \partial_k \frac{1}{\partial_\ell \partial^\ell} h^k + \partial^k (\partial_0 r)_k \frac{1}{\partial_\ell \partial^\ell} h^j - \frac{\mu}{\rho} (\partial_0 r)_k \partial^j r^k + \frac{\mu}{\rho} \partial_0 r^j \partial_k r^k,$$

$$e_* := K \cdot (\partial_0 r)_k \partial_0 r^k + \frac{4\mu}{\rho} \cdot K \partial_\ell r_k \partial^\ell r^k.$$

Remark 4.1. *In the energy estimates* (4.7) *and* (4.15), *the energy density* e_* *shows the dissipative or anti-dissipative property. When the space is expanding, namely,* $\partial_0 a(\cdot) > 0$, *then* K *is a positive function, which shows that* e_* *is positive for non-zero velocity* u *and non-constant displacement* r. *From this fact, we expect the strong dissipative effects on the solutions of the equations by the spatial expansion.*

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